CODIMENSION ONE SYMPLECTIC FOLIATIONS

OMEGAR CALVO, VICENTE MUÑOZ, AND FRANCISCO PRESAS

Abstract. We define the concept of symplectic foliation on a symplectic manifold and provide a method of constructing many examples, by using asymptotically holomorphic techniques.

1. Introduction

During the last three decades there has been an active field of research related to the study of holomorphic foliations over a complex manifold [5, 7, 10, 13]. To define a codimension one holomorphic foliation we need to fix a holomorphic line bundle L over the manifold M. Then we choose a non-zero holomorphic section $\alpha \in H^0(T^*M \otimes L)$, satisfying the integrability condition:

$$\alpha \wedge d\alpha = 0. \tag{1}$$

There is an equivalence relation given by multiplication of α by no-where zero holomorphic functions, and a holomorphic foliation is defined as an equivalence class of such integrable 1-forms. In what follows we restrict ourselves to the case where M is compact, so that the set of foliations is a subset in the projective space $\mathbb{P}H^0(T^*M \otimes L)$.

In this work, we aim to generalize this notion to the symplectic category. We give the following definition

Definition 1.1. *A symplectic foliation α with normal line bundle L on a symplectic manifold (M, ω) is a non-zero element of $\mathcal{C}^\infty(T^*_{\mathbb{C}}M \otimes_{\mathbb{C}} L)$ which satisfies the integrability condition* (1)*. Also we impose that the set of singularities, defined as $S_\alpha = \{x \in M \,\big|\, \alpha(x) = 0\}$, is a finite union of symplectic submanifolds of real codimension greater or equal to four and whose intersections are transverse and symplectic. Finally, we impose that for any $p \in M - S_\alpha$ the subspace $\ker \alpha(p) \subset T_pM$ is symplectic.*

Two symplectic foliations α_1 and α_2 are considered equivalent whenever there is an isomorphism $\psi : L \to L$ as real plane bundles such that $\psi^*\alpha_2 = \alpha_1$.

To understand $\ker \alpha(p)$ as a subspace of T_pM, we look at the isomorphism $T^*_{\mathbb{C}}M \otimes_{\mathbb{C}} L = T^*M \otimes_{\mathbb{R}} L$, where $T^*_{\mathbb{C}}M$ is the complexified cotangent bundle. Therefore we may interpret $\alpha(p) : T_pM \to L_p$ as a real linear map and $\ker \alpha(p) \subset T_pM$ is a codimension two subspace.

Now if α_1 and α_2 are equivalent then $S_{\alpha_1} = S_{\alpha_2}$ and the topological foliations coincide $\ker \alpha_1 = \ker \alpha_2$. Note that the isomorphism $\psi : L \to L$ takes values in $\mathrm{GL}(2, \mathbb{R})$, so in

Date: October, 2001.

Key words and phrases. Foliation, symplectic, asymptotically holomorphic.

Partially supported by The European Contract Human Potential Programme, Research Training Network HPRN-CT-2000-00101. Third author has conducted his research with a grant from Fundación Pedro Barrié de la Maza.

particular if there is a nowhere zero complex function f such that $\alpha_1 = f\alpha_2$ then the foliations are equivalent.

The simplest examples of symplectic foliations are given by the Lefschetz pencils constructed by Donaldson [9]. A chart $\phi : U \subset M \to \mathbb{C}^n$ will be called adapted at the point $x \in U$ if $(\phi_*)_x\omega = \omega_0$, where ω_0 is the standard symplectic form in $\mathbb{C}^n$. A symplectic Lefschetz pencil on a $2n$-dimensional symplectic manifold (M, ω) consists of a codimension 4 symplectic submanifold $N \subset M$ and a map $f : M - N \to \mathbb{CP}^1$ such that locally around N there are adapted coordinates $(z_1, \dots, z_n)$ with values in $\mathbb{C}^n$ where f is written as z_2/z_1. Also f has finitely many isolated critical points around which there are adapted coordinates where $f = z_1^2 + \cdots + z_n^2$. Finally the fibers of f are symplectic off their singularities. These belong a special kind of foliations defined as follows

