

HODGE POLYNOMIALS OF THE MODULI SPACES OF PAIRS

V. MUÑOZ, D. ORTEGA, AND M.J. VÁZQUEZ-GALLO

ABSTRACT. Let X be a smooth projective curve of genus $g \geq 2$ over the complex numbers. A holomorphic pair on X is a couple (E, ϕ) , where E is a holomorphic bundle over X of rank n and degree d , and $\phi \in H^0(E)$ is a holomorphic section. In this paper, we determine the Hodge polynomials of the moduli spaces of rank 2 pairs, using the theory of mixed Hodge structures. We also deal with the case in which E has fixed determinant.

1. INTRODUCTION

Let X be a smooth projective curve of genus $g \geq 2$ over the field of complex numbers. Let $M(n, d)$ be the moduli space of polystable vector bundles over X of rank n and degree d . Similarly, we denote by $M(n, \Lambda)$ the moduli space of polystable vector bundles of rank n and determinant isomorphic to a fixed line bundle Λ of degree d on X . In [11] Desale and Ramanan gave an inductive method to determine the Betti numbers of $M(n, \Lambda)$ for n and d coprime. Later, Earl and Kirwan [12] extended the method to determine the Hodge numbers of $M(n, d)$ and $M(n, \Lambda)$, for n and d coprime.

A *holomorphic pair* on X consists of a couple (E, ϕ) , where E is a holomorphic bundle over X of rank n and degree d , and $\phi \in H^0(E)$ is a non-zero holomorphic section. There is a concept of stability for a pair which depends on the choice of a parameter $\tau \in \mathbb{R}$. Denote by $\mathfrak{M}_\tau(n, d)$ the moduli space of τ -polystable holomorphic pairs, and $\mathfrak{M}_\tau(n, \Lambda)$ the moduli space of τ -polystable holomorphic pairs (E, ϕ) , where the determinant of E is isomorphic to a fixed line bundle Λ of degree d on X . These moduli spaces, studied in [2, 3, 13], are smooth for any rank n at τ -stable points.

In [19] Thaddeus studied how the moduli spaces $\mathfrak{M}_\tau(2, \Lambda)$ change when varying the parameter τ . The range of the parameter τ is an interval $J \subset \mathbb{R}$ split by a finite number of *critical values* τ_c in such a way that, when τ moves without crossing a critical value, $\mathfrak{M}_\tau(2, \Lambda)$ remains unchanged. When τ crosses a critical value, $\mathfrak{M}_\tau(2, \Lambda)$ undergoes a flip (in the sense of Mori theory), consisting of blowing-up an embedded subvariety and then blowing-down the exceptional divisor in a different way. The study of this process is what allows us to obtain information on the topology of all moduli spaces $\mathfrak{M}_\tau(2, \Lambda)$, for any τ , once one knows such information for one particular $\mathfrak{M}_\tau(2, \Lambda)$ (usually the one corresponding to the minimum or maximum possible value of the parameter). This was used in [19] to compute the Poincaré polynomial (i.e., the Betti numbers) of $\mathfrak{M}_\tau(2, \Lambda)$. The argument requires an explicit construction of the blow-up and blow-down centers of the flips, as well as the corresponding normal bundles.

2000 *Mathematics Subject Classification*. Primary: 14F45. Secondary: 14D20, 14H60.

Key words and phrases. Moduli space, complex curve, vector bundle, stable triple, Hodge numbers.

First and third authors partially supported through grant MCyT (Spain) MTM2004-07090-C03-01/02.

In this paper, we determine the Hodge polynomials (i.e., the Hodge numbers) of the moduli spaces $\mathfrak{M}_\tau(2, d)$ and $\mathfrak{M}_\tau(2, \Lambda)$ without describing these blow-ups and blow-downs. We use the theory of mixed Hodge structures introduced by Deligne [8].

In order to set up the theory in a more general framework, we work with triples instead of pairs. A *holomorphic triple* $T = (E_1, E_2, \phi)$ on X consists of two holomorphic vector bundles E_1 and E_2 over X (of ranks n_1 and n_2 , and degrees d_1 and d_2 , respectively) and a holomorphic map $\phi: E_2 \rightarrow E_1$. There is a concept of stability for a triple which depends, as for pairs, on the choice of a parameter $\sigma \in \mathbb{R}$. This gives a collection of moduli spaces $\mathcal{N}_\sigma(n_1, n_2, d_1, d_2)$, which have been studied in [4, 5, 14]. Again, the range of the parameter σ is an interval $I \subset \mathbb{R}$ split by a finite number of critical values. The moduli spaces change when we cross a critical value by the removal of a subvariety and the insertion of a new subvariety. By analogy with the situation of pairs, we call this process a *flip*, although it does not consist of a blow-up and a blow-down in general. The subvarieties that are removed and inserted are called the *flip loci*.

When the rank of E_2 is one, we recover the moduli spaces of pairs. More precisely, there is an isomorphism

$$\mathcal{N}_\sigma(2, 1, d_1, d_2) \cong \mathfrak{M}_\tau(2, d_1 - 2d_2) \times \text{Jac}^{d_2} X,$$

where τ and σ are related by $\tau = \frac{1}{3}(\sigma + d_1 - 2d_2)$.

Our main result is the following:

Theorem 1.1. *Let X be a smooth projective curve of genus $g \geq 2$. Let $\tau \in J$ be a non-critical value for the moduli space $\mathfrak{M}_\tau(2, d)$. Then $\mathfrak{M}_\tau(2, d)$ is a smooth projective variety with Hodge polynomial*

$$e(\mathfrak{M}_\tau(2, d)) = \text{coeff}_{x^0} \left[\frac{(1+u)^g(1+v)^g(1+ux)^g(1+vx)^g}{(1-uv)(1-x)(1-uvx)x^{d-1-[\tau]}} \left(\frac{(uv)^{d-1-[\tau]}}{1-(uv)^{-1}x} - \frac{(uv)^{g+1-d+2[\tau]}}{1-(uv)^2x} \right) \right].$$

For a fixed line bundle Λ on X of degree d and non-critical value $\tau \in J$, the moduli space $\mathfrak{M}_\tau(2, \Lambda)$ is a smooth projective variety with Hodge polynomial

$$e(\mathfrak{M}_\tau(2, \Lambda)) = \text{coeff}_{x^0} \left[\frac{(1+ux)^g(1+vx)^g}{(1-uv)(1-x)(1-uvx)x^{d-1-[\tau]}} \left(\frac{(uv)^{d-1-[\tau]}}{1-(uv)^{-1}x} - \frac{(uv)^{g+1-d+2[\tau]}}{1-(uv)^2x} \right) \right].$$

In Section 2 we review the basics of Hodge-Deligne theory. Sections 3 and 4 recall standard results on σ -stable triples used throughout the paper and give the description of the flip loci. We particularize to triples of rank $(n_1, n_2) = (2, 1)$ in Section 5. In Section 6 we perform the computation of the Hodge polynomials of the moduli spaces of triples of ranks $(2, 1)$ and give a proof of Theorem 1.1. For completeness, Section 7 deals with triples of rank $(1, 2)$. Finally, in Section 8, we use the Hodge polynomial of the moduli spaces of triples to recover the (already known) Hodge polynomial of the moduli space of rank 2 stable vector bundles of odd degree.

Acknowledgements: First author would like to thank Peter Gothen and Marina Logares for discussions. We are grateful to the referee for a careful reading of the manuscript.

2. HODGE POLYNOMIALS

Let us start by recalling the Hodge-Deligne theory of algebraic varieties over \mathbb{C} . Let H be a finite-dimensional complex vector space. A *pure Hodge structure of weight k* on H is a

decomposition

$$H = \bigoplus_{p+q=k} H^{p,q}$$

such that $H^{q,p} = \overline{H^{p,q}}$, the bar denoting complex conjugation in H . (A Hodge structure is usually defined over the field of the rational numbers, meaning that we have a vector space H over \mathbb{Q} whose complexification $H_{\mathbb{C}} = H \otimes \mathbb{C}$ is endowed with a Hodge decomposition as above. Since we shall not need this, we limit ourselves to using Hodge structures over \mathbb{C} .) The k -dimensional cohomology $H^k(Z)$ of a compact Kähler manifold Z (in particular an algebraic smooth projective variety) has a natural Hodge structure of weight k (here we denote $H^k(Z)$ for the cohomology with complex coefficients, although the result also holds for the cohomology with rational coefficients). We denote

$$h^{p,q}(Z) = \dim H^{p,q} ,$$

which is called the Hodge number of type (p, q) . A Hodge structure of weight k on H gives rise to the so-called *Hodge filtration* F on H , where

$$F^p = \bigoplus_{s \geq p} H^{s, k-s} ,$$

which is a descending filtration. Note that $\text{Gr}_F^p H = F^p / F^{p+1} = H^{p,q}$.

Let H be a complex vector space. A (*mixed*) *Hodge structure* over H consists of an ascending weight filtration W on H and a descending Hodge filtration F on H such that F induces a pure Hodge filtration of weight k on each $\text{Gr}_k^W H = W_k / W_{k-1}$. Again we define

$$h^{p,q}(H) = \dim H^{p,q} , \quad \text{where} \quad H^{p,q} = \text{Gr}_F^p \text{Gr}_{p+q}^W H .$$

A morphism between two Hodge structures H, H' is a linear map $f : H \rightarrow H'$ compatible with both the weight and Hodge filtrations. The most relevant fact is that morphisms of Hodge structures are strictly compatible with the filtrations W and F (see [8]). From this it follows that the functor $H \mapsto H^{p,q}$ is exact.

Deligne has shown [8] that, for each complex algebraic variety Z , the cohomology $H^k(Z)$ carries a natural Hodge structure, with weight filtration W and Hodge filtration F , which coincides with the classical (pure) Hodge structure when Z is a smooth projective variety. The associated graded objects of these filtrations are denoted Gr^W and Gr_F , respectively. We define the following Euler characteristics, by using these filtrations:

$$\chi_{p,q}(Z) = \sum_k (-1)^k \dim \text{Gr}_F^p \text{Gr}_{p+q}^W H^k(Z).$$

The cohomology groups with compact support $H_c^k(Z)$ also carry a natural mixed Hodge structure [8], which allows one to define $\chi_{p,q}^c$, using $H_c^k(Z)$ instead of $H^k(Z)$. If Z is smooth of dimension n , then Poincaré duality implies that

$$\chi_{p,q}^c(Z) = \chi_{n-p, n-q}(Z).$$

Definition 2.1. For *any* complex algebraic variety Z (not necessarily smooth, compact or irreducible), we define its *Hodge polynomial* [10] as

$$e(Z) = e(Z)(u, v) = \sum_{p,q} (-1)^{p+q} \chi_{p,q}^c(Z) u^p v^q .$$

Note that if Z is smooth and projective then the mixed Hodge structure on $H_c^k(Z)$ is pure of weight k . This means that $\mathrm{Gr}_k^W H_c^k(Z) = H_c^k(Z) = H^k(Z)$ and the other pieces $\mathrm{Gr}_m^W H_c^k(Z) = 0$, $m \neq k$. So

$$\chi_{p,q}(Z) = \chi_{p,q}^c(Z) = (-1)^{p+q} h^{p,q}(Z),$$

where $h^{p,q}(Z)$ is the usual Hodge number of Z . In this case,

$$e(Z)(u, v) = \sum_{p,q} h^{p,q}(Z) u^p v^q$$

is the (usual) Hodge polynomial of Z . Note that in this case, the Poincaré polynomial of Z is

$$P_t(Z) = \sum_k b^k(Z) t^k = \sum_k \left(\sum_{p+q=k} h^{p,q}(Z) \right) t^k = e(Z)(t, t).$$

where $b^k(Z)$ is the k -th Betti number of Z .

The following result is a slight extension of that of [9].

Theorem 2.2. *Let Z be a complex algebraic variety. Suppose that Z is a finite disjoint union $Z = Z_1 \sqcup \cdots \sqcup Z_n$, where the Z_i are algebraic subvarieties. Then*

$$\chi_{p,q}^c(Z) = \sum_i \chi_{p,q}^c(Z_i).$$

Equivalently, $e(Z) = \sum_i e(Z_i)$.

