Weakly Lefschetz symplectic manifolds

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Abstract

The harmonic cohomology of a Donaldson symplectic submanifold and of an Auroux symplectic submanifold are compared with that of its ambient space. We also study symplectic manifolds satisfying a weakly Lefschetz property, that is, the s–Lefschetz propery. In particular, we consider the symplectic blow-ups $\widetilde{CP}^m$ of the complex projective space CP^m along weakly Lefschetz symplectic submanifolds $M \subset CP^m$. As an application we construct, for each even integer $s \geq 2$, compact symplectic manifolds which are s–Lefschetz but not $(s+1)$–Lefschetz.

1 Introduction

One of the main results of Hodge theory states that any de Rham cohomology class on a compact oriented Riemannian manifold has an unique harmonic representative. In the symplectic setting a notion of harmonicity can be introduced as follows [3]. Let (M, ω) be $2n$–dimensional symplectic manifold. A closed form α on M is called *symplectically harmonic* if $\delta\alpha = 0$, where δ denotes the Koszul differential [16]. However, a symplectic version of the above result does not hold in general. In fact, Mathieu [20] proved that any de Rham cohomology class has a (not necessarily unique) symplectically harmonic representative if and only if (M, ω) satisfies the hard Lefschetz property, i.e. the map

$$L^{n-k}: H^k(M) \longrightarrow H^{2n-k}(M) \qquad (1)$$

given by $L^{n-k}[\alpha] = [\alpha \wedge \omega^{n-k}]$ is onto for all $k \leq n-1$.

In this paper we deal with symplectic manifolds satisfying a weaker property: following [9], we shall say that (M, ω) is an *s–Lefschetz symplectic manifold*, $0 \leq s \leq n-1$, if (1) is an epimorphism for all $k \leq s$. As an obvious fact, whenever (M, ω) is not hard Lefschetz, there is some $s \geq 0$ such that (M, ω) is s–Lefschetz but not $(s+1)$–Lefschetz. So, it seems interesting to understand the way this phenomenon occurs on non-hard Lefschetz symplectic manifolds, in particular if there is some restriction for the possible values of the level s at which the Lefschetz property can be lost, how this affects to other symplectic invariants of the manifold, such as the above mentioned harmonicity, or if the s–Lefschetz property is preserved under the usual constructions of new symplectic manifolds from old ones, for instance the symplectic blowing up [21], the Donaldson symplectic submanifolds [5] and the Auroux symplectic submanifolds [1]. Our purpose in this paper is to explore these questions, as we explain next.

Regarding symplectic harmonicity, in Section 2 we recall some results on the harmonic cohomology of (M, ω) and show how the s–Lefschetz property is related to the existence problem of symplectically harmonic representatives of de Rham classes of M. Let us denote by $H^k_{\rm hr}(M, \omega)$ the space of *harmonic cohomology* in degree k, that is, the subspace of the de Rham cohomology group $H^k(M)$ consisting of all classes which contain at least one symplectically harmonic k–form. In Proposition 2.6 we prove that a $2n$–dimensional symplectic manifold (M,ω) is s-Lefschetz if and only if $H^{2n-k}_{\rm hr}(M,\omega) = H^{2n-k}(M)$ for every $k \leq s$; moreover, the latter condition implies that $H^k_{\rm hr}(M,\omega) = H^k(M)$ for every $k \leq s+2$. In the proof of this proposition, which can be seen as a refinement of the result of Mathieu, we follow the approach by Yan [26] which uses the theory of infinite dimensional $\mathfrak{sl}(2,\mathbb{C})$–representations.

Section 3 is devoted to the study of harmonic cohomology of Donaldson and Auroux symplectic submanifolds. Given a compact symplectic manifold (M,ω) of dimension $2n$ such that $[\omega] \in H^2(M)$ admits a lift to an integral cohomology class, Donaldson proves in [5] the existence of a symplectic submanifold (Z,ω_Z) of codimension 2 in M which realizes the Poincaré dual of $k\,[\omega]$ for any sufficiently large integer k, and such that the inclusion $\jmath\colon Z \hookrightarrow M$ is $(n-1)$–connected. We show the following relation between the harmonic cohomologies $H^*_{\rm hr}(Z,\omega_Z)$ and $H^*_{\rm hr}(M,\omega)$.

Theorem 1.1 *The inclusion $\jmath\colon Z \hookrightarrow M$ induces an isomorphism $\jmath^*\colon H^i_{\rm hr}(M,\omega) \longrightarrow H^i_{\rm hr}(Z,\omega_Z)$ for any $i < n-1$, and a monomorphism for $i = n-1$. Moreover, $H^i_{\rm hr}(Z,\omega_Z)$ and $H^{i+2}_{\rm hr}(M,\omega)$ are isomorphic for every $n \leq i \leq 2(n-1)$.*

Roughly speaking, this result says that a Donaldson symplectic submanifold inherits essentially the same harmonic cohomology as that of its ambient space, with the only possible exception of having less symplectically harmonic forms in the middle degree $n-1$. Auroux has generalized in [1] Donaldson's construction. We show that a result like Theorem 1.1 does not hold in general for the Auroux submanifolds. Moreover, the harmonic cohomology of Auroux symplectic submanifolds has a very different behaviour with respect to its ambient space, and surprisingly there exist submanifolds having strictly more harmonic cohomology classes than their ambient spaces. More concretely, in Example 3.3 we construct an Auroux submanifold (Z,ω_Z) of codimension 2 in a 10-dimensional compact symplectic manifold (M,ω) such that the inclusion $\jmath\colon Z \hookrightarrow M$ induces an isomorphism between the de Rham cohomology groups $H^3(Z)$ and $H^3(M)$, but $\dim H^3_{\rm hr}(Z,\omega_Z) > \dim H^3_{\rm hr}(M,\omega)$.

Given a compact symplectic manifold (M,ω) of dimension $2n$, we can assume, without loss of generality, that the symplectic form ω is integral (by perturbing and rescaling). A theorem of Gromov and Tischler [11, 12, 24] states that there is a symplectic embedding $i\colon(M,\omega) \longrightarrow (CP^m,\omega_0)$, with $m \geq 2n+1$, where ω_0 is the standard Kähler form on CP^m defined by its natural complex structure and the Fubini–Study metric. We consider the symplectic blow-up $\widetilde{CP}^m$ of CP^m along the embedding i (see [21]). Then, $\widetilde{CP}^m$ is a simply connected compact symplectic manifold. In Section 4 we study the s–Lefschetz property of $\widetilde{CP}^m$, $m \geq 2n+1$. More concretely we have the following result.

Theorem 1.2 *If (M,ω) is an s–Lefschetz compact symplectic manifold of dimension $2n$, then the symplectic blow-up $\widetilde{CP}^m$ $(m \geq 2n+1)$ is $(s+2)$–Lefschetz. Moreover, if M is parallelizable and not s–Lefschetz then $\widetilde{CP}^m$ is not $(s+2)$–Lefschetz.*

This will be proved in Theorem 4.2 and Proposition 4.4. Recently Cavalcanti [4] has investigated the hard Lefschetz property of symplectic blow-ups of non-hard Lefschetz symplectic manifolds along hard Lefschetz symplectic submanifolds; in particular, he obtains that the symplectic blow-up of a hard Lefschetz symplectic manifold along a hard Lefschetz submanifold is always hard Lefschetz. Such a result can be also proved with the arguments of Theorem 1.2 as we notice in Remark 3.3.

In [9] examples of compact symplectic manifolds which are s–Lefschetz but not $(s+1)$–Lefschetz are constructed for each $s \leq 2$. As an application of Theorem 1.2 and of the results of Section 2 on the harmonic cohomology of iterated Donaldson submanifolds of symplectic blow-ups, we prove in Section 5 that for each *even* integer number $s \geq 2$, there is a simply connected compact symplectic manifold of dimension $2(s+2)$ which is s–Lefschetz but not $(s+1)$–Lefschetz. Notice that $2(s+2)$ is the lowest possible dimension where such a manifold can live. With the same techniques, we also show a simply-connected symplectic 10–manifold which is 3–Lefschetz but not 4–Lefschetz.

2 Harmonic cohomology of s-Lefschetz symplectic manifolds

We recall some definitions and results about the symplectic codifferential and symplectically harmonic forms. Let (M, ω) be a symplectic manifold, that is, M is a differentiable manifold of dimension $2n$ and ω a closed non-degenerate 2–form on M, the *symplectic form*. Denote by $\Omega^*(M)$, $\mathfrak{X}(M)$ and $\mathcal{F}(M)$ the algebras of differential forms, vector fields and differentiable functions on M, respectively. The isomorphism

$$\natural : \mathfrak{X}(M) \longrightarrow \Omega^1(M)$$

given by $\natural(X) = \iota_X(\omega)$ for $X \in \mathfrak{X}(M)$, where ι_X denotes the contraction by X, extends to an isomorphism of algebras $\natural : \bigoplus_{k\geq 0} \mathfrak{X}^k(M) \longrightarrow \bigoplus_{k\geq 0} \Omega^k(M)$. Then, $G = -\natural^{-1}(\omega)$ is the skew-symmetric bivector field dual to ω. (G is the unique non-degenerate Poisson structure [18] associated with ω.) The Koszul differential $\delta \colon \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$ is defined by

$$\delta = [\iota_G, d].$$

In [3] Brylinski proved that the Koszul differential is a *symplectic codifferential* of the exterior differential with respect to the *symplectic star operator* defined as follows. Denote by $\Lambda^k(G)$, $k \geq 0$, the associated pairing $\Lambda^k(G) : \Omega^k(M) \times \Omega^k(M) \longrightarrow \mathcal{F}(M)$ which is $(-1)^k$–symmetric (i.e. symmetric for even k, anti-symmetric for odd k). Let v_M be the volume form on M given by $v_M = \frac{\omega^n}{n!}$. Imitating the Hodge star operator for Riemannian manifolds, the *symplectic star operator*

$$* : \Omega^k(M) \longrightarrow \Omega^{2n-k}(M)$$

is defined by the condition $\beta \wedge (*\alpha) = \Lambda^k(G)(\beta, \alpha)\, v_M$, for $\alpha, \beta \in \Omega^k(M)$. An easy consequence is that $*^2 = Id$, and if $\alpha \in \Omega^k(M)$ then

$$\delta(\alpha) = (-1)^{k+1}(* \circ d \circ *)(\alpha).$$

Since ω is a closed form, for any $p, k \geq 0$ the homomorphism

$$L^p : \Omega^k(M) \longrightarrow \Omega^{2p+k}(M)$$

given by $L^p(\alpha) = \alpha \wedge \omega^p$ for $\alpha \in \Omega^k(M)$, satisfies that $[L^p, d] = L^p \circ d - d \circ L^p = 0$, and therefore it induces a map $L^p : H^k(M) \longrightarrow H^{2p+k}(M)$ on de Rham cohomology. Relations between the operators ι_G, L, d and δ are proved by Yan in [26]. Here we shall need the following

$$[L, \delta] = d. \tag{2}$$

A k–form $\alpha \in \Omega^k(M)$ is said to be *symplectically harmonic* if $d\alpha = \delta\alpha = 0$. Let $\Omega^k_{\text{hr}}(M, \omega) = \{\alpha \in \Omega^k(M) \mid d\alpha = \delta\alpha = 0\}$ be the space of the symplectically harmonic k–forms. Yan proved that for any $k \geq 0$ the map $L^{n-k} : \Omega^k(M) \longrightarrow \Omega^{2n-k}(M)$ is an isomorphism. This also induces an isomorphism when restricted to the subspaces of harmonic forms, as follows from (2).