Definition 1.2. *A symplectic foliation α on a $2n$-dimensional symplectic manifold (M, ω) is of Kupka type if the singular set S_α is a disjoint union of*

i. *isolated points where there are adapted charts $(z_1, \dots, z_n)$ such that $\alpha = z_1dz_1 + \cdots + z_ndz_n$.*
ii. *codimension 4 smooth symplectic submanifolds such that each point has an adapted chart $(z_1, \dots, z_n)$ with $\alpha = \eta(z_1, z_2)$ for a 1-form η of 2 complex variables with $d\eta(0) \neq 0$ and $\eta^{-1}(0) = \{0\}$.*

We want to show a general construction of symplectic foliations

Theorem 1.3. *Let (M, ω) be a symplectic manifold. Then M admits symplectic foliations of Kupka type which are not symplectic Lefschetz pencils. Also M admits symplectic foliations not of Kupka type.*

The method of construction is a generalization of the techniques developed in [12]. The structure of the paper is as follows. In section 2 we give the basic results of the theory of holomorphic foliations. Section 3 reviews the asymptotically holomorphic theory introduced in [8] and used in [2, 12]. Next in section 4 we introduce the notion of foliation in this category and check that asymptotically holomorphic foliations with some property of transversality give symplectic foliations. In the following section we move on to prove that it is possible to obtain asymptotically holomorphic foliations by embedding M into the projective space $\mathbb{CP}^d$ and intersecting the image with a given holomorphic foliation of $\mathbb{CP}^d$. Finally section 6 is devoted to give some examples of foliations constructed with these techniques.

2. Codimension one holomorphic foliations

In this section we discuss briefly the theory of holomorphic foliations on a compact connected complex manifold M. A codimension one holomorphic foliation with singularities in M is an equivalence class of holomorphic $\alpha \in H^0(M, T^*M \otimes L)$, where L is a holomorphic line bundle and $\alpha \wedge d\alpha = 0$.

Given a foliation α, we say that $p \in M$ is a regular point if $\alpha(p) \neq 0$. Otherwise, we say that p is singular. The set

$$S_\alpha = \{p \in M \mid \alpha(p) = 0\}$$

is the singular set. If this set has components of codimension 1, let D be the corresponding divisor. Then there exists a holomorphic section f of $\mathcal{O}(D)$ such that α/f is a foliation whose singularities are of codimension two or more. So we can always suppose that $\operatorname{codim}_{\mathbb{C}} S_\alpha \geq 2$.

For a regular point $p \in M$ there exists an open neighborhood $U \subset M$ of p such that α may be written as

$$\alpha = h\, df$$

in U, where h and f are holomorphic functions in U. Such f is called *first integral* and h an *integrating factor.* The leaves of the foliation in U are the level surfaces of f. Globally, the leaves of the foliation α are the leaves of the foliation defined in $M - S_\alpha$. If V is a compact hypersurface of M such that $V - V \cap S_\alpha$ is a leaf, in general, we have $V \cap S_\alpha \neq \emptyset$. In any case, by abuse of language, we will say that V is a compact leaf of the foliation.

2.1. **Kupka singularities.** In this section, we will consider an important class of singularities which have stability properties under deformations.

Definition 2.1. *The* Kupka singular set *of the foliation α consists of the points*

$$K_\alpha = \{p \in M \mid \alpha(p) = 0,\, d\alpha(p) \neq 0\}.$$

For every connected component $K \subset K_\alpha$, there exists a holomorphic 1-form

$$\eta = A(z_1, z_2)\, dz_1 + B(z_1, z_2)\, dz_2,$$

called the *transversal type* at K, defined on a neighborhood V of $0 \in \mathbb{C}^2$ and vanishing only at 0, an open cover $\{U_i\}$ of a neighborhood of K in M and a family of submersions $\varphi_i : U_i \to \mathbb{C}^2$, such that

$$\varphi_i^{-1}(0) = K \cap U_i, \quad \text{and} \quad \alpha|_{U_i} = \varphi_i^* \eta.$$

A foliation α is of *Kupka type* if K_α is compact and connected.