Proof: It is enough to see that for a complex quasi-projective variety Z , a closed subvariety Y of Z and $U = Z - Y$, we have

$$e(Z) = e(Y) + e(U).$$

To see this, consider the long exact sequence

$$\cdots \rightarrow H_c^k(U) \rightarrow H_c^k(Z) \rightarrow H_c^k(Y) \rightarrow H_c^{k+1}(U) \rightarrow \cdots$$

of cohomology groups with compact supports. The maps in this exact sequence are compatible with the weight and Hodge filtrations. The induced sequence on the (p, q) -pieces of the Hodge decomposition remains exact for all p, q :

$$\cdots \rightarrow \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H_c^k(U) \rightarrow \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H_c^k(Z) \rightarrow \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H_c^k(Y) \rightarrow \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H_c^{k+1}(U) \rightarrow \cdots$$

This means that the Euler characteristics $\chi_{p,q}^c$ satisfy

$$\chi_{p,q}^c(Z) = \chi_{p,q}^c(Y) + \chi_{p,q}^c(U),$$

from where $e(Z) = e(Y) + e(U)$ follows. \square

Let us give a collection of simple examples:

- Let $Z = \mathbb{P}^n$, then $e(Z) = 1 + uv + (uv)^2 + \cdots + (uv)^n = (1 - (uv)^{n+1}) / (1 - uv)$. For future reference, we shall denote

$$e_n := e(\mathbb{P}^{n-1}) = e(\mathbb{P}(\mathbb{C}^n)) = \frac{1 - (uv)^n}{1 - uv}. \quad (2.1)$$

- Let $\text{Jac}^d X$ be the Jacobian of (any) degree d of a (smooth, projective) complex curve X of genus g . Then

$$e(\text{Jac}^d X) = (1 + u)^g(1 + v)^g. \quad (2.2)$$

- Let X be a curve (smooth, projective) complex curve of genus g , and $k \geq 1$. The Hodge polynomial of the symmetric product $\text{Sym}^k X$ is computed in [6],

$$e(\text{Sym}^k X) = \text{coeff}_{x^0} \frac{(1 + ux)^g(1 + vx)^g}{(1 - x)(1 - uvx)x^k}. \quad (2.3)$$

Lemma 2.3. *Suppose that $\pi : Z \rightarrow Y$ is an algebraic fiber bundle with fiber F which is locally trivial in the Zariski topology, then $e(Z) = e(F)e(Y)$. (In particular this is true for $Z = F \times Y$.)*

Proof: First we need to see that $e(F \times Y) = e(F)e(Y)$. Using Theorem 2.2, we see that it is enough to see it for quasi-projective smooth varieties F and Y . We may prove it by induction on the dimension of $F \times Y$. First, if both F and Y are smooth and projective, then $e(F \times Y) = e(F)e(Y)$ by usual Hodge theory. Second, assume that F is smooth projective and Y is smooth quasi-projective. Then write $Y = Y_1 - Y_2$, where Y_1 is a smooth projective variety and $Y_2 \subset Y_1$ is a smaller dimensional subvariety (this is possible by Hironaka's result on embedded resolution of singularities). Then $e(F \times Y) = e(F \times Y_1) - e(F \times Y_2) = e(F)e(Y_1) - e(F)e(Y_2) = e(F)(e(Y_1) - e(Y_2)) = e(F)e(Y)$. The general case, when F and Y are both smooth quasi-projective, now follows from the previous case by writing $F = F_1 - F_2$, with F_1 smooth projective variety and $F_2 \subset F_1$ a smaller dimensional subvariety.

Decompose $Y = Y_1 \sqcup \dots \sqcup Y_n$ into a finite union of disjoint locally closed subvarieties such that $Z_i = \pi^{-1}(Y_i) \cong F \times Y_i$ (using the local triviality in the Zariski topology). Then $e(Z) = \sum e(Z_i) = \sum e(F)e(Y_i) = e(F)e(Y)$. \square

Remark 2.4. Consider the abelian group $\mathcal{V}ar$ generated by all algebraic varieties subject to the following equivalence relation: if $Z = Z_1 \sqcup Z_2$, then set $[Z] = [Z_1] + [Z_2]$. The cartesian product of varieties yields an algebra structure for $\mathcal{V}ar$. For instance, for a (locally trivial in the Zariski topology) bundle $Z \rightarrow Y$ with fiber F , we have $[Z] = [F] \cdot [Y]$. The Hodge polynomial $[Z] \mapsto e(Z)$ gives a morphism (of algebras) from $\mathcal{V}ar$ to $\mathbb{Z}[u, v]$.

3. MODULI SPACES OF TRIPLES

Let X be a smooth projective curve over the complex numbers of genus $g \geq 2$. A *holomorphic triple* $T = (E_1, E_2, \phi)$ on X consists of two holomorphic vector bundles E_1 and E_2 over X , of ranks n_1 and n_2 and degrees d_1 and d_2 , respectively, and a holomorphic map $\phi: E_2 \rightarrow E_1$. We refer to (n_1, n_2, d_1, d_2) as the *type* of T , to (n_1, n_2) as the *rank* of T , and to (d_1, d_2) as the *degree* of T .

A homomorphism from $T' = (E'_1, E'_2, \phi')$ to $T = (E_1, E_2, \phi)$ is a commutative diagram

$$\begin{array}{ccc} E'_2 & \xrightarrow{\phi'} & E'_1 \\ \downarrow & & \downarrow \\ E_2 & \xrightarrow{\phi} & E_1, \end{array}$$

where the vertical arrows are homomorphisms. A triple $T' = (E'_1, E'_2, \phi')$ is a subtriple of $T = (E_1, E_2, \phi)$ if $E'_1 \subset E_1$ and $E'_2 \subset E_2$ are subbundles, $\phi(E'_2) \subset E'_1$ and $\phi' = \phi|_{E'_2}$. A subtriple $T' \subset T$ is called *proper* if $T' \neq 0$ and $T' \neq T$. The quotient triple $T'' = T/T'$ is given by $E''_1 = E_1/E'_1$, $E''_2 = E_2/E'_2$ and $\phi'' : E''_2 \rightarrow E''_1$ being the map induced by ϕ . We usually denote by (n'_1, n'_2, d'_1, d'_2) and $(n''_1, n''_2, d''_1, d''_2)$, the types of the subtriple T' and the quotient triple T'' .

Definition 3.1. For any $\sigma \in \mathbb{R}$ the σ -slope of T is defined by

$$\mu_\sigma(T) = \frac{d_1 + d_2}{n_1 + n_2} + \sigma \frac{n_2}{n_1 + n_2} .$$

To shorten the notation, we define the μ -slope and λ -slope of the triple T as $\mu = \mu(E_1 \oplus E_2) = \frac{d_1 + d_2}{n_1 + n_2}$ and $\lambda = \frac{n_2}{n_1 + n_2}$, so that $\mu_\sigma(T) = \mu + \sigma\lambda$.

Remark 3.2. For any triple $T = (E_1, E_2, \phi)$, if $T' = (E'_1, E'_2, \phi') \subset T$ is a proper subtriple and $T'' = T/T'$ is the corresponding quotient, then only one of the following situations is possible:

- (1) $\mu_\sigma(T) = \mu_\sigma(T') = \mu_\sigma(T'')$;
- (2) $\mu_\sigma(T') < \mu_\sigma(T) < \mu_\sigma(T'')$;
- (3) $\mu_\sigma(T') > \mu_\sigma(T) > \mu_\sigma(T'')$.

In fact, letting $t = \frac{n'_1 + n'_2}{n_1 + n_2} \in (0, 1)$, the next equality holds

$$\mu_\sigma(T) = t \cdot \mu_\sigma(T') + (1 - t) \cdot \mu_\sigma(T'')$$

Also note the following equality of λ -slopes

$$\lambda = t \cdot \lambda' + (1 - t) \cdot \lambda'' , \tag{3.1}$$

where λ , λ' and λ'' are the λ -slopes of T , T' and T'' , respectively.

Definition 3.3. We say that a triple $T = (E_1, E_2, \phi)$ is σ -stable if

$$\mu_\sigma(T') < \mu_\sigma(T) \quad (\text{equivalently } \mu_\sigma(T/T') > \mu_\sigma(T)) ,$$

for any proper subtriple $T' = (E'_1, E'_2, \phi')$. We define σ -semistability by replacing the above strict inequality with a weak inequality. A triple is called σ -polystable if it is the direct sum of σ -stable triples of the same σ -slope. It is σ -unstable if it is not σ -semistable, and *strictly* σ -semistable if it is σ -semistable but not σ -stable. A σ -destabilizing subtriple $T' \subset T$ is a proper subtriple satisfying $\mu_\sigma(T') \geq \mu_\sigma(T)$.

We denote by

$$\mathcal{N}_\sigma = \mathcal{N}_\sigma(n_1, n_2, d_1, d_2)$$

the moduli space of σ -polystable triples $T = (E_1, E_2, \phi)$ of type (n_1, n_2, d_1, d_2) , and drop the type from the notation when it is clear from the context. This moduli space is constructed in [4] by using dimensional reduction. A direct construction is given by Schmitt [18] using geometric invariant theory. The subspace of σ -stable triples is denoted by $\mathcal{N}_\sigma^s = \mathcal{N}_\sigma^s(n_1, n_2, d_1, d_2)$.

Given a triple $T = (E_1, E_2, \phi)$ one has the dual triple $T^* = (E_2^*, E_1^*, \phi^*)$, where E_i^* is the dual of E_i and ϕ^* is the transpose of ϕ . The following is not difficult to prove.

Proposition 3.4. [4, Proposition 3.16] *The σ -(semi)stability of T is equivalent to the σ -(semi)stability of T^* . The map $T \mapsto T^*$ defines an isomorphism*

$$\mathcal{N}_\sigma(n_1, n_2, d_1, d_2) \cong \mathcal{N}_\sigma(n_2, n_1, -d_2, -d_1).$$

□

This can be used to restrict our study to $n_1 \geq n_2$ and appeal to duality for $n_1 < n_2$.

The particular case where $n_1 = 0$ or $n_2 = 0$ reduces to the case of bundles. Let $M(n, d)$ denote the moduli space of polystable vector bundles of rank n and degree d over X . This moduli space is projective. We also denote by $M^s(n, d)$ the open subset of stable bundles, which is smooth of dimension $n^2(g-1) + 1$. If $\gcd(n, d) = 1$, then $M(n, d) = M^s(n, d)$.

Lemma 3.5. *There are isomorphisms $\mathcal{N}_\sigma(n, 0, d, 0) \cong M(n, d)$ and $\mathcal{N}_\sigma^s(n, 0, d, 0) \cong M^s(n, d)$, for all $\sigma \in \mathbb{R}$. In particular, $\mathcal{N}_\sigma(1, 0, d, 0) = \mathcal{N}_\sigma^s(1, 0, d, 0) \cong \text{Jac}^d X$, for any $\sigma \in \mathbb{R}$.*

There are certain necessary conditions in order for σ -semistable triples to exist. Let $\mu_i = \mu(E_i) = d_i/n_i$ stand for the slope of E_i , for $i = 1, 2$. We write

$$\begin{aligned} \sigma_m &= \mu_1 - \mu_2, \\ \sigma_M &= \left(1 + \frac{n_1 + n_2}{|n_1 - n_2|}\right) (\mu_1 - \mu_2), \quad \text{if } n_1 \neq n_2. \end{aligned}$$

Proposition 3.6. [5] *The moduli space $\mathcal{N}_\sigma(n_1, n_2, d_1, d_2)$ is a complex projective variety. A necessary condition for $\mathcal{N}_\sigma(n_1, n_2, d_1, d_2)$ to be non-empty is*

$$\begin{aligned} 0 \leq \sigma_m \leq \sigma \leq \sigma_M, & \quad \text{if } n_1 \neq n_2, n_1 \neq 0, n_2 \neq 0, \\ 0 \leq \sigma_m \leq \sigma, & \quad \text{if } n_1 = n_2 \neq 0. \end{aligned}$$

We shall denote by $I \subset \mathbb{R}$ the following interval

$$I = \begin{cases} [\sigma_m, \sigma_M], & \text{if } n_1 \neq n_2, n_1 \neq 0, n_2 \neq 0, \\ [\sigma_m, \infty), & \text{if } n_1 = n_2 \neq 0, \\ \mathbb{R}, & \text{if } n_1 = 0 \text{ or } n_2 = 0. \end{cases}$$

To study the dependence of the moduli spaces \mathcal{N}_σ on the parameter, we need to introduce the concept of critical value.