Lemma 2.1 [26] *(Duality on harmonic forms). The map*

$$L^{n-k} \colon \Omega^k_{\rm hr}(M,\omega) \longrightarrow \Omega^{2n-k}_{\rm hr}(M,\omega)$$

is an isomorphism for $k \geq 0$.

For the de Rham cohomology classes of M, we consider the vector space

$$H^k_{\rm hr}(M,\omega) = \frac{\Omega^k_{\rm hr}(M,\omega)}{\Omega^k_{\rm hr}(M,\omega) \cap {\rm Im}\, d}$$

consisting of the cohomology classes in $H^k(M)$ containing at least one symplectically harmonic form. Lemma 2.1 implies that the homomorphism

$$L^{n-k} \colon H^k_{\rm hr}(M,\omega) \longrightarrow H^{2n-k}_{\rm hr}(M,\omega)$$

is surjective. (Notice that the duality on harmonic forms may not be satisfied at the level of the spaces $H^*_{\rm hr}(M,\omega)$.) Since $H^{2n-k}_{\rm hr}(M,\omega)$ is a subspace of the de Rham cohomology $H^{2n-k}(M)$, we conclude (see [14, Corollary 1.7])

$$H^{2n-k}_{\rm hr}(M,\omega) = {\rm Im}\,(L^{n-k} \colon H^k_{\rm hr}(M,\omega) \longrightarrow H^{2n-k}(M)). \tag{3}$$

A nonzero k–form α, with $k \leq n$, is called *effective* if $L^{n-k+1}(\alpha) = 0$. A cohomology class $a \in H^k(M)$ is said to be *primitive* if $L^{n-k+1}(a) = 0$ in $H^{2n-k+2}(M)$. In [17, page 46] the following result is proved.

Lemma 2.2 [17]. *If α is an effective k–form, then there is a constant c such that its symplectic star operator $*\alpha$ satisfies $*\alpha = c\, L^{n-k}(\alpha)$.*

Therefore, any closed effective k–form α on (M,ω) is symplectically harmonic because $d{*}\alpha = c\, d\, L^{n-k}(\alpha) = 0$ by Lemma 2.2. In particular, every closed 1–form is symplectically harmonic since it is effective, so $H^1_{\rm hr}(M,\omega) = H^1(M)$. Moreover, we have

Proposition 2.3 *Let (M,ω) be a symplectic manifold of dimension $2n$. Suppose that there exists some integer $k \leq n$ with $H^{2n-k+2}(M) = 0$. Then, for any closed k–form $\alpha \in \Omega^k(M)$, there is a closed k–form $\widetilde{\alpha}$ such that $\widetilde{\alpha}$ is cohomologous to α and is symplectically harmonic. In particular, $H^k_{\rm hr}(M,\omega) = H^k(M)$.*

Proof : Let $a = [\alpha] \in H^k(M)$. We will find a symplectically harmonic representative of the cohomology class a. Since $L^{n-k+1}(\alpha)$ is a closed $(2n-k+2)$–form and $H^{2n-k+2}(M)$ is zero, there is some $\beta \in \Omega^{2n-k+1}(M)$ such that $L^{n-k+1}(\alpha) = d\beta$. But the map $L^{n-k+1} \colon \Omega^{k-1}(M) \longrightarrow \Omega^{2n-k+1}(M)$ is surjective, so there exists $\gamma \in \Omega^{k-1}(M)$ satisfying $\beta = L^{n-k+1}(\gamma)$. Hence $L^{n-k+1}(\alpha) = d\beta = L^{n-k+1}(d\gamma)$, i.e., $L^{n-k+1}(\alpha - d\gamma) = 0$.

Consider $\widetilde{\alpha} = \alpha - d\gamma$, which is cohomologous to α. Using Lemma 2.2, $L^{n-k+1}(\widetilde{\alpha}) = 0$ implies that $*\widetilde{\alpha} = c\, L^{n-k}(\widetilde{\alpha})$ for some constant c. Thus $d * \widetilde{\alpha} = c\, L^{n-k}(d\widetilde{\alpha}) = 0$ and the k–form $\widetilde{\alpha}$ is symplectically harmonic. *QED*

For the de Rham classes in $H^2(M)$, Mathieu proved the following result.

Lemma 2.4 [20]. *Any cohomology class of degree 2 has a symplectically harmonic representative.*

As a consequence of the previous results, if (M,ω) is a simply connected compact symplectic manifold then every class in $H^k(M)$ has a symplectically harmonic representative for $k \leq 3$.

Corollary 2.5 *Let (M,ω) be a compact simply connected symplectic manifold of dimension 6. Then every de Rham cohomology class of degree $k \neq 4$ admits a symplectically harmonic representative.*

Recall that a symplectic manifold (M,ω) of dimension $2n$ is said to be s–*Lefschetz* with $0 \leq s \leq n-1$, if the map $L^{n-k}\colon H^k(M) \longrightarrow H^{2n-k}(M)$ is an epimorphism for all $k \leq s$. In the compact case we actually have that L^{n-k} are isomorphisms because of Poincaré duality. Note that M is $(n-1)$–Lefschetz if M satisfies the hard Lefschetz theorem.

Proposition 2.6 *Let (M,ω) be a symplectic manifold of dimension $2n$ and let $s \leq n-1$. Then the following statements are equivalent:*

(i) (M,ω) *is s–Lefschetz.*

(ii) $H^k_{\mathrm{hr}}(M,\omega) = H^k(M)$ *for every* $k \leq s+2$, *and* $H^{2n-k}_{\mathrm{hr}}(M,\omega) = H^{2n-k}(M)$ *for every* $k \leq s$.

(iii) $H^{2n-k}_{\mathrm{hr}}(M,\omega) = H^{2n-k}(M)$ *for every* $k \leq s$.

Proof : Clearly (ii) implies (iii). Let us see also that (iii) implies (i). Let $k \leq s$. From (3) we have that

$$H^{2n-k}_{\mathrm{hr}}(M,\omega) = \mathrm{Im}\,(L^{n-k}\mid_{H^k_{\mathrm{hr}}(M,\omega)}\colon H^k_{\mathrm{hr}}(M,\omega) \hookrightarrow H^k(M) \longrightarrow H^{2n-k}(M)).$$

If $H^{2n-k}_{\mathrm{hr}}(M,\omega) = H^{2n-k}(M)$, then the map $L^{n-k}\mid_{H^k_{\mathrm{hr}}(M,\omega)}$ is onto, and therefore the homomorphism $L^{n-k}\colon H^k(M) \longrightarrow H^{2n-k}(M)$ must also be onto. So M is s–Lefschetz.

We want to show that (i) implies (ii). It is enough to prove that $H^k_{\mathrm{hr}}(M,\omega) = H^k(M)$ for every $k \leq s+2$, because in this case, for $k \leq s$, we have $H^{2n-k}_{\mathrm{hr}}(M,\omega) = \mathrm{Im}\,(L^{n-k}\colon H^k(M) \longrightarrow H^{2n-k}(M)) = H^{2n-k}(M)$ using the s–Lefschetz property.

Let us see that $H^k_{\mathrm{hr}}(M,\omega) = H^k(M)$ for every $k \leq s+2$, by induction on s. For $s=0$, we recall that M is 0–Lefschetz as this is satisfied by every symplectic manifold. Now for any symplectic manifold, any class of degree 1 admits an harmonic representative by Lemma 2.2, and we also know that any class of degree 2 admits an harmonic representative by Lemma 2.4.

Now take $s>0$, and suppose that if (M,ω) is $(s-1)$–Lefschetz, it holds $H^k_{\mathrm{hr}}(M,\omega) = H^k(M)$ for $k \leq s+1$. We have to prove that $H^{s+2}_{\mathrm{hr}}(M,\omega) = H^{s+2}(M)$ if M is s–Lefschetz. Let α be a closed element of degree $s+2$. Consider the map $L^{n-s-1} : \Omega^{s+2}(M) \longrightarrow \Omega^{2n-s}(M)$. Then $L^{n-s-1}(\alpha)$ is a closed $(2n-s)$–form. By the s–Lefschetz property there is a closed s–form h (which we may suppose to be symplectically harmonic, by induction hypothesis) such that

$$L^{n-s-1}(\alpha) = L^{n-s}(h) + d\beta\,,$$

for some $\beta \in \Omega^{2n-s-1}(M)$. By the surjectivity of $L^{n-s-1}\colon \Omega^{s+1}(M) \longrightarrow \Omega^{2n-s-1}(M)$ we get the existence of some $(s+1)$–form γ with $\beta = L^{n-s-1}(\gamma)$. Therefore $L^{n-s-1}(\alpha) = L^{n-s}(h) + L^{n-s-1}(d\gamma)$ and hence

$$L^{n-s-1}(\alpha - L(h) - d\gamma) = 0. \tag{4}$$

Put $\widetilde{\alpha} = \alpha - L(h) - d\gamma$. By (4) and Lemma 2.2 we have that $*\,\widetilde{\alpha} = c\,L^{n-s-2}(\widetilde{\alpha})$ for some constant c. Therefore $\widetilde{\alpha}$ is symplectically harmonic. On the other hand, since h is symplectically harmonic we see that $L(h)$ is symplectically harmonic using (2). Hence $\alpha - d\gamma$ is symplectically harmonic and cohomologous to the original α. $\boxed{QED}$

Notice that this result implies that every de Rham cohomology class of M admits a symplectically harmonic representative if and only if (M, ω) is hard Lefschetz, which is Mathieu's theorem. Also, if M is a simply connected compact symplectic manifold of dimension 6 then (M, ω) is hard Lefschetz if and only if every cohomology class of degree 4 has a symplectically harmonic representative.

If M is a manifold of finite type, i.e. all the de Rham cohomology groups $H^k(M)$ are finite dimensional, then we shall denote by $b_k^{\mathrm{hr}}(M, \omega)$ the dimension of the space $H_{\mathrm{hr}}^k(M, \omega)$. As usual, the Betti numbers of M will be denoted by $b_k(M) = \dim H^k(M)$.

It is well-known that if (M, ω) is compact and hard Lefschetz, the odd Betti numbers of M are even. When (M, ω) is s–Lefschetz we have the following proposition.

Proposition 2.7 *Let (M, ω) be a compact symplectic manifold of dimension $2n$. Suppose that (M, ω) is s–Lefschetz with $s \leq n-1$. Then the odd Betti numbers $b_{2i-1}(M)$ are even for $2i-1 \leq s$, and $b_{2n-2j+1}^{\mathrm{hr}}(M, \omega)$ is even for $s < 2j-1 \leq s+2$.*

Proof : Put $k = 2i-1 \leq s$. Let us consider the non-singular pairing

$$p\colon H^k(M) \otimes H^{2n-k}(M) \longrightarrow \mathbb{R}$$

given by

$$p([\alpha], [\beta]) = \int_M \alpha \wedge \beta,$$

for $[\alpha] \in H^k(M)$ and $[\beta] \in H^{2n-k}(M)$. Let $\langle\ ,\ \rangle$ be the skew-symmetric bilinear form defined on $H^k(M)$ by

$$\langle [\alpha], [\alpha'] \rangle = p([\alpha], L^{n-k}[\alpha']),$$

for $[\alpha], [\alpha'] \in H^k(M)$. The rank of $\langle\ ,\ \rangle$ is an even number (see page 4, [17]). The non-singularity of p implies that the rank of $\langle\ ,\ \rangle$ equals the rank of the map $L^{n-k}\colon H^k(M) \longrightarrow H^{2n-k}(M)$, that is, rank $\langle\ ,\ \rangle = b_{2n-k}(M)$ since (M, ω) is s–Lefschetz. Hence $b_k(M)$ is even by Poincaré duality.