The main examples of foliations of Kupka type are the following: Let L_1 and L_2 be holomorphic line bundles on M, such that $L_1^{\otimes p} = L_2^{\otimes q}$, where p and q are relatively prime, positive integers. Given f_1 and f_2 holomorphic sections of the line bundles L_1 and L_2 respectively, the holomorphic section

$$\alpha = p f_1\, df_2 - q f_2\, df_1 \in H^0(M, T^*M \otimes L_1 \otimes L_2),$$

is a foliation. Moreover the leaves of the foliation represented by α, are the fibers of the meromorphic map $\phi = f_1^p / f_2^q$. We say that the map ϕ is a *meromorphic first integral* of the foliation represented by α.

A *branched Lefschetz pencil* (a *Lefschetz pencil* if $p = q = 1$) is a meromorphic map satisfying the following conditions:

i. The holomorphic line bundles L_1 and L_2 are positive.
ii. The hypersurfaces $\{f_1 = 0\}$ and $\{f_2 = 0\}$ are smooth, and meet transversely along a codimension two submanifold K.
iii. The subvarieties defined by $\lambda f_1^p - \mu f_2^q = 0$ with $[\lambda : \mu] \in \mathbb{CP}^1$, are smooth on $M - K$, except for a finite set of points, where they have just a non-degenerate critical point.

These foliations are of Kupka type with $K_\alpha = \{f_1 = f_2 = 0\}$.

Theorem 2.2 ([6])**.** *Let α be a foliation of Kupka type in $\mathbb{CP}^n$, $n \geq 3$. K_α is a complete intersection if and only if $\alpha = p f_1\, df_2 - q f_2\, df_1$.*

For foliations on $\mathbb{CP}^n$, $n \geq 6$, it may be shown that any foliation of Kupka type is a branched Lefschetz pencil.

For the unbranched case, we have the following construction involving the fundamental group [4]. Consider a family (E_t, σ_t) of projectively flat bundles of rank two with section such that $(E_0, \sigma_0) = (L_1 \oplus L_1, (f_1, f_2))$ is a Lefschetz pencil. If L_1 is sufficiently ample, we are able to prove that $H^0(E_t) \neq 0$, and then, we consider the foliation $\sigma_t^* \mathcal{H}_t$, where $\mathcal{H}_t$ denotes the flat structure on the $\mathbb{CP}^1$-bundle $\mathbb{P}(E_t)$.

It is an open question whether any foliation of Kupka type with positive normal bundle and transversal type $z_2\, dz_1 - z_1\, dz_2$ may be described as above.

2.2. **Logarithmic foliations.** A *holomorphic integrating factor* of a foliation α is a holomorphic section $\varphi \in H^0(M, L)$ such that the meromorphic 1-form $\Omega = \frac{\alpha}{\varphi}$ is closed.

Theorem 2.3. *Let M be a projective manifold with $H^1(M; \mathbb{C}) = 0$, and let $\varphi = \varphi_1^{r_1} \cdots \varphi_k^{r_k}$ be an integrating factor of a foliation α. Then*

$$\Omega = \frac{\alpha}{\varphi} = \sum_{i=1}^{k} \lambda_i \frac{d\varphi_i}{\varphi_i} + d\left(\frac{\psi}{\varphi_1^{r_1-1} \cdots \varphi_k^{r_k-1}}\right),$$

where $\lambda_i \in \mathbb{C}$ and ψ is a holomorphic section of the line bundle $\mathcal{O}\left(\sum_{i=1}^{k} (r_i - 1)\{\varphi_i = 0\}\right)$.

¿From this equation, we have that the hypersurfaces $D_i = \{\varphi_i = 0\}$ are compact leaves of the foliation α. The residue theorem implies the relation:

$$\sum_{i=1}^{k} \lambda_i \cdot [\{\varphi_i = 0\}] = 0 \in H^2(M; \mathbb{C}).$$

The integrating factor is *reduced* if $r_i = 1$. In this case

$$\alpha = \varphi_1 \cdots \varphi_k \left(\sum_{i=1}^{k} \lambda_i \frac{d\varphi_i}{\varphi_i}\right),$$

we say that the foliation is *logarithmic*. The singular set is the union of $D_i \cap D_j$ for all possible $1 \leq i < j \leq k$. The Kupka set is

$$K_\alpha = S_\alpha - \bigcup_{1 \leq i < j < t \leq k} (D_i \cap D_j \cap D_t),$$

and it is therefore not compact for $k \geq 3$.