Proposition 3.7. *Let $\sigma_0 \in I$ and let $T = (E_1, E_2, \phi) \in \mathcal{N}_{\sigma_0}(n_1, n_2, d_1, d_2)$ be a strictly σ_0 -semistable triple. Then one of the following conditions holds:*

- (1) *For all σ_0 -destabilizing subtriples $T' = (E'_1, E'_2, \phi')$, we have $\lambda' = \frac{n'_2}{n'_1 + n'_2} = \lambda = \frac{n_2}{n_1 + n_2}$. Then T is strictly σ -semistable for $\sigma \in (\sigma_0 - \varepsilon, \sigma_0 + \varepsilon)$, for some small $\varepsilon > 0$.*
- (2) *There exists a σ_0 -destabilizing subtriple $T' = (E'_1, E'_2, \phi')$ with $\lambda' = \frac{n'_2}{n'_1 + n'_2} \neq \lambda = \frac{n_2}{n_1 + n_2}$. Then either:*
 - (i) $\lambda' > \lambda$. Then for any $\sigma > \sigma_0$, T is σ -unstable.
 - (ii) $\lambda' < \lambda$. Then for any $\sigma < \sigma_0$, T is σ -unstable.

Conversely, if T is a triple such that it is σ -semistable for $\sigma < \sigma_0$ and σ -unstable for $\sigma > \sigma_0$ (resp. σ -semistable for $\sigma > \sigma_0$ and σ -unstable for $\sigma < \sigma_0$) then T is strictly σ_0 -semistable and Case (i) (resp. Case (ii)) holds.

Proof: If T' is a σ_0 -destabilizing subtriple, then $\mu_{\sigma_0}(T') = \mu_{\sigma_0}(T)$, i.e., $\mu' + \sigma_0\lambda' = \mu + \sigma_0\lambda$. If (2) happens, then $\lambda' \neq \lambda$ (which is equivalent to $n'_1n_2 \neq n_1n'_2$). We have the following pictures of the σ -slopes of T and T' , as functions of σ .



Wherever the graph of $\mu_{\sigma}(T')$ is above that of $\mu_{\sigma}(T)$, the triple T is σ -unstable. This yields Cases (i) and (ii).

On the other hand, if T' is a σ_0 -destabilizing subtriple and Case (1) happens, then the equality $\mu' + \sigma_0\lambda' = \mu + \sigma_0\lambda$ and $\lambda' = \lambda$ imply $\mu' = \mu$, hence $\mu_{\sigma}(T') = \mu_{\sigma}(T)$, for all σ , so T' is σ -destabilizing. But for any other subtriple $\tilde{T}' \subset T$, either $\mu_{\sigma_0}(\tilde{T}') < \mu_{\sigma_0}(T)$ in which case $\mu_{\sigma}(\tilde{T}') < \mu_{\sigma}(T)$ for σ close to σ_0 , or $\mu_{\sigma_0}(\tilde{T}') = \mu_{\sigma_0}(T)$ in which case $\mu_{\sigma}(\tilde{T}') = \mu_{\sigma}(T)$ by the above argument. So T is strictly σ -semistable for σ close to σ_0 .

The converse statement follows easily from the pictures of the σ -slopes of T and of a subtriple T' with $\mu_{\sigma}(T') > \mu_{\sigma}(T)$ for $\sigma > \sigma_0$ (resp. for $\sigma < \sigma_0$). \square

Remark 3.8. We call the phenomenon in Case (1) in Proposition 3.7 σ -independent semistability. Note that this implies simultaneously that

$$\frac{n'_2}{n'_1 + n'_2} = \frac{n_2}{n_1 + n_2}, \quad \text{and} \quad \frac{d'_1 + d'_2}{n'_1 + n'_2} = \frac{d_1 + d_2}{n_1 + n_2}.$$

Hence this cannot happen if $\gcd(n_1, n_2, d_1 + d_2) = 1$.

Definition 3.9. The values of $\sigma \in I$ for which Case (2) in Proposition 3.7 occurs are called *critical values*. If $\sigma_c \in I$ is a critical value then there exist integers n'_1, n'_2, d'_1 and d'_2 such that

$$\frac{d'_1 + d'_2}{n'_1 + n'_2} + \sigma_c \frac{n'_2}{n'_1 + n'_2} = \frac{d_1 + d_2}{n_1 + n_2} + \sigma_c \frac{n_2}{n_1 + n_2},$$

equivalently,

$$\sigma_c = \frac{(n_1 + n_2)(d'_1 + d'_2) - (n'_1 + n'_2)(d_1 + d_2)}{n'_1n_2 - n_1n'_2}, \quad (3.2)$$

with $0 \leq n'_i \leq n_i$ and $n'_1n_2 \neq n_1n'_2$. We call the numbers (3.2) *virtual critical values*. In general, not all the virtual critical values are critical values. We say that $\sigma \in I$ is *generic* if it is not critical.

Remark 3.10. In the Case (2) of Proposition 3.7, the quotient triple $T'' = T/T'$ satisfies $\mu_{\sigma_c}(T'') = \mu_{\sigma_c}(T)$ by Remark 3.2. By (3.1) the graph of $\mu_{\sigma}(T'')$ is a line which goes on the opposite side of $\mu_{\sigma}(T')$, i.e., it has smaller slope than λ in Case (i), and bigger slope than λ in Case (ii).

In particular, for a σ_c -semistable decomposable triple $T = T' \oplus T''$, with $\mu_{\sigma_c}(T) = \mu_{\sigma_c}(T') = \mu_{\sigma_c}(T'')$ and $n'_1 n_2 \neq n_1 n'_2$, T is σ -unstable for any $\sigma \neq \sigma_c$. For instance, if $T = (E_1, E_2, \phi)$ has $\phi = 0$ (and $n_1 n_2 \neq 0$), then $T = (E_1, 0, 0) \oplus (0, E_2, 0)$, so it is not σ -stable for any value of σ (cf. [4, Lemma 3.5]).

Lemma 3.11. *Given a triple T , let $I_s(T) = \{\sigma \in \mathbb{R} \mid T \text{ is } \sigma\text{-stable}\}$ and $I_{ss}(T) = \{\sigma \in \mathbb{R} \mid T \text{ is } \sigma\text{-semistable}\}$. Then*

- (a) $I_{ss}(T)$ is one of the following: \emptyset , $\{\sigma_c\}$, $[\sigma_c, \sigma'_c]$, $[\sigma_c, \infty)$, $(-\infty, \sigma_c]$, \mathbb{R} . (Here σ_c, σ'_c are virtual critical values.)
- (b) $I_s(T)$ is either empty or it is the interior of $I_{ss}(T)$.

Proof: For each subtriple $T' \subset T$, let $I_{ss}(T)_{T'} = \{\sigma \in \mathbb{R} \mid \mu_{\sigma}(T') \leq \mu_{\sigma}(T)\}$ and $I_s(T)_{T'} = \{\sigma \in \mathbb{R} \mid \mu_{\sigma}(T') < \mu_{\sigma}(T)\}$. Then we have the following possibilities:

- If $\lambda' > \lambda$ then $I_{ss}(T)_{T'} = (-\infty, \sigma_c]$ and $I_s(T)_{T'} = (-\infty, \sigma_c)$.
- If $\lambda' < \lambda$ then $I_{ss}(T)_{T'} = [\sigma_c, \infty)$ and $I_s(T)_{T'} = (\sigma_c, \infty)$.
- If $\lambda' = \lambda$ and $\mu' < \mu$ then $I_{ss}(T)_{T'} = \mathbb{R}$ and $I_s(T)_{T'} = \mathbb{R}$.
- If $\lambda' = \lambda$ and $\mu' = \mu$ then $I_{ss}(T)_{T'} = \mathbb{R}$ and $I_s(T)_{T'} = \emptyset$.
- If $\lambda' = \lambda$ and $\mu' > \mu$ then $I_{ss}(T)_{T'} = \emptyset$ and $I_s(T)_{T'} = \emptyset$.

where σ_c is always a virtual critical value. The set of possible values (3.2) for σ_c is a discrete set of the real line. So the result follows easily from $I_s(T) = \bigcap I_s(T)_{T'}$ and $I_{ss}(T) = \bigcap I_{ss}(T)_{T'}$, where the intersection runs over the subtriples T' of T . \square

Remark 3.12. If $n_1 = 0$ or $n_2 = 0$ then $I_s(T)$ and $I_{ss}(T)$ are either \emptyset or \mathbb{R} . If $n_1 \neq 0$, $n_2 \neq 0$ and $n_1 \neq n_2$ then the last three cases of (a) of Lemma 3.11 do not happen since $I_{ss}(T) \subset I = [\sigma_m, \sigma_M]$. If $n_1 = n_2 \neq 0$ then the last two cases of (a) of Lemma 3.11 do not happen since $I_{ss}(T) \subset I = [\sigma_m, \infty)$.

Proposition 3.13. *Fix (n_1, n_2, d_1, d_2) . Then*

- (1) *The critical values are a finite number of values $\sigma_c \in I$.*
- (2) *The stability and semistability criteria for two values of σ lying between two consecutive critical values are equivalent; thus the corresponding moduli spaces are isomorphic.*
- (3) *If σ is generic and $\gcd(n_1, n_2, d_1 + d_2) = 1$, then σ -semistability is equivalent to σ -stability, i.e., $\mathcal{N}_{\sigma} = \mathcal{N}_{\sigma}^s$.*
- (4) *If $\gcd(n_1, n_2, d_1 + d_2) = 1$ then the moduli spaces \mathcal{N}_{σ}^s are fine moduli spaces, i.e., there is a universal triple $\mathcal{T} \rightarrow X \times \mathcal{N}_{\sigma}^s$.*

Proof: Items (1), (2) and (3) follow from [5, Proposition 2.6]. Item (4) is in [18]. \square

Remark 3.14. Under the isomorphism of Proposition 3.4, the critical values for triples of type (n_1, n_2, d_1, d_2) and of type $(n_2, n_1, -d_2, -d_1)$ correspond. This can be seen at the level of virtual critical values, since the formula (3.2) is unchanged by the transformation $(n_1, n_2, d_1, d_2, n'_1, n'_2, d'_1, d'_2) \mapsto (n_2, n_1, -d_2, -d_1, n_2 - n'_2, n_1 - n'_1, d'_2 - d_2, d'_1 - d_1)$.

Remark 3.15. There are no critical values for triples of rank $(n, 0)$ or $(0, n)$.

The moduli space of triples of rank $(1, 1)$ is easily described as follows:

Lemma 3.16. • *The moduli space $\mathcal{N}_\sigma(1, 1, d_1, d_2)$ is empty if $d_1 < d_2$.*
 • *If $d_1 \geq d_2$ then $\mathcal{N}_\sigma(1, 1, d_1, d_2) = \mathcal{N}_\sigma^s(1, 1, d_1, d_2) \cong \text{Jac}^{d_2} X \times \text{Sym}^{d_1-d_2} X$, for any $\sigma > \sigma_m = d_1 - d_2$; it is empty if $\sigma < \sigma_m$; and $\mathcal{N}_{\sigma_m} = \text{Jac}^{d_1} X \times \text{Jac}^{d_2} X$, $\mathcal{N}_{\sigma_m}^s = \emptyset$.*

Proof: The first item follows from Proposition 3.6 (otherwise, there are no non-trivial maps $E_2 \rightarrow E_1$ and by Remark 3.10 such triples cannot be σ -stable for any σ). For the second item, the moduli space is parametrizing triples of the form $\phi : E_2 \rightarrow E_1$, where both E_1 and E_2 are line bundles. The only possible proper subtriple is $0 \rightarrow E_1$ and this violates σ -stability for $\sigma \leq \sigma_m$. If $\sigma = \sigma_m$, then the triple is strictly σ_m -semistable and, being polystable, it is $(0, E_2, 0) \oplus (E_1, 0, 0)$. (Another consequence is that σ_m is the only critical value.) If $\sigma > \sigma_m$ then it cannot happen that $\phi = 0$. So the triples $T = (E_1, E_2, \phi)$ are parametrized by $(E_2, Z(\phi)) \in \text{Jac}^{d_2} X \times \text{Sym}^{d_1-d_2} X$. \square