For the final part, take $k = 2j-1$ with $s < k \leq s+2$. Now, the previous argument also shows that $b_{2n-k}^{\mathrm{hr}}(M, \omega)$ is even because the s–Lefschetz property implies $H^k(M) = H_{\mathrm{hr}}^k(M, \omega)$ by Proposition 2.6 and, on other hand, $H_{\mathrm{hr}}^{2n-k}(M, \omega) = \mathrm{Im}\,(L^{n-k})$, therefore the rank of $\langle\ ,\ \rangle$ is an even number which equals $\dim \mathrm{Im}\,(L^{n-k}) = b_{2n-k}^{\mathrm{hr}}(M, \omega)$. $\boxed{QED}$

3 Harmonic cohomology of Donaldson and Auroux symplectic submanifolds

In this section we study the relation between the harmonic cohomology of Donaldson and Auroux symplectic submanifolds and that of the ambient space.

Recall that given a compact symplectic manifold (M, ω) of dimension $2n$ such that $[\omega] \in H^2(M)$ admits a lift to an integral cohomology class, Donaldson proves [5] the existence of a symplectic submanifold Z of codimension 2 in M that realizes the Poincaré dual of $k\,[\omega]$ for any sufficiently large integer k. Moreover, the inclusion $\jmath\colon Z \hookrightarrow M$ is $(n-1)$–connected, that is, $\jmath^*\colon H^i(M) \longrightarrow H^i(Z)$ is an isomorphism for $i < (n-1)$, and a monomorphism for $i = (n-1)$. Let us denote by $\omega_Z = \jmath^*\omega$ the symplectic form on Z.

Proof of Theorem 1.1 : We use here the following property given in [25, Lemma 4.3]: If (M, ω) is a symplectic manifold of dimension $2n$, then for any $2 \leq i \leq n$ the subspace $H^i_{\rm hr}(M,\omega)$ of $H^i(M)$ is given by

$$H^i_{\rm hr}(M,\omega) = P_i(M,\omega) + L_{[\omega]}\left(H^{i-2}_{\rm hr}(M,\omega)\right), \tag{5}$$

where $P_i(M,\omega) = \{\, a \in H^i(M) \;\mid\; L^{n-i+1}_{[\omega]}(a) = 0\}$.

Similarly, for the Donaldson symplectic submanifold Z we have

$$H^i_{\rm hr}(Z,\omega_Z) = P_i(Z,\omega_Z) + L_{[\omega_Z]}\left(H^{i-2}_{\rm hr}(Z,\omega_Z)\right),$$

for any $2 \leq i \leq n-1$, where $P_i(Z,\omega_Z) = \{\, b \in H^i(Z) \;\mid\; L^{n-i}_{[\omega_Z]}(b) = 0\}$.

On the other hand, in [9] it is proved that for any $i \geq n$, a cohomology class $a \in H^i(M)$ satisfies $\jmath^* a = 0$ if and only if $a \cup [\omega] = 0$.

Let us prove first that $\jmath^*(P_i(M,\omega)) \subset P_i(Z,\omega_Z)$ for any $2 \leq i \leq n-1$. Given $a \in P_i(M,\omega)$, let us consider $b = \jmath^* a \in H^i(Z)$. Since $0 = L^{n-i+1}_{[\omega]}(a) = a \cup [\omega]^{n-i+1}$, and $n+1 \leq 2n-i$ (because $n-1 \geq i$), the cohomology class $L^{n-i}_{[\omega]} a \in H^{2n-i}(M)$ satisfies $\jmath^*(L^{n-i}_{[\omega]} a) = 0$. But $\jmath^* \circ L_{[\omega]} = L_{[\omega_Z]} \circ \jmath^*$, which implies $L^{n-i}_{[\omega_Z]}(b) = \jmath^*(L^{n-i}_{[\omega]} a) = 0$, that is, $b \in P_i(Z,\omega_Z)$. Now it is easy to see that $\jmath^* \colon P_i(M,\omega) \longrightarrow P_i(Z,\omega_Z)$ is an isomorphism for $i < (n-1)$ and a monomorphism for $i = (n-1)$, because $\jmath^* \colon H^i(M) \longrightarrow H^i(Z)$ is.

Now, we prove by induction that $\jmath^*(H^i_{\rm hr}(M,\omega)) \subset H^i_{\rm hr}(Z,\omega_Z)$ for any $i \leq (n-1)$. This is clear for $i = 0, 1$, because $H^i_{\rm hr} = H^i$. Let us fix i with $2 \leq i \leq (n-1)$, and suppose that the inclusion holds in any degree $< i$. Since $\jmath^* \circ L_{[\omega]} = L_{[\omega_Z]} \circ \jmath^*$, and $\jmath^*$ takes the primitive classes of degree i on M to primitive classes of degree i on the submanifold Z, the induction hypothesis and (5) imply that

$$\begin{aligned} \jmath^*(H^i_{\rm hr}(M,\omega)) &= \jmath^*(P_i(M,\omega)) + L_{[\omega_Z]}\left(\jmath^*(H^{i-2}_{\rm hr}(M,\omega))\right) \\ &\subset P_i(Z,\omega_Z) + L_{[\omega_Z]}\left(H^{i-2}_{\rm hr}(Z,\omega_Z)\right) \\ &= H^i_{\rm hr}(Z,\omega_Z). \end{aligned}$$

Therefore, for any $i \leq (n-1)$, we have the map $\jmath^* \colon H^i_{\rm hr}(M,\omega) \longrightarrow H^i_{\rm hr}(Z,\omega_Z)$, which is just the restriction to the space of harmonic cohomology classes of the homomorphism $\jmath^* \colon H^i(M) \longrightarrow H^i(Z)$. Thus, $\jmath^*$ is injective for $i \leq (n-1)$. Finally, an inductive argument as above allows us to conclude that $\jmath^*$ is surjective for $i < (n-1)$.

To complete the proof, it remains to see that $H^i_{\rm hr}(Z,\omega_Z)$ and $H^{i+2}_{\rm hr}(M,\omega)$ are isomorphic for every $n \leq i \leq 2(n-1)$. Let us consider the spaces A and B given by

$$\begin{aligned} A &= \ker(L^{i-n+2}_{[\omega]} \colon H^{2n-i-2}_{\rm hr}(M,\omega) \longrightarrow H^{i+2}(M)), \\ B &= \ker(L^{i-n+1}_{[\omega_Z]} \colon H^{2n-i-2}_{\rm hr}(Z,\omega_Z) \longrightarrow H^{i}(Z)), \end{aligned}$$

where $n \leq i \leq 2n-2$. Next we see that $\jmath^*$ induces an isomorphism between A and B. Given $a \in A$, we denote $b = \jmath^*(a) \in H^{2n-i-2}(Z)$. Since $2n-i-2 < n-1$, from the first part of the proof it follows that $b \in H^{2n-i-2}_{\rm hr}(Z,\omega_Z)$. Moreover, since $a \cup [\omega]^{i-n+2} = 0$ if and only if $\jmath^*(a \cup [\omega]^{i-n+1}) = b \cup [\omega_Z]^{i-n+1} = 0$, we have that $b \in B$, that is, $\jmath^*(A) \subset B$. Again, from the first part of the proof we conclude that the map $\jmath^* \colon A \longrightarrow B$ is an isomorphism, because $(2n-i-2) < (n-1)$.

Finally, as an immediate consequence of (3) we get

$$\begin{aligned} H^i_{\rm hr}(Z,\omega_Z) &= \mathrm{Im}\,(L^{i-n+1}_{[\omega_Z]}\colon H^{2n-i-2}_{\rm hr}(Z,\omega_Z) \longrightarrow H^i(Z)) \;\cong\; H^{2n-i-2}_{\rm hr}(Z,\omega_Z)/B \\ &\cong\; H^{2n-i-2}_{\rm hr}(M,\omega)/A \;\cong\; \mathrm{Im}\,(L^{i-n+2}_{[\omega]}\colon H^{2n-i-2}_{\rm hr}(M,\omega) \longrightarrow H^{i+2}(M)) \\ &= H^{i+2}_{\rm hr}(M,\omega), \end{aligned}$$

for any $n \le i \le (2n-2)$, so $b^{\rm hr}_i(Z,\omega_Z) = b^{\rm hr}_{i+2}(M,\omega)$ for any such i. QED

From now on, by an *iterated* Donaldson symplectic submanifold (Z_l,ω_l) of (M,ω) we shall mean a symplectic manifold obtained as

$$(Z_l,\omega_l) \subset (Z_{l-1},\omega_{l-1}) \subset \cdots \subset (Z_1,\omega_1) \subset (Z_0 = M, \omega_0 = \omega),$$

where (Z_i,ω_i) is a Donaldson symplectic submanifold of (Z_{i-1},ω_{i-1}), for any $1 \le i \le l$.

Corollary 3.1 *If (Z_l,ω_l) is an iterated Donaldson symplectic submanifold of (M^{2n},ω), then $b^{\rm hr}_{n-l}(Z_l,\omega_l) \ge b^{\rm hr}_{n-l}(M,\omega)$ and*

$$b_i(Z_l) - b^{\rm hr}_i(Z_l,\omega_l) = b_i(M) - b^{\rm hr}_i(M,\omega)\,, \qquad \textit{for } i \le n-l-1\,,$$

$$b_i(Z_l) - b^{\rm hr}_i(Z_l,\omega_l) = b_{i+2l}(M) - b^{\rm hr}_{i+2l}(M,\omega)\,, \qquad \textit{for } i \ge n-l+1\,.$$

Proof : Applying l times Theorem 1.1 we have $b^{\rm hr}_i(Z_l,\omega_l) = b^{\rm hr}_{i+2l}(M,\omega)$ for any $i \ge n-l+1$. Since $b_{2n-2l-i}(Z_l) = b_{2n-2l-i}(M)$, the Poincaré duality for Z_l and M implies that $b_i(Z_l) = b_{i+2l}(M)$. This proves the corollary for any $i \ge n-l+1$. For the remaining values of i, the result follows directly from Theorem 1.1. QED

Next we want to show that a result like Theorem 1.1 for the Auroux submanifolds does not hold in general.

