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Hilbert-Schmidt Composition Operators on Dirichlet Spaces

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ABSTRACT. In this note we show that analytic self-maps φ of the unit disk inducing Hilbert-Schmidt composition operators C_φ on the weighted Dirichlet space $\mathcal{D}_\alpha$ satisfy that the set $E_\varphi = \{e^{i\theta} \in \partial\mathbb{D} : \ |\varphi(e^{i\theta})| = 1\}$ has zero α-capacity.

1. Introduction

Let $\mathbb{D}$ denote the open unit disc of the complex plane. If φ is an analytic function on $\mathbb{D}$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$, the composition operator induced by φ is defined by

$$C_\varphi f = f \circ \varphi$$

for all holomorphic function f on $\mathbb{D}$. Composition operators have been extensively studied in last four decades. One of the main points is that they connect two important areas of mathematics: operator theory and analytic function theory. Actually, the aim of the works on composition operators has been to discover geometric properties of φ that allow to determine functional analytic properties of C_φ and viceversa. In particular, if $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$, the analysis of the set

$$E_\varphi = \{e^{i\theta} \in \partial\mathbb{D} : \ |\varphi(e^{i\theta})| = 1\}$$

has played an important role in the results on composition operators. Here $\varphi(e^{i\theta}) = \lim_{r\to 1^-} \varphi(re^{i\theta})$, where the limit exists a. e. by Fatou's radial limit theorem (see [**4**], for instance).

The first result related to the set E_φ traces back to 1969 with H. J. Schwartz's work [**8**]. He proved that a necessary condition for C_φ to be compact on the Hardy space $\mathcal{H}^2$ is that E_φ has Lebesgue measure zero. After that, other authors have studied problems relating operators properties of C_φ to the size of E_φ. For a comprehensive treatment of such problems, we refer to C. C. Cowen and B. D. MacCluer's book [**3**].

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The focus of this work is the size of E_φ whenever φ induces a Hilbert-Schmidt composition operator on weighted Dirichlet spaces $\mathcal{D}_\alpha$. These spaces along with the concept of α-capacity of a set are introduced in Section 2. After discussing that there is a close connection between the functions in $\mathcal{D}_\alpha$ and α-capacity, we show the main result

Theorem: *If C_φ is Hilbert-Schmidt on $\mathcal{D}_\alpha$, then the α-capacity of E_φ is zero.*

The above result generalizes a previous one proved by the authors for the Dirichlet space (see [**5**]).

2. Preliminaries

2.1. Weighted Dirichlet spaces. Let A denotes the normalized Lebesgue area measure on the unit disc $\mathbb{D}$, i. e., $dA(z) = \frac{1}{\pi}\, dx\, dy$. If $-1 < \alpha < 1$ the weighted Dirichlet space $\mathcal{D}_\alpha$ consists of analytic functions f on $\mathbb{D}$ such that the norm

$$\|f\|_\alpha^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^\alpha dA(z)$$

is finite. Note that $\mathcal{D}_0$ is the Dirichlet space $\mathcal{D}$, and allowing $\alpha = 1$ the norm obtained is equivalent to the usual one in the Hardy space $\mathcal{H}^2$, so $\mathcal{D}_1$ is the Hardy space.

Observe that as α decreases the functions in $\mathcal{D}_\alpha$ are of slower growth when approaching to the boundary of the unit disc. Moreover, if $\alpha_1 < \alpha_2$, then $\mathcal{D}_{\alpha_1}$ is strictly contained in $\mathcal{D}_{\alpha_2}$.

2.2. Capacity of a set. Let μ be a measure of bounded support S_μ. The α-potential of the measure μ is defined by

$$u_\alpha^\mu(x) = \begin{cases} \displaystyle\int \log \frac{1}{|x-y|}\, d\mu(y), & \alpha = 0; \\[2ex] \displaystyle\int \frac{d\mu(y)}{|x-y|^\alpha}, & 0 < \alpha < 1. \end{cases}$$

If $I_\alpha(\mu)$ denotes the energy integral of μ,

$$I_\alpha(\mu) = \int u_\alpha^\mu d\,\mu(x),$$

then the α-capacity of a bounded Borel set E is defined by

$$C_\alpha(E) = \{\inf I(\mu)\}^{-1}$$

where the infimum is taken over all positive measures μ with total mass 1 and support S_μ contained in E. For $\alpha = 0$, the α-capacity is also called logarithmic capacity.

Note that there exist Borel sets E of Lebesgue measure zero and α-capacity positive for any $0 \le \alpha < 1$. In addition, observe that if $C_{\alpha_1}(E) = 0$ then $C_{\alpha_2}(E) = 0$ for any $\alpha_2 > \alpha_1$. For more about capacities see [**2**] and [**6**].