3. Asymptotically holomorphic theory

Let (M, ω) be a symplectic manifold with $[\omega]/2\pi \in H^2(M; \mathbb{R})$ an integer cohomology class. Such a symplectic manifold will be called *of integer class.* Fix an almost complex structure J compatible with ω and denote $g(u, v) = \omega(u, Jv)$ the associated metric. Let $L \to M$ be the hermitian line bundle with connection whose curvature is $-i\omega$. The key for a search of symplectic objects is to look for objects which are close to be J-holomorphic. The asymptotically holomorphic techniques introduced by Donaldson [8] give a method of construction of such objects by means of increasing the positivity of the curvature of the bundles, which is achieved by twisting with $L^{\otimes k}$ for large k. Let us introduce the main notations following [2, 12].

Definition 3.1. *A sequence of sections s_k of hermitian bundles E_k with connections on M is called asymptotically holomorphic if $|\nabla^p s_k| = O(1)$ for $p \geq 0$ and $|\nabla^{p-1}\bar{\partial} s_k| = O(k^{-1/2})$ for $p \geq 1$. The norms are evaluated with respect to the metrics $g_k = kg$.*

Definition 3.2. *A section s_k of the bundle E_k is η-transverse to 0 if for every $x \in M$ such that $|s_k(x)| < \eta$ then $\nabla s_k(x)$ has a right inverse θ_k such that $|\theta_k| < \eta^{-1}$.*

This means that at a point x close to the zero set of s_k the differential $\nabla s_k(x) : T_xM \to (E_k)_x$ is surjective and that, in the orthogonal to the kernel, this map multiplies the length of the vectors at least by η. This guarantees that $Z_k = Z(s_k)$ is a submanifold with bounded curvature R_{Z_k} (in the metric g_k) uniformly on k, and that T_xZ_k is within distance $O(k^{-1/2})$ of being a complex subspace of T_xM. The condition of s_k being asymptotically holomorphic implies that Z_k is symplectic for large k.

Definition 3.3. *A sequence of submanifolds $S_k \subset M$ is called asymptotically holomorphic if*

$$\angle_M(TS_k, JTS_k) = O(k^{-1/2}), \ |R_{S_k}| = O(1).$$

The angle $\angle_M$ measures the distance, in the grassmannian, between two subspaces [12, definition 3.1]. Thus for k large, any element of a sequence of asymptotically holomorphic submanifolds is symplectic.

Our objective will be to define and construct asymptotically holomorphic foliations. This will be done by embedding our manifold M into a projective space and intersecting it with a holomorphic foliation in it.

Definition 3.4. *A sequence of embeddings $\phi_k : M \to \mathbb{CP}^d$ is γ-asymptotically holomorphic for some $\gamma > 0$ if it satisfies the following conditions:*

i. *$d\phi_k : T_xM \to T_{\phi_k(x)}X$ has a left inverse θ_k of norm less than γ^{-1} at every point $x \in M$, i.e., $d\phi_k$ multiplies the length of vectors at least by γ.*
ii. *$|(\phi_k)_*J - J_0|_{g_k} = O(k^{-1/2})$ on the subspace $(\phi_k)_*T_xM$.*
iii. *$|\nabla^p\phi_k|_{g_k} = O(1)$ and $|\nabla^{p-1}\bar{\partial}\phi_k|_{g_k} = O(k^{-1/2})$, for all $p \geq 1$.*