Suppose that (M,ω) is a compact symplectic manifold of dimension $2n$ with $[\omega] \in H^2(M)$ admitting a lift to an integral cohomology class, and let E be any hermitian vector bundle over M of rank r. Then, in [1] Auroux constructs symplectic submanifolds $(Z_r,\omega_{Z_r}) \hookrightarrow (M,\omega)$ of dimension $2(n-r)$ whose Poincaré duals are $\mathrm{PD}[Z_r] = c_r(E \otimes L^{\otimes k}) = k^r[\omega]^r + k^{r-1}c_1(E)[\omega]^{r-1} + \cdots + c_r(E)$ for any integer number k large enough, where we denote by $c_i(E)$ the i^{th} Chern class of the vector bundle E, and by L the complex line bundle over M with first Chern class $c_1(L) = [\omega]$. These submanifolds also satisfy a Lefschetz theorem on hyperplane sections, that is, the inclusion $\jmath\colon Z_r \hookrightarrow M$ induces $\jmath^*\colon H^i(M) \to H^i(Z_r)$ which is an isomorphism for $i < (n-r)$ and a monomorphism for $i = (n-r)$.

The strongest result in the direction of Theorem 1.1 for the Auroux submanifolds follows from [8, Theorem 4.4]. There it is proved that, for an Auroux submanifold $Z_r \hookrightarrow M$, for large enough k, and for each $s \le (n-r-1)$, if M is s–Lefschetz then Z_r is also s–Lefschetz. In this situation, we have, thanks to Proposition 2.6, that $H^i_{\rm hr}(M,\omega) \cong H^i(M)$ and $H^i_{\rm hr}(Z_r,\omega_{Z_r}) \cong H^i(Z_r)$, for $i \le s+2$. Therefore it follows that there is an isomorphism $\jmath^*\colon H^i_{\rm hr}(M,\omega) \longrightarrow H^i_{\rm hr}(Z_r,\omega_{Z_r})$, for any $i \le \min\{s+2, n-r-1\}$, and a monomorphism in the case $i = (n-r) \le (s+2)$.

To disprove a result like Theorem 1.1 for Auroux submanifolds, we shall see examples of different behaviours in the simplest case, i.e., when M is not 1–Lefschetz. By the above,

$H^i_{\rm hr}(M,\omega)\cong H^i_{\rm hr}(Z_r,\omega_{Z_r})$, for $i=1,2$. So the first case to look at is the study of the relation between

$$H^3_{\rm hr}(M,\omega) \quad \text{and} \quad H^3_{\rm hr}(Z_r,\omega_{Z_r}).$$

In general, to compare them, we are going to assume $n-r>3$, so that there is an isomorphism $j^*: H^3(M)\to H^3(Z_r)$. We need the following lemma.

Lemma 3.2 *Suppose that $(Z_r,\omega_{Z_r})\hookrightarrow (M,\omega)$ is an Auroux symplectic submanifold, and $n-r>3$. In the situation above,*

(i) $b_3^{\rm hr}(M,\omega)=b_3(M)+\dim\ker\left(L^{n-2}_{[\omega]}:H^1(M)\to H^{2n-3}(M)\right)$
$-\dim\ker\left(L^{n-1}_{[\omega]}:H^1(M)\to H^{2n-1}(M)\right)$,

(ii) $b_3^{\rm hr}(Z_r,\omega_{Z_r})=b_3(M)+\dim\ker\left(L^{n-r-2}_{[\omega]}\cup c_r(E\otimes L^{\otimes k}):H^1(M)\to H^{2n-3}(M)\right)$
$-\dim\ker\left(L^{n-r-1}_{[\omega]}\cup c_r(E\otimes L^{\otimes k}):H^1(M)\to H^{2n-1}(M)\right)$,
where $\cup c_r(E\otimes L^{\otimes k}):H^*(M)\to H^{*+2r}(M)$ is interpreted as a map in cohomology.

Proof : Let us start by computing $H^3_{\rm hr}(M,\omega)$. By (5),

$$H^3_{\rm hr}(M,\omega)=P_3(M,\omega)+L_{[\omega]}\left(H^1_{\rm hr}(M,\omega)\right),$$

where $P_3(M,\omega)=\{\,a\in H^3(M)\ \mid\ L^{n-2}_{[\omega]}(a)=0\}$. In the case $i=1$, we have that $H^1_{\rm hr}(M,\omega)=H^1(M)$. Clearly

$$P_3(M,\omega)\cap L_{[\omega]}\left(H^1(M)\right)=L_{[\omega]}\left(\ker(L^{n-1}_{[\omega]}:H^1(M)\to H^{2n-1}(M))\right).$$

On the other hand, $P_3(M,\omega)=\ker\left(L^{n-2}_{[\omega]}:H^3(M)\to H^{2n-1}(M)\right)$ is dual, via Poincaré duality, to coker $\left(L^{n-2}_{[\omega]}:H^1(M)\to H^{2n-3}(M)\right)$. Therefore

$$\begin{aligned}
b_3^{\rm hr}(M,\omega) &= \dim\operatorname{coker}\left(L^{n-2}_{[\omega]}:H^1(M)\to H^{2n-3}(M)\right)\\
&\quad +\dim L_{[\omega]}(H^1(M))-\dim L_{[\omega]}\left(\ker(L^{n-1}_{[\omega]}:H^1(M)\to H^{2n-1}(M))\right)\\
&= b_3(M)-b_1(M)+\dim\ker\left(L^{n-2}_{[\omega]}:H^1(M)\to H^{2n-3}(M)\right)\\
&\quad +b_1(M)-\dim\ker\left(L^{n-1}_{[\omega]}:H^1(M)\to H^{2n-1}(M)\right)\\
&= b_3(M)+\dim\ker\left(L^{n-2}_{[\omega]}:H^1(M)\to H^{2n-3}(M)\right)\\
&\quad -\dim\ker\left(L^{n-1}_{[\omega]}:H^1(M)\to H^{2n-1}(M)\right).
\end{aligned}$$

This proves *(i)*. Now we move on to compute $H^3_{\rm hr}(Z_r,\omega_{Z_r})$. First, note that for $i<n-r$, if $a\in H^{2n-2r-i}(M)$, we have that

$$j^*(a)=0 \iff a\cup c_r(E\otimes L^{\otimes k})=0.$$

Certainly, $j^*(a)=0$ is equivalent to

$$0=\int_{Z_r} j^*(a)\cup j^*(b)=\int_M a\cup b\cup c_r(E\otimes L^{\otimes k}),$$

for any $b \in H^i(M) \cong H^i(Z_r)$. We use that $\mathrm{PD}[Z_r] = c_r(E \otimes L^{\otimes k})$ for the second inequality. This is equivalent to $a \cup c_r(E \otimes L^{\otimes k}) = 0$. With the aid of this, and using *(i)*, we have

$$\begin{aligned} b_3^{\mathrm{hr}}(Z_r, \omega_{Z_r}) &= b_3(Z_r) + \dim\ker\left(L^{n-r-2}_{[\omega_{Z_r}]} : H^1(Z_r) \to H^{2n-2r-3}(Z_r)\right) \\ &\quad - \dim\ker\left(L^{n-r-1}_{[\omega_{Z_r}]} : H^1(Z_r) \to H^{2n-2r-1}(Z_r)\right) \\ &= b_3(M) + \dim\ker\left(L^{n-r-2}_{[\omega]} \cup c_r(E \otimes L^{\otimes k}) : H^1(M) \to H^{2n-3}(M)\right) \\ &\quad - \dim\ker\left(L^{n-r-1}_{[\omega]} \cup c_r(E \otimes L^{\otimes k}) : H^1(M) \to H^{2n-1}(M)\right). \end{aligned}$$

QED

Next we exhibit examples of compact symplectic manifolds (X, Ω) having Auroux submanifolds (Z_r, Ω_{Z_r}) such that $b_3^{\mathrm{hr}}(Z_r, \Omega_{Z_r}) \neq b_3^{\mathrm{hr}}(X, \Omega)$. To define X, first we consider the simply connected nilpotent Lie group G of dimension 6 consisting of all the matrices of the form

$$\begin{pmatrix} 1 & y & t+z & \frac{t}{2} & u+\frac{y^2}{2} & v \\ 0 & 1 & x & \frac{x}{2} & y+\frac{x^2}{2} & xy+\frac{x^3}{6} \\ 0 & 0 & 1 & 0 & 0 & y \\ 0 & 0 & 0 & 1 & 2x & x^2 \\ 0 & 0 & 0 & 0 & 1 & x \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $x, y, z, t, u, v \in \mathbb{R}$. With respect to this global system of coordinates, the forms

$$\alpha_1 = dx,\ \alpha_2 = dy,\ \alpha_3 = dz,\ \alpha_4 = dt - y dx,\ \alpha_5 = du - t dx,\ \alpha_6 = dv - (z+t)dy - \left(u + \frac{y^2}{2}\right) dx$$

constitute a basis of left invariant 1–forms on G, and they satisfy

$$d\alpha_1 = d\alpha_2 = d\alpha_3 = 0, \quad d\alpha_4 = \alpha_{12}, \quad d\alpha_5 = \alpha_{14}, \quad d\alpha_6 = \alpha_{15} + \alpha_{23} + \alpha_{24},$$

where we denote $\alpha_{ij\cdots k} = \alpha_i \wedge \alpha_j \wedge \cdots \wedge \alpha_k$. Because the structure constants are rational numbers, Mal'cev Theorem [19] implies the existence of a discrete subgroup Γ of G such that the quotient space $M = \Gamma \backslash G$ is compact. The cohomology of M is given by

$$\begin{aligned} H^0(M) &= \langle 1 \rangle, \\ H^1(M) &= \langle [\alpha_1], [\alpha_2], [\alpha_3] \rangle, \\ H^2(M) &= \langle [\alpha_{13}], [\alpha_{23}], [\alpha_{24}], [\alpha_{16} + \alpha_{25} - \alpha_{34}], [\alpha_{26} - \alpha_{45}] \rangle, \\ H^3(M) &= \langle [\alpha_{126}], [\alpha_{135}], [\alpha_{136} + \alpha_{146}], [\alpha_{136} + \alpha_{235}], [\alpha_{236} + \alpha_{345}], [\alpha_{156} - \alpha_{246} + \alpha_{345}] \rangle, \\ H^4(M) &= \langle [\alpha_{2345}], [\alpha_{1236}], [\alpha_{2456}], [\alpha_{1456} + \alpha_{2346}], [\alpha_{1356} + \alpha_{1456}] \rangle, \\ H^5(M) &= \langle [\alpha_{23456}], [\alpha_{13456}], [\alpha_{12456}] \rangle, \\ H^6(M) &= \langle [\alpha_{123456}] \rangle. \end{aligned}$$

Therefore M is a symplectic manifold with symplectic form $\omega = \alpha_{16} + \alpha_{25} - \alpha_{34}$, and $b_3(M) = 6$. It is simple to check that $L^2_{[\omega]} : H^1(M) \to H^5(M)$ is the zero map. On the other hand, $L_{[\omega]} : H^1(M) \to H^3(M)$ has kernel of dimension 1 and generated by $[\alpha_1]$. This follows from $\omega \wedge \alpha_1 = d(\alpha_{45} + \alpha_{35})$, so $[\alpha_1]$ is in the kernel, and $[\omega \wedge \alpha_2 \wedge \alpha_3] \neq 0$, so $[\alpha_2], [\alpha_3]$ are not in the kernel. By Lemma 3.2, $b_3^{\mathrm{hr}}(M, \omega) = 6 + 1 - 3 = 4$.