4. EXTENSIONS OF TRIPLES

The homological algebra of triples is controlled by the hypercohomology of a certain complex of sheaves which appears when studying infinitesimal deformations [5, Section 3]. Let $T' = (E'_1, E'_2, \phi')$ and $T'' = (E''_1, E''_2, \phi'')$ be two triples of types (n'_1, n'_2, d'_1, d'_2) and $(n''_1, n''_2, d''_1, d''_2)$, respectively. Let $\text{Hom}(T'', T')$ denote the linear space of homomorphisms from T'' to T' , and let $\text{Ext}^1(T'', T')$ denote the linear space of equivalence classes of extensions of the form

$$0 \longrightarrow T' \longrightarrow T \longrightarrow T'' \longrightarrow 0,$$

where by this we mean a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & E'_1 & \longrightarrow & E_1 & \longrightarrow & E''_1 & \longrightarrow & 0 \\ & & \phi' \uparrow & & \phi \uparrow & & \phi'' \uparrow & & \\ 0 & \longrightarrow & E'_2 & \longrightarrow & E_2 & \longrightarrow & E''_2 & \longrightarrow & 0. \end{array}$$

To analyze $\text{Ext}^1(T'', T')$ one considers the complex of sheaves

$$C^\bullet(T'', T'): (E''_1{}^* \otimes E'_1) \oplus (E''_2{}^* \otimes E'_2) \xrightarrow{c} E''_2{}^* \otimes E'_1, \quad (4.1)$$

where the map c is defined by

$$c(\psi_1, \psi_2) = \phi' \psi_2 - \psi_1 \phi''.$$

Proposition 4.1 ([5, Proposition 3.1]). *There are natural isomorphisms*

$$\begin{aligned} \text{Hom}(T'', T') &\cong \mathbb{H}^0(C^\bullet(T'', T')), \\ \text{Ext}^1(T'', T') &\cong \mathbb{H}^1(C^\bullet(T'', T')), \end{aligned}$$

and a long exact sequence associated to the complex $C^\bullet(T'', T')$:

$$\begin{aligned} 0 &\longrightarrow \mathbb{H}^0(C^\bullet(T'', T')) \longrightarrow H^0((E''_1{}^* \otimes E'_1) \oplus (E''_2{}^* \otimes E'_2)) \longrightarrow H^0(E''_2{}^* \otimes E'_1) \\ &\longrightarrow \mathbb{H}^1(C^\bullet(T'', T')) \longrightarrow H^1((E''_1{}^* \otimes E'_1) \oplus (E''_2{}^* \otimes E'_2)) \longrightarrow H^1(E''_2{}^* \otimes E'_1) \\ &\longrightarrow \mathbb{H}^2(C^\bullet(T'', T')) \longrightarrow 0. \end{aligned}$$

We introduce the following notation:

$$\begin{aligned} h^i(T'', T') &= \dim \mathbb{H}^i(C^\bullet(T'', T')), \\ \chi(T'', T') &= h^0(T'', T') - h^1(T'', T') + h^2(T'', T'). \end{aligned}$$

Proposition 4.2 ([5, Proposition 3.2]). *For any holomorphic triples T' and T'' we have*

$$\begin{aligned} \chi(T'', T') &= \chi(E_1''^* \otimes E_1') + \chi(E_2''^* \otimes E_2') - \chi(E_2''^* \otimes E_1') \\ &= (1-g)(n_1''n_1' + n_2''n_2' - n_2''n_1') + n_1'd_1'' - n_1'd_1' + n_2'd_2'' - n_2'd_2' - n_2'd_1'' + n_1'd_2'', \end{aligned}$$

where $\chi(E) = \dim H^0(E) - \dim H^1(E)$ is the Euler characteristic of E .

Proposition 4.3 ([5, Proposition 3.5]). *Suppose that T' and T'' are σ -semistable, for some value of σ .*

- (1) *If $\mu_\sigma(T') < \mu_\sigma(T'')$ then $\mathbb{H}^0(C^\bullet(T'', T')) = 0$.*
- (2) *If $\mu_\sigma(T') = \mu_\sigma(T'')$ and T'' is σ -stable, then*

$$\mathbb{H}^0(C^\bullet(T'', T')) \cong \begin{cases} \mathbb{C} & \text{if } T' \cong T'' \\ 0 & \text{if } T' \not\cong T''. \end{cases}$$

Since the space of infinitesimal deformations of T is isomorphic to $\mathbb{H}^1(C^\bullet(T, T))$, the previous results also apply to studying deformations of a holomorphic triple T .

Theorem 4.4 ([5, Theorem 3.8]). *Let $T = (E_1, E_2, \phi)$ be an σ -stable triple of type (n_1, n_2, d_1, d_2) .*

- (1) *The Zariski tangent space at the point defined by T in the moduli space of stable triples is isomorphic to $\mathbb{H}^1(C^\bullet(T, T))$.*
- (2) *If $\mathbb{H}^2(C^\bullet(T, T)) = 0$, then the moduli space of σ -stable triples is smooth in a neighbourhood of the point defined by T .*
- (3) *At a smooth point $T \in \mathcal{N}_\sigma^s(n_1, n_2, d_1, d_2)$ the dimension of the moduli space of σ -stable triples is*

$$\begin{aligned} \dim \mathcal{N}_\sigma^s(n_1, n_2, d_1, d_2) &= h^1(T, T) = 1 - \chi(T, T) \\ &= (g-1)(n_1^2 + n_2^2 - n_1n_2) - n_1d_2 + n_2d_1 + 1. \end{aligned}$$

- (4) *Let $T = (E_1, E_2, \phi)$ be a σ -stable triple. If T is injective or surjective (meaning that $\phi : E_2 \rightarrow E_1$ is injective or surjective) then the moduli space is smooth at T .*

Description of the flip loci. Fix the type (n_1, n_2, d_1, d_2) for the moduli spaces of holomorphic triples. Now we want to describe the differences between two spaces $\mathcal{N}_{\sigma_1}^s$ and $\mathcal{N}_{\sigma_2}^s$ when σ_1 and σ_2 are separated by a critical value. Let $\sigma_c \in I$ be a critical value and set

$$\sigma_c^+ = \sigma_c + \varepsilon, \quad \sigma_c^- = \sigma_c - \varepsilon,$$

where $\varepsilon > 0$ is small enough so that σ_c is the only critical value in the interval (σ_c^-, σ_c^+) .

Definition 4.5. We define the *flip loci* as

$$\begin{aligned} \mathcal{S}_{\sigma_c^+} &= \{T \in \mathcal{N}_{\sigma_c^+} \mid T \text{ is } \sigma_c^- \text{-unstable}\} \subset \mathcal{N}_{\sigma_c^+}, \\ \mathcal{S}_{\sigma_c^-} &= \{T \in \mathcal{N}_{\sigma_c^-} \mid T \text{ is } \sigma_c^+ \text{-unstable}\} \subset \mathcal{N}_{\sigma_c^-}. \end{aligned}$$

and $\mathcal{S}_{\sigma_c^\pm}^s = \mathcal{S}_{\sigma_c^\pm} \cap \mathcal{N}_{\sigma_c^\pm}^s$ for the stable part of the flip loci.

Note that for $\sigma_c = \sigma_m$, $\mathcal{N}_{\sigma_m^-}$ is empty, hence $\mathcal{N}_{\sigma_m^+} = \mathcal{S}_{\sigma_m^+}$. Also $\mathcal{N}_{\sigma_m^s} = \emptyset$, by the last part of Proposition 3.7. Analogously, when $n_1 \neq n_2$, $\mathcal{N}_{\sigma_M^+}$ is empty, $\mathcal{N}_{\sigma_M^-} = \mathcal{S}_{\sigma_M^-}$ and $\mathcal{N}_{\sigma_M^s}$ is empty.

Lemma 4.6. *Let σ_c be a critical value. Then*

- (1) $\mathcal{N}_{\sigma_c^+} - \mathcal{S}_{\sigma_c^+} = \mathcal{N}_{\sigma_c^-} - \mathcal{S}_{\sigma_c^-}$.
- (2) $\mathcal{N}_{\sigma_c^+}^s - \mathcal{S}_{\sigma_c^+}^s = \mathcal{N}_{\sigma_c^-}^s - \mathcal{S}_{\sigma_c^-}^s = \mathcal{N}_{\sigma_c^s}^s$.

Proof: Item (1) is an easy consequence of the definition of flip loci. Item (2) is the content of [5, Lemma 5.3]. \square

Now we want to describe the flip loci $\mathcal{S}_{\sigma_c^\pm}$. Let σ_c be a critical value, and let (n'_1, n'_2, d'_1, d'_2) such that $\lambda' \neq \lambda$ and (3.2) holds. Put $(n''_1, n''_2, d''_1, d''_2) = (n_1 - n'_1, n_2 - n'_2, d_1 - d'_1, d_2 - d'_2)$. Denote $\mathcal{N}'_\sigma = \mathcal{N}_\sigma(n'_1, n'_2, d'_1, d'_2)$ and $\mathcal{N}''_\sigma = \mathcal{N}_\sigma(n''_1, n''_2, d''_1, d''_2)$.

Lemma 4.7. *Let $T \in \mathcal{S}_{\sigma_c^+}$ (resp. $T \in \mathcal{S}_{\sigma_c^-}$). Then T is a non-trivial extension*

$$0 \rightarrow T' \rightarrow T \rightarrow T'' \rightarrow 0, \quad (4.2)$$

where $\mu_{\sigma_c}(T') = \mu_{\sigma_c}(T) = \mu_{\sigma_c}(T'')$, $\lambda' < \lambda$ (resp. $\lambda' > \lambda$) and T' and T'' are both σ_c -semistable.

Conversely, suppose $T' \in \mathcal{N}'_{\sigma_c}$ and $T'' \in \mathcal{N}''_{\sigma_c}$ are both σ_c -stable, and $\lambda' < \lambda$ (resp. $\lambda' > \lambda$). Then for any non-trivial extension (4.2), T lies in $\mathcal{S}_{\sigma_c^+}^s$ (resp. in $\mathcal{S}_{\sigma_c^-}^s$). Moreover, such T can be written uniquely as an extension (4.2) with $\mu_{\sigma_c}(T') = \mu_{\sigma_c}(T)$.

In particular, suppose σ_c is not a critical value for the moduli spaces of triples of types (n'_1, n'_2, d'_1, d'_2) and $(n''_1, n''_2, d''_1, d''_2)$, $\gcd(n'_1, n'_2, d'_1 + d'_2) = 1$ and $\gcd(n''_1, n''_2, d''_1 + d''_2) = 1$. Then if $\lambda' < \lambda$ (resp. $\lambda' > \lambda$), there is a bijective correspondence between non-trivial extensions (4.2), with $T' \in \mathcal{N}'_{\sigma_c}$ and $T'' \in \mathcal{N}''_{\sigma_c}$ and triples $T \in \mathcal{S}_{\sigma_c^+}$ (resp. $\mathcal{S}_{\sigma_c^-}$).

Proof: Let T be a triple which is σ_c^+ -semistable but σ_c^- -unstable. Then Proposition 3.7 implies the existence of a subtriple T' and a quotient triple $T'' = T/T'$ such that $\mu_{\sigma_c}(T') = \mu_{\sigma_c}(T) = \mu_{\sigma_c}(T'')$ and $\lambda' < \lambda$. By Remark 3.10, T defines a non-trivial extension in $\text{Ext}^1(T'', T')$. Now suppose T' is σ_c -unstable; then there exists $\tilde{T} \subset T'$ with $\mu_{\sigma_c}(\tilde{T}) > \mu_{\sigma_c}(T') = \mu_{\sigma_c}(T)$, contradicting the σ_c -semistability of T . So T' is σ_c -semistable. Also, if T'' is σ_c -unstable, then there exists a quotient $T'' \rightarrow \tilde{T}''$ with $\mu_{\sigma_c}(T'') = \mu_{\sigma_c}(T) > \mu_{\sigma_c}(\tilde{T}'')$, contradicting again the σ_c -semistability of T . Hence T'' is σ_c -semistable.