Theorem 3.5 ([12])**.** *Let (M, ω) be a closed symplectic $2n$-dimensional manifold of integer class endowed with a compatible almost complex structure J and let s_k be an asymptotically holomorphic sequence of sections of the vector bundles $\mathbb{C}^{d+1} \otimes L^{\otimes k}$ with $d \geq 2n+1$. Then for any $\alpha > 0$ there exists another asymptotically holomorphic sequence σ_k and $\gamma > 0$ such that:*

i. *$|s_k - \sigma_k|_{C^2, g_k} < \alpha$.*
ii. *σ_k is γ-projectizable, i.e., $|\sigma_k| \geq \gamma$ on all of M, and for all k.*
iii. *$\mathbb{P}(\sigma_k)$ is a γ-asymptotically holomorphic sequence of embeddings in $\mathbb{CP}^d$ for k large enough.*
iv. *$\phi_k^*[\omega_{FS}] = k[\omega]$, where ω_{FS} is the Fubini-Study symplectic form on $\mathbb{CP}^d$.*

Moreover, let us have two asymptotically holomorphic sequences ϕ_k^0 and ϕ_k^1 of embeddings in $\mathbb{CP}^d$, with respect to two compatible almost complex structures. Then for k large enough, there exists an isotopy of asymptotically holomorphic embeddings ϕ_k^t connecting ϕ_k^0 and ϕ_k^1.

The difference with the result stated in [12] is that here we use C^2-close perturbations, but this makes no real difference. To be able to intersect the embedded manifold with a complex submanifold of $\mathbb{CP}^d$ having control of the resulting submanifold, we need a notion of estimated transversality

Definition 3.6. *Let $N \subset \mathbb{CP}^d$ be a complex smooth submanifold and choose a distribution of complex subspaces $D_N(y) \subset T_y\mathbb{CP}^d$, in a neighborhood of N, which extends the tangent distribution to N. An embedding $\phi_k : M \to \mathbb{CP}^d$ is σ-transverse to N, with $\sigma > 0$ small enough, if for all $x \in M$,*

$$d(\phi_k(x), N) < \sigma \Rightarrow \angle_m((\phi_k)_*(T_xM), D_N(\phi_k(x))) > \sigma.$$

This angle $\angle_m$ measures the amount of transversality between two intersecting vector subspaces [12, definition 3.3]. The condition above assures that the intersection $\phi_k(M) \cap N$ is a sequence of asymptotically holomorphic submanifolds of M (see [12, proposition 3.10]). Moreover, this condition may be achieved.

Theorem 3.7 ([12])**.** *Let $\phi_k = \mathbb{P}(s_k)$, where s_k is a γ-projectizable asymptotically holomorphic sequence of sections of $\mathbb{C}^{d+1} \otimes L^{\otimes k}$, $d \geq 2n+1$, such that ϕ_k is a γ-asymptotically holomorphic sequence of embeddings, for some $\gamma > 0$. Let N be a complex submanifold in $\mathbb{CP}^d$. Then for any $\delta > 0$ there exists an asymptotically holomorphic sequence of sections σ_k of $\mathbb{C}^{d+1} \otimes L^{\otimes k}$ such that*

i. $|\sigma_k - s_k|_{C^2, g_k} < \delta$.
ii. *$\psi_k = \mathbb{P}(\sigma_k)$ is a η-asymptotically holomorphic embedding in $\mathbb{CP}^d$ which is ε-transverse to N, for some $\eta > 0$ and $\varepsilon > 0$, for k large enough.*

Moreover, let us have two asymptotically holomorphic sequences ϕ_k^0 and ϕ_k^1 of embeddings in $\mathbb{CP}^d$, with respect to two compatible almost complex structures, which are ε-transverse to N. Then for k large enough, there exists an isotopy of asymptotically holomorphic embeddings ϕ_k^t which are ε'-transverse to N, connecting ϕ_k^0 and ϕ_k^1.

4. Asymptotically holomorphic foliations

Consider a foliation α of M as an application $\alpha : TM \to L$. The line bundle L is called the normal bundle of the foliation. Using the almost complex structure on TM, we decompose α in complex linear and complex anti-linear parts,

$$\alpha = \alpha_{1,0} + \alpha_{0,1}.$$

When $\alpha_{0,1}(x) = 0$ the subspace $\ker \alpha(x) \subset T_xM$ is complex. Still when $|\alpha_{0,1}(x)| < |\alpha_{1,0}(x)|$ the subspace $\ker \alpha(x)$ is symplectic.