But M is of dimension 6, and we need a manifold of dimension $2n$, where $n-r>3$. We shall fix $2n=8+2r$ and define the $2n$–dimensional manifold

$$X = M \times \mathbb{CP}^{r+1}.$$

Let ω_0 be the Fubini-Study symplectic form of $\mathbb{CP}^{r+1}$, so X is a sympletic manifold with symplectic form $\Omega = \omega + \omega_0$. Now

$$\begin{aligned} H^1(X) &= H^1(M), \\ H^3(X) &= H^3(M) \oplus (H^1(M) \otimes H^2(\mathbb{CP}^{r+1})), \\ &\vdots \\ H^{2n-3}(X) &= (H^5(M) \otimes H^{2r}(\mathbb{CP}^{r+1})) \oplus (H^3(M) \otimes H^{2r+2}(\mathbb{CP}^{r+1})), \\ H^{2n-1}(X) &= H^5(M) \otimes H^{2r+2}(\mathbb{CP}^{r+1}). \end{aligned}$$

First we will compute $b_3^{\mathrm{hr}}(X,\Omega)$ by using Lemma 3.2. Clearly $b_3(X) = 6+3 = 9$. The map $L_{[\Omega]} = L_{[\omega]} + L_{[\omega_0]}$, so $L_{[\Omega]}^{n-1} : H^1(X) \to H^{2n-1}(X) = H^5(M) \otimes H^{2r+2}(\mathbb{CP}^{r+1})$ equals

$$L_{[\Omega]}^{n-1} = \left(L_{[\omega]} + L_{[\omega_0]}\right)^{n-1} = \sum_j \binom{n-1}{j} L_{[\omega]}^j L_{[\omega_0]}^{n-1-j} = 0,$$

since $L_{[\omega]}^j = 0$ for $j>1$ and $L_{[\omega_0]}^{n-1-j} = 0$ for $n-1-j > r+1$, i.e., for $j<2$. The map $L_{[\Omega]}^{n-2} : H^1(X) \to H^{2n-3}(X) = \left(H^5(M) \otimes H^{2r}(\mathbb{CP}^{r+1})\right) \oplus \left(H^3(M) \otimes H^{2r+2}(\mathbb{CP}^{r+1})\right)$ equals

$$L_{[\Omega]}^{n-2} = \sum_j \binom{n-2}{j} L_{[\omega]}^j L_{[\omega_0]}^{n-2-j} = L_{[\omega]} L_{[\omega_0]}^{r+1}.$$

So $\ker\left(L_{[\Omega]}^{n-2} : H^1(X) \to H^{2n-3}(X)\right) = \ker\left(L_{[\omega]} : H^1(M) \to H^3(M)\right) = \langle [\alpha_1] \rangle$. Lemma 3.2 yields

$$b_3^{\mathrm{hr}}(X,\Omega) = 9 + 1 - 3 = 7,$$

for any value of r. With these preliminaries at hand, we are ready to start with our examples.

Example 3.3 *The compact symplectic manifold* $(X = M \times \mathbb{CP}^2, \Omega)$ *has an Auroux submanifold* $Z_1 \subset (X,\Omega)$ *such that* $b_3^{\mathrm{hr}}(Z_1, \Omega_{Z_1}) > b_3^{\mathrm{hr}}(X,\Omega)$.

Proof : Let $A = [\alpha_{26} - \alpha_{45}] \in H^2(M)$. To define an Auroux submanifold $Z_1 \subset (X,\Omega)$ in the conditions required, we consider a rank 1 bundle E with first Chern class $c_1(E) = A \in H^2(M) \subset H^2(X)$. Note that $n=5$, $r=1$ in this case. Hence the Auroux submanifold $Z_1 \subset X$ has $PD[Z_1] = k[\Omega] + A$. To apply the part *(ii)* of Lemma 3.2, we need to compute the map $L_{[\Omega]}^3 (kL_{[\Omega]} + L_A) : H^1(X) \to H^9(X) = H^5(M) \otimes H^4(\mathbb{CP}^2)$, where L_A is the map in cohomology given by cup product with the class A. This is

$$L_{[\Omega]}^3 (kL_{[\Omega]} + L_A) = L_{[\Omega]}^3 L_A = L_{[\omega]} L_A L_{[\omega_0]}^2,$$

since $L_{[\Omega]}^4 = 0$, by the above calculation. This map has kernel of dimension 1, generated by $[\alpha_1]$, since $L_{[\omega]}([\alpha_1]) = 0$, but

(6) $$[\alpha_2] \cup [\alpha_3] \cup [\omega] \cup A \neq 0.$$

The map $L^2_{[\Omega]}(kL_{[\Omega]}+A) : H^1(X) \to H^7(X) = (H^5(M)\otimes H^2(\mathbb{CP}^2)) \oplus (H^3(M)\otimes H^4(\mathbb{CP}^2))$ equals

$$L^2_{[\Omega]}(kL_{[\Omega]}+L_A) = kL_{[\omega]}L^2_{[\omega_0]} + L_{[\omega]}L_AL_{[\omega_0]}.$$

The first component has kernel generated by $[\alpha_1]$, by what we have seen above. The second component has the same kernel again, so $\dim\ker\Big(L^2_{[\Omega]}(kL_{[\Omega]}+L_A) : H^1(X)\to H^7(X)\Big) = 1$. Now Lemma 3.2 gives

$$b_3^{\rm hr}(Z_1,\Omega_{Z_1}) = 9+1-1 = 9.$$

Therefore, $b_3^{\rm hr}(Z_1,\Omega_{Z_1}) > b_3^{\rm hr}(X,\Omega)$. *QED*

Notice that in the example above, all the calculation hinges in (6). In fact, we have

Example 3.4 *The compact symplectic manifold* $(X = M\times\mathbb{CP}^2,\Omega)$ *has an Auroux submanifold* $Z_1'\subset(X,\Omega)$ *such that* $b_3^{\rm hr}(Z_1',\Omega_{Z_1'}) = b_3^{\rm hr}(X,\Omega)$.

Proof : We take a class $A\in H^2(M)$ such that $[\alpha_2]\cup[\alpha_3]\cup[\omega]\cup A = 0$; for instance, use $A = [\alpha_{13}]$. Then we obtain an Auroux submanifold $(Z_1',\Omega_{Z_1'})$ of (X,Ω) with $b_3^{\rm hr}(Z_1',\Omega_{Z_1'}) = b_3^{\rm hr}(X,\Omega) = 7$. *QED*

Finally we give an example where the Auroux submanifold has less harmonic cohomology than the ambient submanifold.

Example 3.5 *There are Auroux submanifolds* $(Z_3,\Omega_{Z_3})\subset(Z_1,\Omega_{Z_1})\subset(X = M\times\mathbb{CP}^4,\Omega)$ *such that* $b_3^{\rm hr}(Z_3,\Omega_{Z_3}) < b_3^{\rm hr}(Z_1,\Omega_{Z_1})$.

Proof : Consider the manifold $(X = M\times\mathbb{CP}^4,\Omega)$ of dimension 14 (now $n = 7$ and $r = 3$). Take again $A = [\alpha_{26}-\alpha_{45}]\in H^2(M)$ and let E be a rank 1 bundle with $c_1(E) = A$. There is another bundle F such that $E\oplus F$ is a trivial bundle. Actually, one may take F to have rank 2 and Chern classes $c_1(F) = -A$ and $c_2(F) = A^2$. Let $Z_3\subset(X = M\times\mathbb{CP}^4,\Omega)$ be the Auroux submanifold (of codimension 6) associated to the (trivial) bundle $E\oplus F$. Since the Chern classes of $E\oplus F$ are all zero, we have that

$$b_3^{\rm hr}(Z_3,\Omega_{Z_3}) = b_3^{\rm hr}(X,\Omega) = 7,$$

as in Example 3.4.

Now let $Z_1\subset(X = M\times\mathbb{CP}^4,\Omega)$ be the Auroux submanifold associated to the bundle E. By Example 3.3, we have that $b_3^{\rm hr}(Z_1,\Omega_{Z_1}) = 9$. But, the construction in [1] is carried out in such a way that Z_3 is also an Auroux submanifold of Z_1 (of codimension 4), and

$$b_3^{\rm hr}(Z_3,\Omega_{Z_3}) < b_3^{\rm hr}(Z_1,\Omega_{Z_1}).$$

QED

4 Symplectic blow-ups

This section is devoted to the study of the s–Lefschetz property for the symplectic blow-up $\widetilde{CP^m}$ of the complex projective space CP^m along a symplectic submanifold $M \hookrightarrow CP^m$.

Let (M, ω) be a *compact* symplectic manifold of dimension $2n$. Without loss of generality we can assume that the symplectic form ω is integral (by perturbing it to make it rational and then rescaling), i.e., $[\omega] \in H^2(M; \mathbb{Z})$. A theorem of Gromov and Tischler [11, 24] states that there is a symplectic embedding $i\colon (M, \omega) \longrightarrow (CP^m, \omega_0)$, with $m \geq 2n+1$, where ω_0 is the standard Kähler form on CP^m defined by its natural complex structure and the Fubini–Study metric. We take the symplectic blow-up $\widetilde{CP^m}$ of CP^m along the embedding i (see [21]). Then $\widetilde{CP^m}$ is a simply connected compact symplectic manifold.

Recall that $i^*\omega_0 = \omega$. We will denote also by ω_0 the pull back of ω_0 to $\widetilde{CP^m}$ under the natural projection $\widetilde{CP^m} \to CP^m$. Let $\widetilde{M}$ be the projectivization of the normal bundle of the embedding $M \hookrightarrow CP^m$. Then $\pi\colon \widetilde{M} \longrightarrow M$ is a locally trivial bundle with fiber CP^{m-n-1}. We will denote by ν the Thom form of the submanifold $\widetilde{M} \subset \widetilde{CP^m}$. The class $[\nu]$ is called the Thom class of the blow-up. Then $\widetilde{CP^m}$ has a symplectic form Ω whose cohomology class is $[\Omega] = [\omega_0] + \epsilon\,[\nu]$ for $\epsilon > 0$ small enough.

Let us consider a closed tubular neighborhood $\widetilde{W}$ of $\widetilde{M}$ in $\widetilde{CP^m}$. By the tubular neighborhood theorem we know that the normal bundle of $\widetilde{M} \hookrightarrow \widetilde{CP^m}$ contains a disk subbundle which is diffeomorphic to $\widetilde{W}$. Denote by $p\colon \widetilde{W} \longrightarrow \widetilde{M}$ the natural map. There is a map $q : \Omega^*(M) \to \Omega^{*+2}(\widetilde{CP^m})$ given by pull-back by $\pi : \widetilde{M} \to M$, followed by extending to a neighborhood of $\widetilde{M}$ using $p : \widetilde{W} \to \widetilde{M}$ and then wedging by ν, i.e., $q(\alpha) = p^*\pi^*(\alpha) \wedge \nu$. We shall denote $q(\alpha) = \alpha \wedge \nu$ for short. Note that

$$(\alpha \wedge \nu) \wedge (\beta \wedge \nu) = (\alpha \wedge \beta \wedge \nu) \wedge \nu\,,$$

for $\alpha, \beta \in \Omega^*(M)$. This makes notations of the type $\alpha \wedge \beta \wedge \nu^2$ unambiguous. Also remark that $[\omega_0 \wedge \nu] = [\omega \wedge \nu]$ although $\omega_0 \wedge \nu \neq \omega \wedge \nu$ as forms.