The second statement is basically due to the uniqueness of Jordan-Hölder filtrations. We provide the argument for completeness. Suppose T is such a non-trivial extension and let us check that it is σ_c^+ -stable. For this, let $\tilde{T} \subset T$ be a subtriple. Consider the subtriple \tilde{T}' given as the saturation of $\tilde{T} \cap T'$ (which is a triple of torsion-free subsheaves). Let $\tilde{T}'' = \tilde{T}/\tilde{T}'$, so that there is an exact sequence

$$\begin{array}{ccccccc} 0 & \rightarrow & \tilde{T}' & \rightarrow & \tilde{T} & \rightarrow & \tilde{T}'' & \rightarrow & 0 \\ & & \cap & & \cap & & \cap & & \\ 0 & \rightarrow & T' & \rightarrow & T & \rightarrow & T'' & \rightarrow & 0. \end{array}$$

Clearly $\mu_{\sigma_c}(\tilde{T}') \leq \mu_{\sigma_c}(T') = \mu_{\sigma_c}(T)$ and $\mu_{\sigma_c}(\tilde{T}'') \leq \mu_{\sigma_c}(T'') = \mu_{\sigma_c}(T)$, with a strict inequality if either of the subtriples

$$\tilde{T}' \subset T', \quad \tilde{T}'' \subset T'' \quad (4.3)$$

is proper. In this case, Remark 3.2 gives that $\mu_{\sigma_c}(\tilde{T}) < \mu_{\sigma_c}(T)$. By changing σ_c to $\sigma_c^+ = \sigma_c + \varepsilon$, we still have $\mu_{\sigma_c^+}(\tilde{T}) < \mu_{\sigma_c^+}(T)$.

If both subtriples (4.3) are not proper, then either $\tilde{T}' = T'$, $\tilde{T}'' = 0$ in which case $\tilde{T} = T'$ and hence $\mu_{\sigma_c^+}(\tilde{T}) < \mu_{\sigma_c^+}(T)$ since $\lambda' < \lambda$; or $\tilde{T}' = 0$, $\tilde{T}'' = T''$ in which case there is a splitting $T'' = \tilde{T}'' = \tilde{T} \hookrightarrow T$ of the exact sequence $0 \rightarrow T' \rightarrow T \rightarrow T'' \rightarrow 0$, and the extension is trivial.

The argument also shows that for such $T \in \mathcal{S}_{\sigma_c^+}$, the only subtriple with equal σ_c -slope is T' . This produces the uniqueness of the defining exact sequence.

The last part follows since the conditions σ_c not being a critical value for the moduli spaces of triples of types (n'_1, n'_2, d'_1, d'_2) and $(n''_1, n''_2, d''_1, d''_2)$, $\gcd(n'_1, n'_2, d'_1 + d'_2) = 1$ and $\gcd(n''_1, n''_2, d''_1 + d''_2) = 1$ imply that \mathcal{N}'_{σ_c} and \mathcal{N}''_{σ_c} do not contain properly σ_c -semistable triples. \square

Theorem 4.8. *Let σ_c be a critical value with $\lambda' < \lambda$ (resp. $\lambda' > \lambda$). Assume*

- (i) σ_c is not a critical value for the moduli spaces of triples of types (n'_1, n'_2, d'_1, d'_2) and $(n''_1, n''_2, d''_1, d''_2)$, $\gcd(n'_1, n'_2, d'_1 + d'_2) = 1$ and $\gcd(n''_1, n''_2, d''_1 + d''_2) = 1$.
- (ii) $\mathbb{H}^0(C^\bullet(T'', T')) = \mathbb{H}^2(C^\bullet(T'', T')) = 0$, for every $(T', T'') \in \mathcal{N}'_{\sigma_c} \times \mathcal{N}''_{\sigma_c}$.
- (iii) There are universal triples for \mathcal{N}'_{σ_c} and \mathcal{N}''_{σ_c} .

Then $\mathcal{S}_{\sigma_c^+}$ (resp. $\mathcal{S}_{\sigma_c^-}$) is the projectivization of a bundle of rank $-\chi(T'', T')$ over $\mathcal{N}'_{\sigma_c} \times \mathcal{N}''_{\sigma_c}$.

Proof: Let $T' = (\mathcal{E}'_1, \mathcal{E}'_2, \Phi') \rightarrow \mathcal{N}'_{\sigma_c} \times X$ and $T'' = (\mathcal{E}''_1, \mathcal{E}''_2, \Phi'') \rightarrow \mathcal{N}''_{\sigma_c} \times X$ denote universal triples, provided by Proposition 3.13 (4). Let $B = \mathcal{N}'_{\sigma_c} \times \mathcal{N}''_{\sigma_c}$ and pull back T' and T'' to $B \times X$. Consider the complex $C^\bullet(T'', T')$ as defined in (4.1) and take relative hypercohomology $\mathbb{H}^i_\pi(C^\bullet(T'', T'))$ with respect to the projection $\pi : B \times X \rightarrow B$. Setting

$$W = \mathbb{H}^1_\pi(C^\bullet(T'', T')),$$

we have an identification $\mathcal{S}_{\sigma_c^+} \cong \mathbb{P}W$, by Lemma 4.7 (that is, a bijection). By (ii), W is a bundle of rank $h^1(T'', T') = -\chi(T'', T')$.

We construct a universal triple for $\mathbb{P}W$. Let $q : \mathbb{P}W \rightarrow B$ be the projection and consider the extension

$$\begin{aligned} \xi \in \text{Ext}^1((q \times 1_X)^* T'', (q \times 1_X)^* T' \otimes \mathcal{O}_{\mathbb{P}W}(1)) \\ &= \mathbb{H}^1(C^\bullet((q \times 1_X)^* T'', (q \times 1_X)^* T') \otimes \mathcal{O}_{\mathbb{P}W}(1)) \\ &= H^0(\mathbb{H}^1_\pi(C^\bullet(T'', T')) \otimes q_* \mathcal{O}_{\mathbb{P}W}(1)) \\ &= H^0(W \otimes W^*) = \text{End}(W) \end{aligned}$$

corresponding to the identity homomorphism. (We have used that $\mathbb{H}^0_\pi(C^\bullet(T'', T')) = 0$ in the third line of the above chain of equalities.) This gives an extension

$$0 \rightarrow (q \times 1_X)^* T' \otimes \mathcal{O}_{\mathbb{P}W}(1) \rightarrow \mathcal{T} \rightarrow (q \times 1_X)^* T'' \rightarrow 0,$$

which produces the required universal triple $\mathcal{T} \rightarrow \mathbb{P}W \times X$. This defines an algebraic map $\mathbb{P}W \rightarrow \mathcal{S}_{\sigma_c^+}$, which is the required isomorphism. \square

Remark 4.9. This result is analogous to the results in [14, Section 5] which deal with the moduli space of parabolic triples. In that case, a suitable choice of parabolic weights ensure that condition (i) in Theorem 4.8 is always satisfied.

Condition (iii) in Theorem 4.8 is necessary for this proof, and it is not automatic. For instance the moduli space of stable bundles $M^s(n, d)$ does not have a universal bundle when $\gcd(n, d) \neq 1$.

The construction of the flip loci can be used for the critical value $\sigma_c = \sigma_m = \mu_1 - \mu_2$, which allows us to describe the moduli space $\mathcal{N}_{\sigma_m^+}$. We refer to the value of σ given by $\sigma = \sigma_m^+ = \sigma_m + \varepsilon$ as *small*.

Proposition 4.10. *We have $\mathcal{N}_{\sigma_m} \cong M(n_1, d_1) \times M(n_2, d_2)$ and only contains strictly σ_m -semistable triples. There is a map*

$$\pi : \mathcal{N}_{\sigma_m^+}(n_1, n_2, d_1, d_2) \rightarrow M(n_1, d_1) \times M(n_2, d_2)$$

which sends $T = (E_1, E_2, \phi)$ to (E_1, E_2) .

- (i) *If $\gcd(n_1, d_1) = 1$, $\gcd(n_2, d_2) = 1$ and $\mu_1 - \mu_2 > 2g - 2$, then $\mathcal{N}_{\sigma_m^+}^s = \mathcal{N}_{\sigma_m^+}$ and it is the projectivization of a bundle over $M(n_1, d_1) \times M(n_2, d_2)$, of rank $n_2d_1 - n_1d_2 - n_1n_2(g - 1)$.*
- (ii) *In general, if $\mu_1 - \mu_2 > 2g - 2$, the subset*

$$\pi^{-1}(M^s(n_1, d_1) \times M^s(n_2, d_2)) \subset \mathcal{N}_{\sigma_m^+}$$

is a projective bundle over $M^s(n_1, d_1) \times M^s(n_2, d_2)$, whose fibers are projective spaces of dimension $n_2d_1 - n_1d_2 - n_1n_2(g - 1) - 1$.

Proof: Let $T = (E_1, E_2, \phi) \in \mathcal{N}_{\sigma_m}$. Note that $0 \rightarrow E_1$ is a σ_m -destabilizing subtriple. As T is polystable, $T = (E_1, 0, 0) \oplus (0, E_2, 0)$. (Here also E_1 and E_2 should be polystable bundles.) Conversely if the triple $T = (E_1, E_2, \phi)$ has $\phi = 0$ (and $n_1n_2 \neq 0$), then automatically $T = (E_1, 0, 0) \oplus (0, E_2, 0)$ and it is σ -unstable for any $\sigma \neq \sigma_m$, by Remark 3.10. If E_1 and E_2 are polystable bundles then $T \in \mathcal{N}_{\sigma_m}$. This proves the first statement.

Next, the map π is well-defined since a σ_m^+ -semistable triple is automatically σ_m -semistable.

Now let us see item (i). Note that all the triples in $\mathcal{N}_{\sigma_m^+}$ are non-trivial extensions of $T' = (E_1, 0, 0)$ by $T'' = (0, E_2, 0)$ by Lemma 4.7. The long exact sequence in Proposition 4.1 yields $\mathbb{H}^0 = 0$, $\mathbb{H}^1 = \text{Hom}(E_2, E_1)$ and $\mathbb{H}^2 = H^1(E_2^* \otimes E_1) \cong \text{Hom}(E_1, E_2 \otimes K)$, where we have abbreviated $\mathbb{H}^i = \mathbb{H}^i(C^\bullet(T'', T'))$. Since both E_1 and E_2 are stable bundles and $\mu_1 > \mu_2 + 2g - 2$, we get that $\mathbb{H}^2 = 0$. Therefore the conditions of Theorem 4.8 are satisfied and $-\chi(T'', T') = -(n_1n_2(g - 1) - n_2d_1 + n_1d_2)$.

Item (ii) is similar, but now we do not have a universal bundle for $M^s(n_1, d_1)$ or $M^s(n_2, d_2)$ at our disposal. Working in the étale topology, we have (locally) a universal bundle which yields that $\pi^{-1}(M^s(n_1, d_1) \times M^s(n_2, d_2)) \rightarrow M^s(n_1, d_1) \times M^s(n_2, d_2)$ is a fibration. The fiber over (E_1, E_2) is the projective space $\mathbb{P}\mathbb{H}^1(C^\bullet(T'', T')) = \mathbb{P}\text{Hom}(E_2, E_1)$, using Lemma 4.7. \square

5. MODULI OF TRIPLES OF RANK (2, 1) AND PAIRS

Let X be a smooth projective curve of genus $g \geq 2$. In this section, we shall consider triples $T = (E_1, E_2, \phi)$ where E_1 is a vector bundle of degree d_1 and rank 2 and E_2 is a line

bundle of degree d_2 . Let $\mathcal{N}_\sigma = \mathcal{N}_\sigma(2, 1, d_1, d_2)$ denote the moduli space of σ -polystable triples of such type. By Proposition 3.6, σ is in the interval

$$I = [\sigma_m, \sigma_M] = [\mu_1 - \mu_2, 4(\mu_1 - \mu_2)] = [d_1/2 - d_2, 2d_1 - 4d_2], \quad (5.1)$$

where $\mu_1 - \mu_2 \geq 0$. Otherwise \mathcal{N}_σ is empty.