Definition 4.1. *A sequence of foliations α_k with hermitian normal bundles E_k is asymptotically holomorphic if*

$$|\nabla^p \alpha_k| = O(1), \quad |\nabla^p (\alpha_k)_{0,1}| = O(k^{-1/2}), \qquad p = 0, 1, 2,$$
$$|\nabla^{p-1} \bar{\partial} (\alpha_k)_{1,0}| = O(k^{-1/2}), \qquad p = 1, 2.$$

Also we need a measure of transversality for foliations. This is provided by the following definition. As in the holomorphic case, there is a subset of the singular case, $K_{\alpha_k} \subset S_{\alpha_k}$ which is easily controlled, and which in some cases it reduces to the Kupka set of α_k.

Definition 4.2. *Let $\gamma, \varepsilon > 0$. A sequence of foliations α_k with hermitian normal bundles E_k is (γ, ε)-regular if there is a subset K_{α_k} of S_{α_k} such that*

i. *K_{α_k} is a union of (closed) asymptotically holomorphic submanifolds whose intersections are transverse and asymptotically holomorphic submanifolds.*
ii. *Let B^k_γ be the tubular neighborhood of radius γ of K_{α_k} in g_k-norm. Then α_k defines a regular foliation in $B^k_\gamma - K_{\alpha_k}$ such that for any point $x \in B^k_\gamma - K_{\alpha_k}$ it is satisfied that*
$$\angle_M(\ker \alpha_k, J \ker \alpha_k) = O(k^{-1/2}),$$
i.e., the leaves in $B^k_\gamma - K_{\alpha_k}$ are asymptotically holomorphic.
iii. *$(\alpha_k)_{1,0}$ is ε-transverse to zero as a section of $T^{1,0}M^* \otimes E_k$ over $M - B^k_\gamma$.*
iv. *For any point $x \in M - B^k_\gamma$, there is a uniform number $r > 0$ such that in the ball of g_k-radius r centered at x, the foliation can be written as $\alpha_k = h_k\, df_k$, for some trivialization of E_k, where h_k and f_k are asymptotically holomorphic and h_k is bounded above and below by a uniform constant (independent of k and x).*

Proposition 4.3. *Let α_k be a sequence of asymptotically holomorphic foliations with hermitian normal bundles E_k which are (γ, ε)-regular. Then there exists an arbitrarily small C^1-perturbation of α_k which is a symplectic foliation for k large enough.*

Proof. For k large enough, α_k defines a symplectic foliation in B^k_γ as a simple corollary of the definition (see [12, section 3.2]). In the complementary $M - B^k_\gamma$ we study the set of bad points
$$B(\alpha_k) = \{x \in M - B^k_\gamma \mid |(\alpha_k)_{1,0}(x)| \leq |(\alpha_k)_{0,1}(x)|\}.$$
Notice that this is the set of points where the distribution is singular or is not symplectic. We will say that $x \in M - B^k_\gamma$ is a critical point of α_k if $(\alpha_k)_{1,0}(x) = 0$. We want to modify the foliation so that $B(\alpha_k)$ only consists of finitely many isolated critical points in which the foliation has a standard model of the form $z_1 dz_1 + \ldots + z_n dz_n$.

Using lemma 4.4 below we can take k large enough so that the set of bad points is included in a disjoint collection of balls of uniform size centered at the critical points. Then we perturb the foliation in a small neighborhood of the set of critical points, by using proposition 4.5, to obtain a new foliation with $B(\alpha_k)$ equal to the set of critical points and such that in a neighborhood of a critical point it has the form $\alpha_k = \sum z_i dz_i$. □

Lemma 4.4. *For k large enough the set $B(\alpha_k)$ is contained in a finite set of disjoint balls $B(x_j, c)$ of uniform g_k-radius c around the critical points x_j, such that the balls $B(x_j, 2c)$ are disjoint and contained in $M - B^k_{\gamma/2}$.*