The cohomology of $\widetilde{CP^m}$ was studied by McDuff [21]. There she proved that there is a short exact sequence

$$0 \longrightarrow H^*(CP^m) \longrightarrow H^*(\widetilde{CP^m}) \longrightarrow A^* \longrightarrow 0, \tag{7}$$

where A^* is a free module over $H^*(M)$ generated by $\{[\nu], [\nu^2], \cdots, [\nu^{m-n-1}]\}$.

Before going on to the study of the s–Lefschetz property for $\widetilde{CP^m}$, we need to recall the splitting of the cohomology groups in terms of the primitive classes proved by Yan [26] for hard Lefschetz symplectic manifolds. His proof also works for s–Lefschetz symplectic manifolds.

Lemma 4.1 *Let (M, ω) be a compact symplectic manifold of dimension $2n$ satisfying the s–Lefschetz property for $s \leq n-1$. Then, there is a splitting*

$$H^k(M) = P_k(M) \oplus L(H^{k-2}(M)),$$

where $P_k(M)$ is given by

$$P_k(M) = \{v \in H^k(M) \ \mid \ L^{n-k+1}(v) = 0\},$$

for $k \leq s$. The elements in $P_k(M)$ are called primitive cohomology classes of degree k.

Proof : First, let us see that $P_k(M) \cap \operatorname{Im} L = 0$. Take $x \in P_k(M)$ with $x = L(y)$, $y \in H^{k-2}(M)$. Then $L^{n-k+2}(y) = L^{n-k+1}(x) = 0$. By the $(k-2)$–Lefschetz property, $y = 0$ and hence $x = 0$.

Now let us consider $a \in H^k(M)$ with $k \leq s$, and take the element $L^{n-k+1}(a) \in H^{2n-k+2}(M)$. If $L^{n-k+1}(a)$ is the zero class, then $a \in P_k(M)$ and the lemma is proved. If $L^{n-k+1}(a)$ is non-zero, then there exists $b \in H^{k-2}(M)$ such that $L^{n-k+1}(a) = L^{n-k+2}(b)$ since (M, ω) is s–Lefschetz and so the map $L^{n-k+2} : H^{k-2}(M) \longrightarrow H^{2n-k+2}(M)$ is an isomorphism. Hence $a - L(b) \in P_k(M)$. But $a = (a - L(b)) + L(b)$ which lies in $P_k(M) \oplus \operatorname{Im} L$. $\boxed{QED}$

According Lemma 4.1 we can write

$$H^k(M) = P_k(M) \oplus (P_{k-2}(M) \cup [\omega]) \oplus \cdots \oplus (P_{k-2\lambda}(M) \cup [\omega^\lambda]), \tag{8}$$

with $\lambda = [\frac{k}{2}]$.

Theorem 4.2 *For any $s \leq n-1$, if (M, ω) is s–Lefschetz then there exists $\epsilon_0 > 0$ such that $(\widetilde{CP}^m, \Omega = \omega_0 + \epsilon\nu)$ is $(s+2)$–Lefschetz, for any $\epsilon \in (0, \epsilon_0]$. In particular, for $\epsilon \in \mathbb{Q} \cap (0, \epsilon_0]$, we have that $[\Omega_\epsilon]$ a rational class (and hence a multiple of it is integral).*

Proof : Following the notation stated at the beginning of this Section, we must prove that the map $[\omega_0 + \epsilon\nu]^{m-k} \colon H^k(\widetilde{CP}^m) \longrightarrow H^{2m-k}(\widetilde{CP}^m)$ is an isomorphism for any $k \leq s+2 \leq n+1$. First, using (7) and (8), we notice that for $k \leq s+2$ the cohomology group $H^k(\widetilde{CP}^m)$ is generated by the classes:

$$\begin{cases} [\omega_0]^{\frac{k}{2}}, & \text{if } k \text{ is even,} \\ \\ [p_{k-2i-2t} \wedge \omega_0^t \wedge \nu^i], & \text{where } [p_{k-2i-2t}] \in P_{k-2i-2t}(M), i > 0, t \geq 0 \text{ and } i + t \leq [\frac{k}{2}]. \end{cases}$$

Suppose that k is even (the proof is similar when k is odd). We prove that the map $[\omega_0 + \epsilon\nu]^{m-k}$ is injective by computing each one of the following cohomology classes in $H^{2m}(\widetilde{CP}^m)$: $[\omega_0 + \epsilon\nu]^{m-k} \cup [\omega_0]^{\frac{k}{2}} \cup [\omega_0]^{\frac{k}{2}}$, $[\omega_0 + \epsilon\nu]^{m-k} \cup [p_{k-2i-2t} \wedge \omega_0^t \wedge \nu^i] \cup [\omega_0]^{\frac{k}{2}}$ for $i + t \leq \frac{k}{2}$, and $[\omega_0 + \epsilon\nu]^{m-k} \cup [p_{k-2i-2t} \wedge \omega_0^t \wedge \nu^i] \cup [q_{k-2j-2s} \wedge \omega_0^s \wedge \nu^j]$ if $i + t, j + s \leq \frac{k}{2}$, where $[q_{k-2j-2s}] \in P_{k-2j-2s}(M)$.

We begin by showing that the class $[\omega_0 + \epsilon\nu]^{m-k} \cup [\omega_0]^{\frac{k}{2}} \cup [\omega_0]^{\frac{k}{2}}$ is non-trivial. We have

$$[\omega_0 + \epsilon\nu]^{m-k} \cup [\omega_0]^{\frac{k}{2}} \cup [\omega_0]^{\frac{k}{2}} = \sum_{r=0}^{m-k} \binom{m-k}{r} \epsilon^r [\omega_0^{m-r} \wedge \nu^r] = [\omega_0]^m + \sum_{r=1}^{m-k} \binom{m-k}{r} \epsilon^r [\omega_0^{m-r} \wedge \nu^r].$$

In this sum, the terms $[\omega_0^{m-r} \wedge \nu^r]$ are zero for $1 \leq r \leq m-n-1$ since M has dimension $2n$ and so $[\omega_0^{n+1} \wedge \nu] = [\omega^{n+1} \wedge \nu] = 0$. Then,

$$\begin{aligned} [\omega_0 + \epsilon\nu]^{m-k} \cup [\omega_0]^{\frac{k}{2}} \cup [\omega_0]^{\frac{k}{2}} &= [\omega_0]^m + \binom{m-k}{m-n} \epsilon^{m-n} [\omega_0^n \wedge \nu^{m-n}] \\ &\quad + \sum_{r=m-n+1}^{m-k} \binom{m-k}{r} \epsilon^r [\omega_0^{m-r} \wedge \nu^r] \\ &= [\omega_0]^m + \binom{m-k}{m-n} \epsilon^{m-n} [\omega^n \wedge \nu^{m-n}] + O(\epsilon^{m-n+1}), \end{aligned} \tag{9}$$

which is a non-zero class (for ϵ small enough).

Proceeding in a similar way, let $i+t \leq \frac{k}{2}$, $i>0$, $t \geq 0$, and $[p_{k-2i-2t}] \in P_{k-2i-2t}(M)$. Then

$$
\begin{aligned}
[\omega_0+\epsilon\nu]^{m-k} \cup [p_{k-2i-2t} \wedge \omega_0^t \wedge \nu^i] \cup [\omega_0]^{\frac{k}{2}} &= \sum_{r=0}^{m-k} \binom{m-k}{r} \epsilon^r [p_{k-2i-2t} \wedge \omega_0^{t+m-\frac{k}{2}-r} \wedge \nu^{r+i}] \\
&= \binom{m-k}{m-n-i} \epsilon^{m-n-i} [p_{k-2i-2t} \wedge \omega^{n+i+t-\frac{k}{2}} \wedge \nu^{m-n}] \qquad (10) \\
&\quad + O(\epsilon^{m-n-i+1}),
\end{aligned}
$$

using that for $i < m-n-r$, we have that $[p_{k-2i-2t} \wedge \omega^{t+m-\frac{k}{2}-r} \wedge \nu^{r+i}] = 0$, since $\deg(p_{k-2i-2t} \wedge \omega^{t+m-\frac{k}{2}-r}) > 2n$. Suppose that

$$
x = a[\omega_0]^{\frac{k}{2}} + \sum_{i+t \leq \frac{k}{2}, i>0} [p_{k-2i-2t} \wedge \omega_0^t \wedge \nu^i] \in H^k(\widetilde{CP}^m) \qquad (11)
$$

is an element such that $[\omega_0+\epsilon\nu]^{m-k} \cup x = 0$. Then multiplying by $[\omega_0]^{\frac{k}{2}}$ and using (9) and (10), we get that $a=0$. So

$$
x = \sum_{i+t \leq \frac{k}{2}, i>0} [p_{k-2i-2t} \wedge \omega_0^t \wedge \nu^i]\,. \qquad (12)
$$

Now we compute for $i+t \leq \frac{k}{2}$ and $j+s \leq \frac{k}{2}$ the following product

$$
[\omega_0+\epsilon\nu]^{m-k} \cup [p_{k-2i-2t} \wedge \omega_0^t \wedge \nu^i] \cup [q_{k-2j-2s} \wedge \omega_0^s \wedge \nu^j] = \qquad (13)
$$

$$
= \binom{m-k}{m-n-i-j} \epsilon^{m-n-i-j} [p_{k-2i-2t} \wedge q_{k-2j-2s} \wedge \omega^{n-k+i+t+j+s} \wedge \nu^{m-n}] + O(\epsilon^{m-n-i-j+1}).
$$

Let us concentrate on the leading term. The duality on $H^r(M)$ defines a duality on the space $P_r(M)$ of the primitive cohomology classes:

$$
p^\sharp \colon P_r(M) \otimes P_r(M) \longrightarrow \mathbb{R}
$$

given by

$$
p^\sharp([\alpha],[\beta]) = \int_M \alpha \wedge \beta \wedge \omega^{n-r},
$$

which is nondegenerate, but

$$
p^\sharp \colon P_r(M) \otimes P_{r+2s}(M) \longrightarrow \mathbb{R}
$$

given by

$$
p^\sharp([\alpha],[\beta]) = \int_M \alpha \wedge \beta \wedge \omega^{n-r-s},
$$

is zero if $s \neq 0$, since $[\omega]^{n-r-s}$ maps $P_{r+2s}(M)$ to zero. Thus the matrix $A_{i+t,j+s}$ associated to $p^\sharp : P_{k-2i-2t}(M) \otimes P_{k-2j-2s}(M) \to \mathbb{R}$ is non-singular if $i+t=j+s$ and zero if $i+t \neq j+s$.