Theorem 5.1. *For $\sigma \in I$, \mathcal{N}_σ is a projective variety. It is smooth and of (complex) dimension $3g - 2 + d_1 - 2d_2$ at the stable points \mathcal{N}_σ^s . Moreover, for generic values of σ , $\mathcal{N}_\sigma = \mathcal{N}_\sigma^s$ (hence it is smooth and projective).*

Proof: The first line follows from Proposition 3.6. Now let $T = (E_1, E_2, \phi)$ be any σ -stable triple. Then $\phi : E_2 \rightarrow E_1$ is injective, since E_2 is a line bundle and $\phi \neq 0$ ($\phi = 0$ would imply that T is decomposable, hence not σ -stable). Therefore by Theorem 4.4 (4), \mathcal{N}_σ is smooth at T and of the stated dimension. Finally, since $\gcd(1, 2, d_1 + d_2) = 1$, Proposition 3.13 (3) implies the last assertion. \square

The moduli spaces of σ -stable triples of rank $(2, 1)$ are intimately related to the moduli space of τ -stable pairs of rank 2. A holomorphic pair (E, ϕ) is formed by a rank n holomorphic bundle E and a non-zero holomorphic section $\phi \in H^0(E)$. There is a notion of stability depending on a real parameter $\tau \in \mathbb{R}$ given in [13, Definition 4.7]. A holomorphic pair (E, ϕ) is τ -stable if

- (1) $\mu(E') < \tau$ for every subbundle $E' \subset E$ with $\text{rk}(E') > 0$.
- (2) $\mu(E/E') > \tau$ for every subbundle $E' \subset E$ with $0 < \text{rk}(E') < r$ and $\phi \in H^0(E')$.

There is a moduli space $\mathfrak{M}_\tau(n, d)$ of τ -stable pairs of rank n and degree d , and a moduli space $\mathfrak{M}_\tau(n, \Lambda)$ of τ -stable pairs of rank n and satisfying that the determinant of the bundle E is some fixed line bundle Λ of degree d on X . These moduli spaces have been studied in [2, 3, 13].

Bertram [2] and Thaddeus [19] gave an explicit GIT construction of the moduli spaces $\mathfrak{M}_\tau(2, \Lambda)$ of pairs (E, ϕ) of rank 2 with fixed determinant $\det(E) = \Lambda$, where Λ is a fixed line bundle.

We have the following relationship between the moduli spaces of triples of rank $(2, 1)$ and the moduli spaces of pairs.

Lemma 5.2. *We have an isomorphism $\mathcal{N}_\sigma(2, 1, d_1, d_2) \cong \mathfrak{M}_\tau(2, d_1 - 2d_2) \times \text{Jac}^{d_2}(X)$.*

Proof: In fact, a triple (E_1, E_2, ϕ) is σ -stable if and only if the pair $(E_1 \otimes E_2^*, \phi)$ is τ -stable with $\tau = \frac{1}{3}(\sigma + d_1 - 2d_2)$ (cf. [13, Definition 4.27] and [4, Proposition 3.4]). Therefore there exists a bijective correspondence $\mathcal{N}_\sigma(2, 1, d_1, d_2) \cong \mathfrak{M}_\tau(2, d_1 - 2d_2) \times \text{Jac}^{d_2}(X)$, given by $(E_1, E_2, \phi) \mapsto ((E_1 \otimes E_2^*, \phi), E_2)$. This is clearly an isomorphism. \square

Critical values and flip loci. Let us compute the critical values corresponding to $n_1 = 2$, $n_2 = 1$. Following Definition 3.9, we have the following possibilities:

- (1) $n'_1 = 1$, $n'_2 = 0$. The corresponding σ_c -destabilizing subtriple is of the form $0 \rightarrow E'_1$, where $E'_1 = M$ is a line bundle of degree $\deg(M) = d_M$. The critical value is

$$\sigma_c = 3d_M - d_1 - d_2.$$

- (2) $n'_1 = 1, n'_2 = 1$. The corresponding σ_c -destabilizing subtriple T' is of the form $E_2 \rightarrow E'_1$, where E'_1 is a line bundle. Let $T'' = T/T'$ be the quotient bundle, which is of the form $0 \rightarrow E''_1$, where $E''_1 = M$ is a line bundle, and let $d_M = \deg(M)$ be its degree. Then $d'_2 = d_2, d'_1 = d_1 - d_M$ and

$$\sigma_c = -(3(d_1 - d_M + d_2) - 2(d_1 + d_2)) = 3d_M - d_1 - d_2.$$

- (3) $n'_1 = 2, n'_2 = 0$. In this case, the only possible subtriple is $0 \rightarrow E_1$. This produces the critical value

$$\sigma_c = \frac{d_1 - 2d_2}{2} = \mu_1 - \mu_2 = \sigma_m,$$

i.e., the minimum of the interval I for σ .

- (4) $n'_1 = 0, n'_2 = 1$. The subtriple T' must be of the form $E_2 \rightarrow 0$. This forces $\phi = 0$ in $T = (E_1, E_2, \phi)$. So T is decomposable, of the form $T' \oplus T'' = (0, E_2, 0) \oplus (E_1, 0, 0)$. By Remark 3.10, T is σ -unstable for any $\sigma \neq \sigma_c$, where

$$\sigma_c = \frac{2d_2 - d_1}{-2} = \mu_1 - \mu_2 = \sigma_m.$$

Actually T is σ_m -semistable if and only if E_1 is a rank 2 semistable bundle.

Lemma 5.3. *Let $\sigma_c = 3d_M - d_1 - d_2$ be a critical value. Then*

$$\mu_1 \leq d_M \leq d_1 - d_2,$$

and $\sigma_c = \sigma_m \Leftrightarrow d_M = \mu_1$.

Proof: This simply consists of rewriting the inequalities $\sigma_m \leq \sigma_c \leq \sigma_M$. \square

6. HODGE POLYNOMIALS OF THE MODULI SPACES OF TRIPLES OF RANK $(2, 1)$

In this section we are going to compute the Hodge polynomial of the moduli space of σ -stable triples of type $(2, 1, d_1, d_2)$ without making use of the blow-ups and blow-downs constructed by Thaddeus in [19]. As $\mathcal{N}_\sigma = \mathcal{N}_\sigma(2, 1, d_1, d_2)$ is smooth and projective for non-critical values σ , by Theorem 5.1, the polynomial $e(\mathcal{N}_\sigma)$ is the usual Hodge polynomial of \mathcal{N}_σ .

Let σ_c be a critical value. By Lemma 4.6, we have that

$$\mathcal{N}_{\sigma_c^+} - \mathcal{S}_{\sigma_c^+} = \mathcal{N}_{\sigma_c^-} - \mathcal{S}_{\sigma_c^-}.$$

Now Theorem 2.2 implies that the Hodge polynomials satisfy

$$e(\mathcal{N}_{\sigma_c^+}) - e(\mathcal{S}_{\sigma_c^+}) = e(\mathcal{N}_{\sigma_c^-}) - e(\mathcal{S}_{\sigma_c^-}). \quad (6.1)$$

We have the following computation for the difference $e(\mathcal{S}_{\sigma_c^-}) - e(\mathcal{S}_{\sigma_c^+})$.

Lemma 6.1. *Let $\sigma_c = 3d_M - d_1 - d_2$ be a critical value for $\mathcal{N}_\sigma(2, 1, d_1, d_2)$ and assume that $\sigma_c \neq \sigma_m$. Then $e(\mathcal{S}_{\sigma_c^-}) - e(\mathcal{S}_{\sigma_c^+})$ is*

$$\text{coeff}_{x^0} \left[\frac{((uv)^{d_1 - d_2 - d_M} - (uv)^{2d_M - d_1 + g - 1})(1 + u)^{2g}(1 + v)^{2g}(1 + ux)^g(1 + vx)^g}{(1 - uv)(1 - x)(1 - uvx)x^{d_1 - d_2 - d_M}} \right].$$

Proof: Since $\sigma_c \neq \sigma_m$, the critical value $\sigma_c = 3d_M - d_1 - d_2$ determines the value of d_M and there are only two possibilities for flip loci:

- (1) The subtriple T' is of the form $0 \rightarrow E'_1$ with $E'_1 = M$ a line bundle of degree d_M , and the quotient triple T'' is of the form $E_2 \rightarrow E''_1$ where E''_1 is a line bundle of degree $d_1 - d_M$. Then by Lemmas 3.5 and 3.16,

$$\begin{aligned}\mathcal{N}'_{\sigma_c} &= \mathcal{N}_{\sigma_c}(1, 0, d_M, 0) = \text{Jac}^{d_M} X, \\ \mathcal{N}''_{\sigma_c} &= \mathcal{N}_{\sigma_c}(1, 1, d_1 - d_M, d_2) = \text{Jac}^{d_2} X \times \text{Sym}^{d_1 - d_M - d_2} X.\end{aligned}$$

Here note that the second line needs that $\sigma_c \neq d_1 - d_M - d_2$ (which is the only critical value for the moduli spaces of triples of rank $(1, 1)$. This is true since $\mu_1 < d_M$). Now note that $\lambda' = 0 < \lambda = \frac{1}{3}$, $\mathbb{H}^0(C^\bullet(T'', T')) = 0$ by Proposition 4.3. As $\phi'' : E''_2 \rightarrow E''_1$ is injective, $E''_1^* \otimes E'_1 \rightarrow E''_2^* \otimes E'_1$ is generically surjective, so $H^1(E''_1^* \otimes E'_1) \rightarrow H^1(E''_2^* \otimes E'_1)$ is surjective. Hence the long exact sequence in Proposition 4.1 implies that $\mathbb{H}^2(C^\bullet(T'', T')) = 0$. Therefore, by Theorem 4.8, $\mathcal{S}_{\sigma_c^+}$ is the projectivization of a rank $-\chi(T'', T')$ bundle over $\mathcal{N}'_{\sigma_c} \times \mathcal{N}''_{\sigma_c}$, where

$$-\chi(T'', T') = d_1 - d_M - d_2,$$

using Proposition 4.2. Now Lemma 2.3 yields

$$\begin{aligned}e(\mathcal{S}_{\sigma_c^+}) &= e(\mathbb{P}^{d_1 - d_M - d_2 - 1})e(\mathcal{N}'_{\sigma_c} \times \mathcal{N}''_{\sigma_c}) \\ &= e_{d_1 - d_M - d_2} e(\text{Jac } X)^2 e(\text{Sym}^{d_1 - d_M - d_2} X),\end{aligned}$$

with the notation in (2.1). This formula also holds when $d_1 - d_M - d_2 = 0$, since in such case $\mathcal{S}_{\sigma_c^+} = \emptyset$ and $e_{d_1 - d_M - d_2} = 0$ (note that it cannot happen that $d_1 - d_M - d_2 < 0$).

- (2) The subtriple T' is of the form $E_2 \rightarrow E'_1$ with quotient of the form $0 \rightarrow E''_1$, where $E''_1 = M$ is a line bundle of degree d_M . Now

$$\begin{aligned}\mathcal{N}'_{\sigma_c} &= \mathcal{N}_{\sigma_c}(1, 1, d_1 - d_M, d_2) = \text{Jac}^{d_2} X \times \text{Sym}^{d_1 - d_M - d_2} X, \\ \mathcal{N}''_{\sigma_c} &= \mathcal{N}_{\sigma_c}(1, 0, d_M, 0) = \text{Jac}^{d_M} X.\end{aligned}$$

Again $\mathbb{H}^0(C^\bullet(T'', T')) = 0$ by Proposition 4.3, and $\mathbb{H}^2(C^\bullet(T'', T')) = 0$ by the long exact sequence in Proposition 4.1 and using $E''_2 = 0$. As $\lambda' = \frac{1}{2} > \lambda = \frac{1}{3}$, Theorem 4.8 says that $\mathcal{S}_{\sigma_c^-}$ is the projectivization of a rank $-\chi(T'', T')$ bundle over $\mathcal{N}'_{\sigma_c} \times \mathcal{N}''_{\sigma_c}$, where

$$-\chi(T'', T') = 2d_M - d_1 + g - 1,$$

by Proposition 4.2. So

$$e(\mathcal{S}_{\sigma_c^-}) = e_{2d_M - d_1 + g - 1} e(\text{Jac } X)^2 e(\text{Sym}^{d_1 - d_M - d_2} X).$$