Proof. Let us see first that the minimal g_k-distance between two critical points of α_k in $M - B^k_\gamma$ is bounded below by a uniform constant $c_0 > 0$. Suppose that x_1 is a critical point, i.e., $(\alpha_k)_{1,0}(x_1) = 0$. In a neighborhood of uniform radius, $|(\alpha_k)_{1,0}(x)| < \varepsilon$. The ε-transversality implies that $\nabla(\alpha_k)_{1,0}(x) : TM \to T^*M \otimes E_k$ has an inverse. If there is another critical point x_2 in such a ball, then there is a point y in the segment joining x_1 and x_2 such that $\nabla(\alpha_k)_{1,0}(y)(x_2 - x_1) = 0$, which is a contradiction.

Given a point $x \in B(\alpha_k)$ then the distance of x to the set of critical points is bounded above, for k large enough, by c, where $c > 0$ is an arbitrarily small uniform constant. For

k large enough $|(\alpha_k)_{1,0}(x)| \leq |(\alpha_k)_{0,1}(x)| \leq Ck^{-1/2}$. By the ε-transversality, $\nabla(\alpha_k)_{1,0}(x) : TM \to T^*M \otimes E_k$ has an inverse of norm bounded by ε^{-1}. The bounds in the second derivatives of α_k allows us to control the radius where the inverse function theorem applies (see [9, lemma 8]). Therefore there must be a zero of $(\alpha_k)_{1,0}$ in a neighborhood of some uniform radius c. Moreover this c may be made as small as we please just by increasing k. □

Proposition 4.5. *Let α_k be a sequence of asymptotically holomorphic foliations which is (γ, ε)-regular. Let $x_j \in M - B^k_\gamma$ be a critical point of α_k such that there are asymptotically holomorphic functions h_k and f_k with h_k bounded below and $\alpha_k = h_k\, df_k$ on a neighborhood of uniform g_k-radius $B(x_j, 2c)$. Then there exists an arbitrarily small C^1-perturbation of α_k supported in $B(x_j, 2c)$, which is a symplectic foliation in the annulus $B(x_j, 2c) - B(x_j, c)$ and of the form $z_1 dz_1 + \cdots + z_n dz_n$ in $B(x_j, c)$.*

Proof. Given a critical point $x_j \in M - B^k_\gamma$, i.e., $(\alpha_k)_{1,0}(x_j) = 0$, write $\alpha_k = h_k\, df_k$ as in the statement. Recall that from the proof of lemma 4.4, the constant c can be chosen arbitrarily small, just by increasing the first k satisfying the property. This implies that $\nabla\alpha_k(p)$ is "approximately constant" in that ball (by radial parallel transport). We can use an asymptotically holomorphic chart $\Phi_k : B_{\mathbb{C}^n}(0, 1) \to B_{g_k}(x_j, 2c)$ provided in [2, lemma 3] to trivialize the manifold at a neighborhood of x_j. Consider f_k, h_k then as functions of $z_1, \dots, z_n$. We may suppose without loss of generality that $f_k(x_j) = 0$ and that $h_k(x_j) = 1$.

Since $(\alpha_k)_{1,0}$ is ε-transverse at x_j and α_k is asymptotically holomorphic, then for k large enough

$$\partial(\alpha_k)_{1,0}(0) : TM \to T^*M \otimes E_k$$

has an inverse of norm bounded by $(\varepsilon')^{-1}$, i.e., it multiplies the length of vectors by an amount at least ε', for some ε' slightly smaller than ε. Since $\partial(\alpha_k)_{1,0}(0) = \partial\partial f_k(0)$ is the complex Hessian of f_k, we define

$$H = \frac{1}{2} \sum \frac{\partial^2 f_k}{\partial z_i \partial z_j}(0) z_i z_j.$$

Consider the following foliation in $B(x_j, 2c)$,

$$\tilde{\alpha}_k = h_k\, dH.$$

Here dH is a holomorphic foliation with respect to the standard complex structure J_0 on the ball. Since this asymptotically holomorphic chart satisfies that the Nihenjius tensor has norm $O(k^{-1/2}|z|)$, we have that $(\tilde{\alpha}_k)_{0,1} = O(k^{-1/2}|z|)$. We have that $f_k(0) = H(0) = 0$, $\partial f_k(0) = dH(0) = 0$ and $\partial\partial f_k(0) = \nabla\nabla H(0)$. Since both are asymptotically holomorphic we have that $|f_k - H| = O(|z|^3 + k^{-1/2}|z|)$. Analogously $|\alpha_k - \tilde{\alpha}_k| = O(|z|^2 + k^{-1/2})$.