Consider the spaces

$$
P_\mu := \bigoplus_{i+t=\mu, i>0} P_{k-2i-2t}(M)\,[\omega^t]\,[\nu^i]
$$

and

$$
W = \bigoplus_{1 \leq \mu \leq \frac{k}{2}} P_\mu\,,
$$

so that $H^k(\widetilde{CP}^m) = [\omega_0^{\frac{k}{2}}] \oplus W$. There is a bilinear map

$$p_1^\sharp : W \otimes W \longrightarrow \mathbb{R}$$

given by

$$p_1^\sharp([p_{k-2i-2t} \wedge \omega_0^t \wedge \nu^i], [q_{k-2j-2s} \wedge \omega_0^s \wedge \nu^j]) = \int_{\widetilde{CP}^m} p_{k-2i-2t} \wedge q_{k-2j-2s} \wedge \omega^{n-k+i+t+j+s} \wedge \nu^{m-n} .$$

The matrix B_μ of $p_1^\sharp|_{P_\mu \otimes P_\mu}$ is the block matrix whose block in the place (i,j) with $1 \leq i,j \leq \mu$ is the matrix

$$\binom{m-k}{m-n-i-j} \cdot \epsilon^{m-n-i-j} \cdot A_\mu .$$

Let $d = \dim P_{k-2i-2t}(M)$. The determinant of B_μ is

$$\det(A_\mu)^\mu \cdot \left[\det\left(\epsilon^{m-n-i-j}\binom{m-k}{m-n-i-j}\right)_{1\leq i,j\leq \mu}\right]^d = \tag{14}$$

$$= \det(A_\mu)^\mu \cdot \left[\epsilon^{(m-n)\mu-\mu(\mu+1)} \frac{\binom{m-k+\mu-1}{m-n-\mu-1} \cdots \binom{m-k}{m-n-\mu-1}}{\binom{m-n-2}{m-n-\mu-1} \cdots \binom{m-n-\mu-1}{m-n-\mu-1}}\right]^d ,$$

which is of the form $\lambda_\mu \cdot \epsilon^{a_\mu}$ where $\lambda_\mu \neq 0$. Here we use that $k \leq s+2 \leq n+1 \Rightarrow m-k > m-n-\mu-1$ and $\mu \leq \frac{k}{2} < m-n \Rightarrow m-n-\mu-1 \geq 0$.

The determinant of the matrix of $p_1^\sharp$ is the product of $\det B_\mu$ for $1 \leq \mu \leq \frac{k}{2}$, hence of the form $\lambda \cdot \epsilon^a$ where $\lambda \neq 0$. The matrix associated to the bilinear map $p_2^\sharp : W \otimes W \longrightarrow \mathbb{R}$ given by

$$p_2^\sharp([p_{k-2i-2t} \wedge \omega_0^t \wedge \nu^i], [q_{k-2j-2s} \wedge \omega_0^s \wedge \nu^j]) = [\omega_0 + \epsilon\nu]^{m-k} \cup [p_{k-2i-2t} \wedge \omega_0^t \wedge \nu^i] \cup [q_{k-2j-2s} \wedge \omega_0^s \wedge \nu^j]$$

has at each entry an ϵ–perturbation of the corresponding entry of B_μ, by (13). Hence its determinant is $\lambda \cdot \epsilon^a + O(\epsilon^{a+1})$ and it is nonzero for small $\epsilon > 0$. Therefore $p_2^\sharp$ is a pairing and hence (12) is zero. So $\widetilde{CP}^m$ is $(s+2)$–Lefschetz.

To complete the proof, we must notice that in the conditions of Theorem 4.2, there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$ the manifold $(\widetilde{CP}^m, \Omega_\epsilon = \omega_0 + \epsilon\nu)$ is $(s+2)$-Lefschetz. In particular, if $[\omega_0]$ is an integral 2-cohomology class, then for rational $\epsilon > 0$, we have that $[\Omega_\epsilon]$ is a rational class, hence a multiple of it is an integral class. QED

Remark 4.3 *Cavalcanti [4, Theorem 4.2] has proved that if M is hard Lefschetz then $\widetilde{CP}^m$ is also hard Lefschetz. This also can be proved with the arguments of Theorem 4.2 with few modifications:*

We suppose M is hard-Lefschetz and must prove that $\widetilde{CP}^m$ is k–Lefschetz for any $n+2 \leq k \leq m-1$. In this case, the group $H^k(\widetilde{CP}^m)$ is generated by $[\omega_0]^{\frac{k}{2}}$ (if k is even) and $[p_{k-2i-2t} \wedge \omega_0^t \wedge \nu^i]$, $[p_{k-2i-2t}] \in P_{k-2i-2t}(M)$, $0 < i < m-n$, $k-n \leq t+2i$, $t+i \leq [\frac{k}{2}]$. The rest of the argument is unchanged except at two points: use that $i < m-n$ in (10) to get that $a = 0$ in (11); and use that $2\mu \geq k-n \Rightarrow m-k \geq m-n-\mu-1$ to get that $\lambda_\mu \neq 0$ in (14).

The following result shows that the converse of the previous theorem is also true if M is parallelizable.

Proposition 4.4 *Let (M,ω) be a compact symplectic manifold of dimension $2n$, such that M is parallelizable and (M,ω) is not s–Lefschetz for some $s \geq 1$. Then $\widetilde{CP}^m$ is not $(s+2)$–Lefschetz.*

Proof : Since M is parallelizable, its tangent bundle TM is trivial. Denote by N the normal bunble of $M \hookrightarrow CP^m$. Then the restriction to M of the tangent bundle of CP^m is $TCP^m|_M = TM \oplus N$. The total Chern class of N is given by $c(N) = c(TCP^m|_M) = (1+[\omega])^{m+1}$, so $c_i(N)$ is a multiple of $[\omega]^i$.

Taking into account that (M,ω) is not s–Lefschetz, we know that there is a non-trivial class $[p_s] \in H^s(M)$ such that $[p_s] \in \ker(H^s(M) \times H^s(M) \longrightarrow \mathbb{R})$. This means that for any other element $[q_s] \in H^s(M)$ we have that $[p_s \wedge q_s \wedge \omega^{n-s}] = 0$ in $H^*(M)$. In the cohomology ring $H^*(\widetilde{CP}^m)$ we have the following equality

$$[p_s \wedge \nu \wedge q_s \wedge \omega_0^l \wedge \nu^{m-s-l-1}] = \begin{cases} 0, & \text{if } m-s-l < m-n, \\ [p_s \wedge q_s \wedge \omega^{n-s} \wedge \nu^{m-n}] = 0, & \text{if } m-s-l = m-n, \\ [p_s \wedge q_s \wedge \omega^l \wedge P(c(N)) \wedge \nu^{m-n}] = 0, & \text{if } m-s-l > m-n, \end{cases}$$

since $P(c(N))$ is a polynomial in the Chern classes of N, and hence a multiple of $[\omega]^{n-s-l}$, because the Chern classes of N are multiples of powers of $[\omega]$.

Therefore for any $j+l \leq \frac{s+2}{2}$, $j > 0$, and $[q_{s+2-2j-2l} \wedge \omega_0^l \wedge \nu^j] \in H^{s+2}(\widetilde{CP}^m)$, we have

$$[\omega_0 + \epsilon\nu]^{m-s-2} \cup [p_s \wedge \nu] \cup [q_{s+2-2j-2l} \wedge \omega_0^l \wedge \nu^j] = 0.$$

Also, in the case where $s+2$ is even, we have

$$[\omega_0 + \epsilon\nu]^{m-s-2} \cup [p_s \wedge \nu] \cup [\omega_0]^{\frac{s+2}{2}} = 0.$$

Thus $[p_s \wedge \nu] \in \ker(H^{s+2}(\widetilde{CP}^m) \times H^{s+2}(\widetilde{CP}^m) \longrightarrow \mathbb{R})$, which proves that $\widetilde{CP}^m$ is not $(s+2)$–Lefschetz. QED

5 Examples of s-Lefschetz symplectic manifolds

In this section, examples of compact symplectic manifolds which are s–Lefschetz but not $(s+1)$–Lefschetz are constructed for $s = 3$ and for any *even* integer $s \geq 2$.

First we show the existence of a simply connected compact symplectic manifold M_s, of high dimension, which is s–Lefschetz but not $(s+1)$–Lefschetz, for each even integer value of $s \geq 2$. The idea for the construction of M_s is to follow an iterative procedure starting from an appropriate low dimensional compact symplectic manifold, take a symplectic embedding of it in a complex projective space CP^m and then consider the symplectic blow-up of CP^m along the embedded submanifold in order to get a simply connected compact symplectic manifold which, according to Theorem 4.2, will be Lefschetz up to a strictly higher level.

The starting point to construct M_s will be the Kodaira–Thurston manifold KT [15, 23]. We begin reviewing it. Consider the Heisenberg group H, that is, the connected nilpotent Lie group of dimension 3 consisting of matrices of the form

$$a = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

where $x, y, z \in \mathbb{R}$. A global system of coordinates (x,y,z) for H is given by $x(a) = x$, $y(a) = y$, $z(a) = z$, and a standard calculation shows that $\{dx,\ dy,\ dz - xdy\}$ is a basis for the left invariant

1–forms on H. Let Γ be the discrete subgroup of H consisting of matrices whose entries x, y and z are integer numbers. So the quotient space $\Gamma\backslash H$ is compact, and the forms dx, dy, $dz - xdy$ descend to 1–forms α, β, γ on $\Gamma\backslash H$ such that α and β are closed, and $d\gamma = -\alpha\wedge\beta$.

The *Kodaira–Thurston manifold* KT is the product $KT = \Gamma\backslash H \times S^1$ (see [15, 23]). Now, if η is the standard invariant 1–form on S^1, then $\{\alpha, \beta, \gamma, \eta\}$ constitutes a (global) basis for the 1–forms on KT. Since

$$d\alpha = d\beta = d\eta = 0, \quad d\gamma = -\alpha\wedge\beta,$$

using Nomizu's theorem [22] we compute the real cohomology of KT:

$$\begin{aligned}
H^0(KT) &= \langle 1\rangle,\\
H^1(KT) &= \langle [\alpha], [\beta], [\eta]\rangle,\\
H^2(KT) &= \langle [\alpha\wedge\gamma], [\beta\wedge\gamma], [\alpha\wedge\eta], [\beta\wedge\eta]\rangle,\\
H^3(KT) &= \langle [\alpha\wedge\gamma\wedge\eta], [\beta\wedge\gamma\wedge\eta], [\alpha\wedge\beta\wedge\gamma]\rangle,\\
H^4(KT) &= \langle [\alpha\wedge\beta\wedge\gamma\wedge\eta]\rangle.
\end{aligned}$$

Therefore, KT is a symplectic manifold with the symplectic form $\omega = \alpha\wedge\gamma + \beta\wedge\eta$. It is clear that (KT, ω) is not 1–Lefschetz, which follows directly from its cohomology or from the general result of Benson and Gordon [2]. Moreover, $H^k_{\mathrm{hr}}(KT, \omega) = H^k(KT)$ for any $k \neq 3$, but $b_3^{\mathrm{hr}}(KT, \omega) = 2 < 3 = b_3(KT)$. It is easy to see that the same holds for any other symplectic form on KT.