Using (2.2) for the Hodge polynomial of the Jacobian of X and (2.3) for the Hodge polynomial of the symmetric product of X , we get

$$\begin{aligned}e(\mathcal{S}_{\sigma_c^-}) - e(\mathcal{S}_{\sigma_c^+}) &= (e_{2d_M - d_1 + g - 1} - e_{d_1 - d_M - d_2}) e(\text{Jac } X)^2 e(\text{Sym}^{d_1 - d_M - d_2} X) \\ &= \left(\frac{(uv)^{d_1 - d_2 - d_M} - (uv)^{2d_M - d_1 + g - 1}}{1 - uv} \right) (1 + u)^{2g} (1 + v)^{2g} \\ &\quad \cdot \text{coeff}_{x^0} \frac{(1 + ux)^g (1 + vx)^g}{(1 - x)(1 - uvx)x^{d_1 - d_2 - d_M}},\end{aligned}$$

from which the stated result is obtained. \square

Theorem 6.2. *Let X be a smooth projective curve of genus $g \geq 2$ and consider the moduli space $\mathcal{N}_\sigma = \mathcal{N}_\sigma(2, 1, d_1, d_2)$, for a non-critical value $\sigma > \sigma_m$. Set $d_0 = \left\lceil \frac{1}{3}(\sigma + d_1 + d_2) \right\rceil + 1$. Then the Hodge polynomial of \mathcal{N}_σ is*

$$e(\mathcal{N}_\sigma) = \text{coeff}_{x^0} \left[\frac{(1+u)^{2g}(1+v)^{2g}(1+ux)^g(1+vx)^g}{(1-uv)(1-x)(1-uvx)x^{d_1-d_2-d_0}} \left(\frac{(uv)^{d_1-d_2-d_0}}{1-(uv)^{-1}x} - \frac{(uv)^{-d_1+g-1+2d_0}}{1-(uv)^2x} \right) \right].$$

Proof: By (6.1), we have that $e(\mathcal{N}_{\sigma_c^-}) - e(\mathcal{N}_{\sigma_c^+}) = e(\mathcal{S}_{\sigma_c^-}) - e(\mathcal{S}_{\sigma_c^+})$. Lemma 6.1 implies that this quantity equals

$$\text{coeff}_{x^0} \left[\frac{((uv)^{d_1-d_2-d_M} - (uv)^{2d_M-d_1+g-1})(1+u)^{2g}(1+v)^{2g}(1+ux)^g(1+vx)^g}{(1-uv)(1-x)(1-uvx)x^{d_1-d_2-d_M}} \right].$$

Now we add up all the contributions for $\sigma < \sigma_c \leq \sigma_M$. This corresponds to

$$\frac{1}{3}(\sigma + d_1 + d_2) < d_M \leq d_1 - d_2. \quad (6.2)$$

Note that since σ is not a critical value, we cannot have equality in the left, hence the left hand term is not an integer. Therefore (6.2) is equivalent to $d_0 \leq d_M \leq d_1 - d_2$, with d_0 given as in the statement. Thus

$$e(\mathcal{N}_\sigma) = \text{coeff}_{x^0} \left(\frac{(1+u)^{2g}(1+v)^{2g}(1+ux)^g(1+vx)^g}{(1-uv)(1-x)(1-uvx)x^{d_1-d_2}} \sum_{d_M=d_0}^{d_1-d_2} (uv)^{-d_M} x^{d_M} - \frac{(1+u)^{2g}(1+v)^{2g}(1+ux)^g(1+vx)^g}{(1-uv)(1-x)(1-uvx)x^{d_1-d_2}} \sum_{d_M=d_0}^{d_1-d_2} (uv)^{2d_M} x^{d_M} \right).$$

If we add to the summations terms with $d_M > d_1 - d_2$, then all the new terms contribute positive powers of x to the global expression, so they do not appear after extracting the coefficient of x^0 . This means that we can take the sum from d_0 to ∞ . Using

$$\sum_{d_M=d_0}^{\infty} (uv)^{-d_M} x^{d_M} = \frac{(uv)^{-d_0} x^{d_0}}{1-(uv)^{-1}x} \quad \text{and} \quad \sum_{d_M=d_0}^{\infty} (uv)^{2d_M} x^{d_M} = \frac{(uv)^{2d_0} x^{d_0}}{1-(uv)^2x},$$

we get

$$e(\mathcal{N}_\sigma) = \text{coeff}_{x^0} \left(\frac{(1+u)^{2g}(1+v)^{2g}(1+ux)^g(1+vx)^g}{(1-uv)(1-x)(1-uvx)(1-(uv)^{-1}x)x^{d_1-d_2}} (uv)^{-d_0} x^{d_0} - \frac{(1+u)^{2g}(1+v)^{2g}(1+ux)^g(1+vx)^g}{(1-uv)(1-x)(1-uvx)(1-(uv)^2x)x^{d_1-d_2}} (uv)^{2d_0} x^{d_0} \right),$$

from which the result follows. \square

Obviously, $e(\mathcal{N}_\sigma) = 0$ for $\sigma < \sigma_m$, since in this case \mathcal{N}_σ is empty.

We recover the Poincaré polynomial of the moduli space $\mathcal{N}_\sigma(2, 1, d_1, d_2)$. This agrees with the formula given in [14, Theorem 6.4] for the case of parabolic triples, if we consider in the latter that there are no parabolic points.

Corollary 6.3. *Let X be a smooth projective curve of genus $g \geq 2$ and consider the moduli space $\mathcal{N}_\sigma = \mathcal{N}_\sigma(2, 1, d_1, d_2)$ for a non-critical value $\sigma > \sigma_m$. Set $d_0 = \left\lceil \frac{1}{3}(\sigma + d_1 + d_2) \right\rceil + 1$.*

Then the Poincaré polynomial of $\mathcal{N}_\sigma = \mathcal{N}_\sigma(2, 1, d_1, d_2)$ is

$$P_t(\mathcal{N}_\sigma) = \text{coeff}_{x^0} \left[\frac{(1+t)^{4g}(1+tx)^{2g}}{(1-t^2)(1-x)(1-t^2x)x^{d_1-d_2-d_0}} \left(\frac{t^{2d_1-2d_2-2d_0}}{1-t^{-2}x} - \frac{t^{-2d_1+2g-2+4d_0}}{1-t^4x} \right) \right].$$

Proof: Substitute $u = t$, $v = t$ into the formula of Theorem 6.2. \square

Proof of Theorem 1.1: By Lemma 5.2, we have an isomorphism

$$\mathfrak{M}_\tau(2, d) \times \text{Jac}^{d_2}(X) \cong \mathcal{N}_\sigma(2, 1, d_1, d_2),$$

with $d_1 = d + 2d_2$ and $\sigma = 3\tau - d$. Then

$$e(\mathfrak{M}_\tau(2, d)) = \frac{e(\mathcal{N}_\sigma(2, 1, d_1, d_2))}{(1+u)^g(1+v)^g}.$$

So Theorem 6.2 gives

$$\begin{aligned} e(\mathfrak{M}_\tau(2, d)) &= \text{coeff}_{x^0} \left[\frac{(1+u)^g(1+v)^g(1+ux)^g(1+vx)^g}{(1-uv)(1-x)(1-uvx)x^{d+d_2-d_0}} \left(\frac{(uv)^{d+d_2-d_0}}{1-(uv)^{-1}x} - \frac{(uv)^{g-1-d-2d_2+2d_0}}{1-(uv)^2x} \right) \right] \\ &= \text{coeff}_{x^0} \left[\frac{(1+u)^g(1+v)^g(1+ux)^g(1+vx)^g}{(1-uv)(1-x)(1-uvx)x^{d-1-[\tau]}} \left(\frac{(uv)^{d-1-[\tau]}}{1-(uv)^{-1}x} - \frac{(uv)^{g+1-d+2[\tau]}}{1-(uv)^2x} \right) \right], \end{aligned}$$

since $d_0 = \lceil \frac{1}{3}(\sigma + d_1 + d_2) \rceil + 1 = [\tau] + d_2 + 1$. Note that τ is not an integer and

$$\tau \in J = [d/2, d],$$

using (5.1), $d_1 = d + 2d_2$ and $\tau = \frac{1}{3}(\sigma + d)$. The critical values for $\mathfrak{M}_\tau(2, d)$ are the integers in J .

For the second part, note that the determinant map gives a fibration

$$\mathfrak{M}_\tau(2, d) \rightarrow \text{Jac}^d X \tag{6.3}$$

with fibers isomorphic to $\mathfrak{M}_\tau(2, \Lambda)$. Note that all the moduli spaces $\mathfrak{M}_\tau(2, \Lambda')$ for any $\Lambda' \in \text{Jac}^d X$ are isomorphic to each other and smooth (for non-critical value τ). The Serre spectral sequence of the fibration has E_2 term isomorphic to $H^*(\mathfrak{M}_\tau(2, \Lambda)) \otimes H^*(\text{Jac}^d X)$ and converges to $H^*(\mathfrak{M}_\tau(2, d))$. By [19, formula (4.1)] (upon the substitution $i = d - 1 - [\tau]$),

$$P_t(\mathfrak{M}_\tau(2, \Lambda)) = \text{coeff}_{x^0} \left[\frac{(1+tx)^{2g}}{(1-t^2)(1-x)(1-t^2x)x^{d-1-[\tau]}} \left(\frac{t^{2d-2-2[\tau]}}{1-t^{-2}x} - \frac{t^{2g+2-2d+4[\tau]}}{1-t^4x} \right) \right],$$

and by our above calculation

$$P_t(\mathfrak{M}_\tau(2, d)) = \text{coeff}_{x^0} \left[\frac{(1+t)^{2g}(1+tx)^{2g}}{(1-t^2)(1-x)(1-t^2x)x^{d-1-[\tau]}} \left(\frac{t^{2d-2-2[\tau]}}{1-t^{-2}x} - \frac{t^{2g+2-2d+4[\tau]}}{1-t^4x} \right) \right],$$

from where we obtain $P_t(\mathfrak{M}_\tau(2, d)) = P_t(\mathfrak{M}_\tau(2, \Lambda))P_t(\text{Jac}^d X)$. Therefore the Serre spectral sequence degenerates, and so the fibration (6.3) is a rationally cohomologically trivial fibration. This implies that there is an isomorphism of Hodge structures

$$H^*(\mathfrak{M}_\tau(2, d)) \cong H^*(\mathfrak{M}_\tau(2, \Lambda)) \otimes H^*(\text{Jac}^d X).$$

So

$$e(\mathfrak{M}_\tau(2, d)) = e(\text{Jac}^d X)e(\mathfrak{M}_\tau(2, \Lambda)).$$

The result follows from this. \square

7. HODGE POLYNOMIAL OF THE MODULI SPACE OF TRIPLES OF RANK (1, 2)

For completeness, we include the computation of the Hodge polynomial of the moduli of triples of rank (1, 2). Such triples are of the form $\phi : E_2 \rightarrow E_1$, where E_2 is a rank 2 bundle and E_1 is a line bundle. By Proposition 3.6, σ is in the interval

$$I = [\sigma_m, \sigma_M] = [\mu_1 - \mu_2, 4(\mu_1 - \mu_2)] = [d_1 - d_2/2, 4d_1 - 2d_2],$$

where $\mu_1 - \mu_2 \geq 0$. Otherwise \mathcal{N}_σ is empty. By the duality result of Proposition 3.4, one has an isomorphism

$$\mathcal{N}_\sigma(1, 2, d_1, d_2) \cong \mathcal{N}_\sigma(2, 1, -d_2, -d_1).$$

So the study of these triples reduces to the case of triples of rank (2, 1). From Theorem 5.1 one immediately obtains

Theorem 7.1. *For $\sigma \in I$, \mathcal{N}_σ is a projective variety. It is smooth and of (complex) dimension $3g - 2 + 2d_1 - d_2$ at the stable points \mathcal{N}_σ^s . Moreover, for generic values of σ , $\mathcal{N}_\sigma = \mathcal{N}_\sigma^s$ (hence it is smooth and projective).*

From Lemma 5.3, we obtain

Lemma 7.2. *The critical values for $\mathcal{N}_\sigma(1, 2, d_1, d_2)$ are the numbers $\sigma_c = 3d_M + d_1 + d_2$, where $-\mu_2 \leq d_M \leq d_1 - d_2$. Also $\sigma_c = \sigma_m \Leftrightarrow d_M = -\mu_2$.*

Theorem 7.3. *Consider $\mathcal{N}_\sigma = \mathcal{N}_\sigma(1, 2, d_1, d_2)$. Let $\sigma > \sigma_m$ be a non-critical value. Set $d_0 = \left\lceil \frac{1}{3}(\sigma - d_1 - d_2) \right\rceil + 1$. Then the Hodge polynomial of \mathcal{N}_σ is*

$$e(\mathcal{N}_\sigma) = \text{coeff}_{x^0} \left[\frac{(1+u)^{2g}(1+v)^{2g}(1+ux)^g(1+vx)^g}{(1-uv)(1-x)(1-uvx)x^{d_1-d_2-d_0}} \left(\frac{(uv)^{d_1-d_2-d_0}}{1-(uv)^{-1}x} - \frac{(uv)^{d_2+g-1+2d_0}}{1-(uv)^2x} \right) \right].$$

Proof: We use that $e(\mathcal{N}_\sigma(1, 2, d_1, d_2)) = e(\mathcal{N}_\sigma(2, 1, -d_2, -d_1))$ and the formula in Theorem 6.2, where d_1 and d_2 are substituted by $-d_2, -d_1$ and

$$d_0 = \left\lceil \frac{1}{3}(\sigma - d_2 - d_1) \right\rceil + 1. \quad \square$$

8. HODGE POLYNOMIAL OF THE MODULI OF BUNDLES

Let $M(2, d)$ denote the moduli space of polystable vector bundles of rank 2 and degree d over X . Note that $M(2, d) \cong M(2, d + 2k)$, for any integer k , so there are basically two moduli spaces, depending on whether the degree is even or odd. We are going to apply the results of the Section 6 to compute the Hodge polynomials of $M(2, d)$ and $M(2, \Lambda)$ with d odd, and Λ a fixed line bundle of degree d on X , recovering the results of [12] for $M(n, d)$ and for $M(n, \Lambda)$, when $n = 2$. Note that the Hodge polynomial of the moduli space $M(2, \Lambda)$ was also found by del Baño [7, §3] using motivic techniques. The formula for the Hodge polynomial of $M(2, \Lambda)$ can also be obtained from the previous paper [17], where an explicit basis for the cohomology groups of $M(2, \Lambda)$ is given and the elements of that basis are of pure Hodge type provided one chooses the underlying basis of $H^1(X)$ to consist of elements of pure Hodge type.