On the other hand, if $0 \le \alpha < 1$ there is a close relation between functions in $\mathcal{D}_\alpha$ and α-capacity. In 1939, Beurling [**1**] proved that if f belongs to the Dirichlet space $\mathcal{D}_0$, $\alpha = 0$, then the radial limits $f(e^{i\theta}) = \lim_{r\to 1^-} f(re^{i\theta})$ exist except on a set of logarithmic capacity zero (see also [**2**]). In 1965 H. Wallin [**9**] generalized this result for $0 < \alpha < 1$, that is, if $f \in \mathcal{D}_\alpha$ then $f(e^{i\theta})$ is defined except on a set of α-capacity zero.

2.3. Hilbert-Schmidt composition operators. Recall that a linear operator T on a Hilbert space $\mathcal{H}$ is said to be Hilbert-Schmidt if there exists an orthonormal basis $\{e_n\}$ in $\mathcal{H}$ such that the series

$$\sum_{n=1}^{\infty} \|Te_n\|^2 \tag{1}$$

is convergent. It is easy to see that, in such a case, condition (1) does not depend on the particular choice of the orthonormal basis $\{e_n\}$. Clearly, every Hilbert-Schmidt operator is bounded.

First, let us give the following characterižation of Hilbert-Schmidt composition operators on weighted Dirichlet spaces

PROPOSITION 2.1. *Let φ be an analytic self-map of $\mathbb{D}$. Then C_φ is Hilbert-Schmidt on $\mathcal{D}_\alpha$ if and only if*

$$\int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{\alpha+2}}(1-|z|^2)^\alpha dA(z) < \infty.$$

The proof of proposition 2.1 just follows by considering that $\{z^n/(n+1)^{(1-\alpha)/2}\}$ is an orthonormal basis in $\mathcal{D}_\alpha$ and Stirling's formula.

3. Proof of the Main Result

Let us suppose that φ induces a Hilbert-Schmidt composition operator on $\mathcal{D}_\alpha$. If $-1 < \alpha < 0$ then the supremum norm $\|\varphi\|_\infty < 1$ (see [**3**], Theorem 4.5) and therefore, E_φ is the empty set and there is nothing to discuss.

Suppose that $0 \le \alpha < 1$. Let us consider u the positive harmonic function

$$u(z) = \mathrm{Re}\frac{1+\varphi(z)}{1-\varphi(z)}.$$

If $z = x + iy$ and $\nabla u(z)$ denotes the the gradient vector $\nabla u(z) = (\partial u/\partial x, \partial u/\partial y)$, a computation using proposition 2.1 yields that the integral

$$\int_{\mathbb{D}} \frac{|\nabla u(z)|^2}{u(z)^{\alpha+2}} \frac{(1-|z|^2)^\alpha}{|1-\varphi(z)|^{2\alpha}} dA(z)$$

is finite. Therefore, the integral

$$\int_{\mathbb{D}} \frac{|\nabla u(z)|^2}{u(z)^{\alpha+2}}(1-|z|^2)^\alpha dA(z)$$

is also finite, or equivalently,

$$\int_{\mathbb{D}} |\nabla(u^{-\alpha/2})|^2(1-|z|^2)^\alpha dA < \infty. \tag{2}$$

From here we construct an analytic function f belonging to $\mathcal{D}_\alpha$ to which is applied Wallin's result.

Let $P_r(\theta)$ denote the Poisson kernel and consider v the harmonic function

$$v(z) = v(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u^{-\alpha/2}(t)\, P_r(\theta - t)\, dt.$$

Since harmonic functions minimize the energy integral, we deduce that the integral

$$\int_{\mathbb{D}} |\nabla v(z)|^2 (1 - |z|^2)^\alpha dA(z)$$

is finite. Let f be an analytic function on the unit disc such that $f(0) = 0$ and Re $f = v$. From what we have just showed, it follows that f belongs to the weighted Dirichlet space $\mathcal{D}_\alpha$. Moreover, the radial limits satisfy

$$\lim_{r \to 1^{-1}} \text{Re } f(re^{i\theta}) = u^{-\alpha/2}(e^{i\theta}).$$

Now Wallin's Theorem states that the radial limits $f(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta})$ exist except on a set of α-capacity zero, and so do Re $f(e^{i\theta})$. Therefore, the set

$$\{e^{i\theta} \in \partial\mathbb{D} : \ u^{-\alpha/2}(e^{i\theta}) = \infty\}$$

has α-capacity zero. The desired result follows since this set coincides with E_φ.

REMARK 3.1. *Main theorem is only a necessary condition. In fact, the function $\varphi(z) = (1 + z)/2$ induces a non Hilbert-Schmidt composition operator on any weighted Dirichlet space $\mathcal{D}_\alpha$ and the set E_φ only has one element.*

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