Denote $M_0 = KT$. By Gromov–Tischler theorem [11, 24] there exists a symplectic embedding of (KT, ω) in the complex projective space CP^{m_0}, with $m_0 = 5$, endowed with its standard Kähler form. Let us denote by $(M_2 = \widetilde{CP}^{m_0}, \Omega_2)$ the blow-up of CP^{m_0} along M_0. By Theorem 4.2 we can consider Ω_2 an integral form. We may again embed symplectically (M_2, Ω_2) into CP^{m_2} with $m_2 = 11$ and blow-up CP^{m_2} along M_2 to obtain $(M_4 = \widetilde{CP}^{m_2}, \Omega_4)$. So in this fashion we get a simply connected compact symplectic manifold (M_s, Ω_s) for any even integer $s \geq 2$ obtained as the symplectic blow-up $\widetilde{CP}^{m_{s-2}}$ of $CP^{m_{s-2}}$ along (M_{s-2}, Ω_{s-2}) symplectically embedded into $CP^{m_{s-2}}$, where $m_{s-2} = 2m_{s-4} + 1$. Notice that the dimension of the manifold M_{s+2} is equal to $2m_s$, where

$$m_s = 6\cdot 2^r - 1,$$

for $s = 2r \geq 0$.

Proposition 5.1 *For any even integer $s \geq 2$, the simply connected compact symplectic manifold $M_s = \widetilde{CP}^{m_{s-2}}$ is s–Lefschetz but not $(s+1)$–Lefschetz.*

Proof : Since $M_0 = KT$ is 0–Lefschetz (any symplectic manifold is), we can apply Theorem 4.2 r times, with $2r = s$, to conclude that the manifold M_s is s–Lefschetz. To show that M_s is not $(s+1)$–Lefschetz we note the following fact. Consider (M, ω) a compact symplectic manifold and embed symplectically $M \hookrightarrow CP^m$ with $m \geq 2n+1$, where $2n$ is the dimension of M. As usual we write $\widetilde{CP}^m$ for the symplectic blow-up of CP^m along M. By (7), the Betti number $b_i(\widetilde{CP}^m)$ is given by

$$b_i(\widetilde{CP}^m) = b_{i-2}(M) + b_{i-4}(M) + \cdots + b_1(M)$$

if $i > 1$ is odd. Therefore, $b_3(M_2) = b_1(KT) = 3$. For M_4, we have $b_1(M_4) = b_3(M_4) = 0$ and $b_5(M_4) = 3$. In general, for any manifold M_s the odd Betti numbers $b_{2j-1}(M_s)$ vanish for $j \leq r$, and $b_{s+1}(M_s) = b_1(KT) = 3$. This proves that M_s is not $(s+1)$–Lefschetz using Proposition 2.7.

QED

In the following result we decrease as much as possible the dimension of the examples constructed in Proposition 5.1 by using iterated Donaldson symplectic submanifolds.

Proposition 5.2 *Let $s \geq 2$ be an even integer, and let M_s be the simply connected compact symplectic manifold constructed in Proposition* 5.1. *Then, there is a symplectic submanifold $W_s \hookrightarrow M_s$ of dimension $2(s+2)$ which is s–Lefschetz but not $(s+1)$–Lefschetz, and every de Rham cohomology class in $H^i(W_s)$ admits a symplectically harmonic representative for any $i \neq s+3$.*

Proof : According to Theorem 4.2, we can assume that the symplectic form Ω_s of M_s is an integral form and (M_s, Ω_s) is s–Lefschetz. Therefore, we can consider an iterated Donaldson symplectic submanifold $Z_l \hookrightarrow M_s$ of codimension $2l$, i.e. $\dim Z_l = 2(m_{s-2} - l)$. In particular, if $s = 2r$ then we take $l_s = m_{s-2} - s - 2 = 6 \cdot 2^{r-1} - 2r - 3$, and denote by W_s the corresponding simply connected compact symplectic manifold Z_{l_s} of dimension $2(s+2)$.

Since $6 \cdot 2^r - 2r - 3 = 2m_{s-2} - s - 1$, Poincaré duality implies that $b_{6\cdot 2^r - 2r-3}(M_s) = b_{s+1}(M_s)$, which equals $b_1(KT) = 3$ as shown in the proof of Proposition 5.1.

Notice that $6 \cdot 2^r - 2r - 3 = s + 3 + 2l_s$. Therefore, $b_{s+3}(W_s) = b_{s+3+2l_s}(M_s) = 3$. Moreover, Corollary 3.1 implies that $b_i(W_s) - b_i^{\rm hr}(W_s) = 0$ for $i > (s+3)$, and $b_{s+3}(W_s) - b_{s+3}^{\rm hr}(W_s) = b_{s+3+2l_s}(M_s) - b_{s+3+2l_s}^{\rm hr}(M_s) \equiv 1 \pmod 2$, by Proposition 2.7. From Proposition 2.6 we conclude that W_s is s–Lefschetz but not $(s+1)$–Lefschetz.

QED

Remark 5.3 *If we begin with* any *symplectic* 4*–manifold N whose first Betti number is $b_1(N) =$* 1 *(see* [10]*), then we obtain a symplectic manifold W_s' satisfying the conditions of Proposition 5.2, but with $b_{s+3}^{\rm hr}(W_s') = 0$.*

Corollary 5.4 *Let n and s be integer numbers such that $s \geq 2$ is even, and $n \geq s+2$. Then there exists a simply connected compact symplectic manifold of dimension $2n$ which is s–Lefschetz but not $(s+1)$–Lefschetz.*

It is worthy to remark that Proposition 5.2 and Corollary 5.4 also hold in the *non-simply connected* setting. For any even integer $s \geq 2$, it suffices to take the product of the symplectic manifold W_s constructed in Proposition 5.2 by a 2–dimensional torus $\mathbb{T}^2$, and then consider a Donaldson symplectic submanifold to reduce the dimension.

One can also address the problem of constructing examples of symplectic manifolds M_s which are s–Lefschetz and not $(s+1)$–Lefschetz for *odd* integer numbers $s \geq 1$. We do the cases $s = 1$ and $s = 3$. Consider the connected completely solvable Lie group G of dimension 6 consisting of matrices of the form

$$a = \begin{pmatrix} e^t & 0 & xe^t & 0 & 0 & y_1 \\ 0 & e^{-t} & 0 & xe^{-t} & 0 & y_2 \\ 0 & 0 & e^t & 0 & 0 & z_1 \\ 0 & 0 & 0 & e^{-t} & 0 & z_2 \\ 0 & 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $t, x, y_i, z_i \in \mathbb{R}$ $(i = 1, 2)$. A global system of coordinates $(t, x, y_1, y_2, z_1, z_2)$ for G is defined by $t(a) = t$, $x(a) = x$, $y_i(a) = y_i$, $z_i(a) = z_i$, and a standard calculation shows that a basis for the left invariant 1–forms on G consists of

$$\{dt,\ dx,\ e^{-t}dy_1 - xe^{-t}dz_1,\ e^t dy_2 - xe^t dz_2,\ e^{-t}dz_1,\ e^t dz_2\}.$$

Let Γ be a discrete subgroup of G such that the quotient space $M = \Gamma\backslash G$ is compact. (Such a subgroup exists, see [7].) Hence the forms dt, dx, $e^{-t}dy_1 - xe^{-t}dz_1$, $e^t dy_2 - xe^t dz_1$, $e^{-t}dz_1$, $e^t dz_2$ descend to 1–forms α, β, γ_1, γ_2, δ_1, δ_2 on M satisfying

$$d\alpha = d\beta = 0, \quad d\gamma_1 = -\alpha\wedge\gamma_1 - \beta\wedge\delta_1, \quad d\gamma_2 = \alpha\wedge\gamma_2 - \beta\wedge\delta_2, \quad d\delta_1 = -\alpha\wedge\delta_1, \quad d\delta_2 = \alpha\wedge\delta_2,$$

and such that $\{\alpha, \beta, \gamma_1, \gamma_2, \delta_1, \delta_2\}$ is a global basis for the 1–forms on M. Using Hattori's theorem [13] we compute the real cohomology of M:

$$\begin{aligned}
H^0(M) &= \langle 1\rangle,\\
H^1(M) &= \langle [\alpha], [\beta]\rangle,\\
H^2(M) &= \langle [\alpha\wedge\beta], [\delta_1\wedge\delta_2], [\gamma_1\wedge\delta_2 + \gamma_2\wedge\delta_1]\rangle,\\
H^3(M) &= \langle [\alpha\wedge\delta_1\wedge\delta_2], [\beta\wedge\gamma_1\wedge\gamma_2], [\beta\wedge(\gamma_1\wedge\delta_2+\gamma_2\wedge\delta_1)], [\alpha\wedge(\gamma_1\wedge\delta_2+\gamma_2\wedge\delta_1)]\rangle,\\
H^4(M) &= \langle [\alpha\wedge\beta\wedge\gamma_1\wedge\gamma_2], [\alpha\wedge\beta\wedge\gamma_1\wedge\delta_2], [\gamma_1\wedge\gamma_2\wedge\delta_1\wedge\delta_2]\rangle,\\
H^5(M) &= \langle [\alpha\wedge\gamma_1\wedge\gamma_2\wedge\delta_1\wedge\delta_2], [\beta\wedge\gamma_1\wedge\gamma_2\wedge\delta_1\wedge\delta_2]\rangle,\\
H^6(M) &= \langle [\alpha\wedge\beta\wedge\gamma_1\wedge\gamma_2\wedge\delta_1\wedge\delta_2]\rangle.
\end{aligned}$$

Consider the symplectic form ω on M given by $\omega = \alpha\wedge\beta+\gamma_1\wedge\delta_2+\gamma_2\wedge\delta_1$. Then $[\omega]\cup[\delta_1\wedge\delta_2] = 0$ in $H^4(M)$, which means that M is not 2–Lefschetz. But a simple computation shows that the cup product by $[\omega]^2$ is an isomorphism between $H^1(M)$ and $H^5(M)$. Therefore, (M, ω) is 1–Lefschetz, but not 2–Lefschetz. Moreover, $b_k^{\rm hr}(M,\omega) = b_k(M)$ for $k \neq 4$, and $b_4^{\rm hr}(M,\omega) = 2 < 3 = b_4(M)$ (compare with Corollary 2.5). The same holds for any symplectic form on M [14]. Therefore, (M, ω) is 1–Lefschetz, but not 2–Lefschetz.

Now we deal with the case $s = 3$. Consider a symplectic embedding of $(M_1, \Omega_1) = (M, \omega)$ in the complex projective space CP^{m_1}, with $m_1 = 7$, endowed with its standard symplectic form. We define $(M_3 = \widetilde{CP}^{m_1}, \Omega_3)$ as the symplectic blow-up of CP^{m_1} along M_1.

Proposition 5.5 *The simply connected compact symplectic manifold* (M_3, Ω_3) *is* 3*–Lefschetz but not* 4*–Lefschetz. Moreover, there is a symplectic submanifold* $W_3 \hookrightarrow M_3$ *of dimension* 10 *which is* 3*–Lefschetz but not* 4*–Lefschetz, and every de Rham cohomology class in* $H^i(W_3)$ *admits a symplectically harmonic representative for any* $i \neq 6$*.*

Proof : Since (M, ω) is 1–Lefschetz but not 2–Lefschetz, Theorem 4.2 and Proposition 4.4 imply that $M_3 = \widetilde{CP}^7$ is 3–Lefschetz and not 4–Lefschetz. As in the proof of Proposition 5.2, an iterated Donaldson submanifold Z_l, $l = 2$, of M_3 provides an example W_3 in dimension 10 which is 3–Lefschetz and not 4–Lefschetz. *QED*

Note also that there exists simply connected compact symplectic manifolds of dimension 6 which are 1–Lefschetz but not 2–Lefschetz [10, Theorem 7.1].

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