Proposition 8.1. *The Hodge polynomial of $M(2, d)$, for odd degree d , is*

$$e(M(2, d)) = \frac{(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^{2g}(1+v)^{2g}}{(1-uv)(1-(uv)^2)}.$$

Suppose that Λ is a fixed line bundle of odd degree on X . Then the Hodge polynomial of $M(2, \Lambda)$ is

$$e(M(2, \Lambda)) = \frac{(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^g(1+v)^g}{(1-uv)(1-(uv)^2)}.$$

Proof: Atiyah and Bott [1, Proposition 9.7] show that the determinant map $M(n, d) \rightarrow \text{Jac}^d X$ with fibre $M(n, \Lambda)$ is a cohomologically trivial fibration (for the case $n = 2$, the cohomological triviality of this fibration was proved in [16]). This implies [12, Lemma 3] an isomorphism of Hodge structures

$$H^*(M(n, d)) \cong H^*(M(n, \Lambda)) \otimes H^*(\text{Jac}^d X),$$

from where, in particular,

$$e(M(2, d)) = e(M(2, \Lambda))e(\text{Jac} X).$$

Therefore the second formula is a consequence of the first. We shall prove the first one.

Choose $(n_1, d_1) = (2, d)$ and $(n_2, d_2) = (1, d_2)$. If d_2 is very negative then $\mu_1 - \mu_2 = d/2 - d_2 > 2g - 2$. We shall choose the minimum possible value $d - 2d_2 = 4g - 3$. Then Proposition 4.10 (i) applies and $\mathcal{N}_{\sigma_m^+}$ is the projectivization of a vector bundle over $M(2, d) \times \text{Jac}^{d_2} X$, of rank $d - 2d_2 - 2(g - 1) = 2g - 1$. Lemma 2.3 implies that

$$e(\mathcal{N}_{\sigma_m^+}) = e(\text{Jac}^{d_2} X) e(M(2, d)) e_{2g-1}, \quad (8.1)$$

where

$$e(\text{Jac}^{d_2} X) = (1+u)^g(1+v)^g \quad \text{and} \quad e_{2g-1} = \frac{1 - (uv)^{2g-1}}{1 - uv}. \quad (8.2)$$

To compute the left hand side of (8.1), we use Theorem 6.2 for $\sigma = \sigma_m^+ = \mu_1 - \mu_2 + \varepsilon$, $\varepsilon > 0$ small. Clearly,

$$d_0 = \left\lceil \frac{1}{3}(\mu_1 - \mu_2 + \varepsilon + 2\mu_1 + \mu_2) \right\rceil + 1 = \lceil \mu_1 \rceil + 1 = \frac{d+1}{2}.$$

The Hodge polynomial of $\mathcal{N}_{\sigma_m^+}$ is thus

$$e(\mathcal{N}_{\sigma_m^+}) = \text{coeff}_{x^0} \left[\frac{(1+u)^{2g}(1+v)^{2g}(1+ux)^g(1+vx)^g}{(1-uv)(1-x)(1-uvx)x^{d-d_2-(d+1)/2}} \left(\frac{(uv)^{d-d_2-d_0}}{1-(uv)^{-1}x} - \frac{(uv)^{-d+g-1+d_0}}{1-(uv)^2x} \right) \right].$$

Plugging this into (8.1) and using (8.2), we get

$$e(M(2, d)) = \frac{(1+u)^g(1+v)^g}{1-(uv)^{2g-1}} \text{coeff}_{x^0} \left[\frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)x^{2g-2}} \left(\frac{(uv)^{2g-2}}{1-(uv)^{-1}x} - \frac{(uv)^g}{1-(uv)^2x} \right) \right].$$

To compute the coefficient of x^0 , we work as follows. Denote

$$f(x) = (1+ux)^g(1+vx)^g x^{1-2g},$$

$$F(a, b, c) = \text{coeff}_{x^0} \frac{xf(x)}{(1-ax)(1-bx)(1-cx)},$$

with a, b, c distinct and different from zero. Then

$$e(M(2, d)) = \frac{(1+u)^g(1+v)^g}{1-(uv)^{2g-1}} \left[F(1, uv, \frac{1}{uv}) (uv)^{2g-2} - F(1, uv, (uv)^2) (uv)^g \right]. \quad (8.3)$$

On the other hand, we have the equality

$$\frac{1}{(1-ax)(1-bx)(1-cx)} = \frac{A}{1-ax} + \frac{B}{1-bx} + \frac{C}{1-cx},$$

where

$$A = \frac{a^2}{(a-b)(a-c)}, \quad B = \frac{b^2}{(b-a)(b-c)}, \quad C = \frac{c^2}{(c-a)(c-b)},$$

and so, by the residue theorem applied to $\frac{f(x)}{(1-ax)(1-bx)(1-cx)}$,

$$F(a, b, c) = -\operatorname{Res} \left\{ \frac{A f(x)}{1-ax}; \frac{1}{a} \right\} - \operatorname{Res} \left\{ \frac{B f(x)}{1-bx}; \frac{1}{b} \right\} - \operatorname{Res} \left\{ \frac{C f(x)}{1-cx}; \frac{1}{c} \right\}, \quad (8.4)$$

because $f(x)$ is holomorphic in $\mathbb{C} - \{0\}$ and $\frac{f(x)}{(1-ax)(1-bx)(1-cx)}$ has no pole at ∞ . We use the basic identity

$$-\operatorname{Res} \left\{ \frac{f(x)}{1-tx}; \frac{1}{t} \right\} = \frac{1}{t} f \left(\frac{1}{t} \right), \quad (8.5)$$

with $t = a, b, c$ respectively, for the calculation of residues at a simple pole. Substituting (8.5) into (8.4) we get

$$F(a, b, c) = \frac{(a+u)^g(a+v)^g}{(a-b)(a-c)} + \frac{(b+u)^g(b+v)^g}{(b-a)(b-c)} + \frac{(c+u)^g(c+v)^g}{(c-a)(c-b)}.$$

Putting this into (8.3), we get

$$e(M(2, d)) = \frac{(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^{2g}(1+v)^{2g}}{(1-uv)(1-(uv)^2)}.$$

The computation only works for generic values of u, v (because of the restriction that we have imposed on a, b, c above), but this is enough to get the equality, since both terms are polynomials in u and v . \square

REFERENCES

- [1] ATIYAH, M. F.; BOTT, R.: The Yang-Mills equations over Riemann surfaces. *Philos. Trans. Roy. Soc. London Ser. A* **308** (1982) 523–615.
- [2] BERTRAM, A.: Stable pairs and stable parabolic pairs. *J. Algebraic Geom.* **3** (1994), no. 4, 703–724.
- [3] BRADLOW, S. B.; DASKALOPOULOS: Moduli of stable pairs for holomorphic bundles over Riemann surfaces. *Int. J. Math.* **2** (1991), 477–513.
- [4] BRADLOW, S. B.; GARCÍA-PRADA, O.: Stable triples, equivariant bundles and dimensional reduction. *Math. Ann.* **304** (1996), no. 2, 225–252.
- [5] BRADLOW, S. B.; GARCÍA-PRADA, O.; GOTHEN, P.B: Moduli spaces of holomorphic triples over compact Riemann surfaces. *Math. Ann.* **328** (2004), no. 1-2, 299–351.
- [6] BURILLO, J.: El polinomio de Poincaré–Hodge de un producto simétrico de variedades kählerianas compactas. *Collect. Math.* **41** (1990), no. 1, 59–69.
- [7] DEL BAÑO, S.: On the motive of moduli spaces of rank two vector bundles over a curve. *Compositio Math.* **131** (2002), no. 1, 1–30.
- [8] DELIGNE, P.: Théorie de Hodge I,II,III. In *Proc. I.C.M.*, vol. 1, 1970, pp. 425–430; in *Publ. Math. I.H.E.S.* **40** (1971), 5–58; *ibid.* **44** (1974), 5–77.
- [9] DURFEE, A.H.: *Algebraic varieties which are a disjoint union of subvarieties*, *Lecture Notes in Pure Appl. Math.* **105**, Marcel Dekker, 1987, pp. 99–102.

- [10] DANIVOL, V.I.; KHOVANSKIĬ, A.G.: Newton polyhedra and an algorithm for computing Hodge-Deligne numbers, *Math. U.S.S.R. Izvestiya* **29** (1987), 279–298.
- [11] DESALE, U. V.; RAMANAN, S.: Poincaré polynomials of the variety of stable bundles. *Math. Ann.* **216** (1975), no. 3, 233–244.
- [12] EARL, R.; KIRWAN, F.: The Hodge numbers of the moduli spaces of vector bundles over a Riemann surface. *Q. J. Math.* **51** (2000), no. 4, 465–483.
- [13] GARCÍA-PRADA, O.: Dimensional reduction of stable bundles, vortices and stable pairs. *Internat. J. Math.* **5** (1994), no. 1, 1–52.
- [14] GARCÍA-PRADA, O.; GOTHEN, P.B.; MUÑOZ, V.: Betti numbers of the moduli space of rank 3 parabolic Higgs bundles. *Memoirs Amer. Math. Soc.* In press.
- [15] HARDER, G.; NARASIMHAN, M. S.: On the cohomology groups of moduli spaces of vector bundles on curves. *Math. Ann.* **212** (1974/75), 215–248.
- [16] NEWSTEAD, P.E.: Topological properties of some spaces of stable bundles. *Topology* **6** (1967) 241–262.
- [17] NEWSTEAD, P.E.: Characteristic classes of stable bundles of rank 2 over an algebraic curve. *Trans. Amer. Math. Soc.* **169** (1972), 337–345.
- [18] SCHMITT, A.: A universal construction for the moduli spaces of decorated vector bundles. *Transform. Groups* **9** (2004), no. 2, 167–209.
- [19] THADDEUS, M.: Stable pairs, linear systems and the Verlinde formula. *Invent. Math.* **117** (1994), no. 2, 317–353.

Vicente Muñoz

Departamento de Matemáticas

Consejo Superior de Investigaciones Científicas

Serrano 113 bis, 28006 Madrid, Spain

`vicente.munoz@imaff.cfmac.csic.es`

Daniel Ortega

Departamento de Matemáticas

Facultad de Ciencias

Universidad Autónoma de Madrid

28049 Madrid, Spain

`daniel.ortega@uam.es`

Maria-Jesús Vázquez-Gallo

Departamento de Ingeniería Civil: Servicios Urbanos

Unidad Docente: Matemáticas

Escuela de Ingeniería de Obras Públicas

Universidad Politécnica de Madrid

Alfonso XII 3 y 5, 28014 Madrid, Spain

`mariajesus.vazquez@upm.es`