Now let β be a bump function such that $\beta(x) = 1$ for $x \in B(x_j, c)$, $\beta(x) = 0$ for $x \notin B(x_j, 3c/2)$ and $|\nabla\beta| = O(c^{-1})$. Define the foliation in the whole of M,

$$\hat{\alpha}_k = h_k\, d\,(\beta f_k + (1 - \beta)H) = \beta\alpha_k + (1 - \beta)\tilde{\alpha}_k + \nabla\beta(f_k - H).$$

We want to prove that outside x_j,

$$|(\hat{\alpha}_k)_{1,0}| > |(\hat{\alpha}_k)_{0,1}|. \tag{2}$$

In $B(x_j, c)$, $\hat{\alpha}_k = \tilde{\alpha}_k = h_k\, dH = h_k(\sum a_{ij} z_i dz_j)$. The ε-transversality of $(\alpha_k)_{1,0}$ implies that all the eigenvalues of the symmetric matrix (a_{ij}) have norm bigger than ε'. Therefore $|\tilde{\alpha}_k| \geq \varepsilon'|z|/2$ (for c small). Also $|(\tilde{\alpha}_k)_{0,1}| \leq Ck^{-1/2}|z|$, for some constant C, so (2) holds in $B(x_j, c)$. On $B(x_j, 3c/2) - B(x_j, c)$ we have

$$|\hat{\alpha}_k - \tilde{\alpha}_k| \leq |\alpha_k - \tilde{\alpha}_k| + |\nabla\beta||f_k - H| = O(c^2 + k^{-1/2}).$$

In particular, $|(\hat{\alpha}_k)_{0,1}| = O(c^2 + k^{-1/2})$ and $|(\hat{\alpha}_k)_{1,0}| \geq \varepsilon' c/2 - O(c^2 + k^{-1/2})$. Taking c small (but uniformly on k) and then k large enough we have (2).

Finally take $\tilde{h}_k$ to be equal to h_k in $B(x_j, 2c) - B(x_j, 3c/2)$ and equal to 1 in $B(x_j, c)$. Then

$$\tilde{h}_k\, d\left(\beta f_k + (1-\beta)H\right)$$

satisfies the required properties and in $B(x_j, c)$ it is of the form $\sum a_{ij} z_i dz_j$. A suitable orthonormal change of coordinates transforms this into $\sum z_i dz_i$. □

Remark 4.6. *Note that the perturbed foliation in the proof above is of the form $\alpha_k = h_k\, df_k$ in $B(x_j, 2c)$, so the integral submanifolds are the level sets $f_k = \lambda$. In a small neighborhood of the singularity, the leaves of the foliation are of the form $\sum z_i^2 = \lambda$.*

5. Construction of asymptotically holomorphic foliations

Once introduced all the asymptotically holomorphic machinery, we are ready to perform our main construction of asymptotically holomorphic foliations, generalizing the ideas contained in [12]. Let (M, ω, J) be a $2n$-dimensional symplectic manifold of integer class with a fixed compatible almost complex structure. Let $L \to M$ be the hermitian line bundle with connection whose curvature is $-i\omega$.

Take any asymptotically holomorphic sequence of sections $s_k = (s_k^0, \ldots, s_k^d)$ of $\mathbb{C}^{d+1} \otimes L^{\otimes k}$ such that $\phi_k = \mathbb{P}(s_k) : M \to \mathbb{CP}^d$ is a sequence of asymptotically holomorphic embeddings with $\phi_k^* \mathcal{O}(1) = L^{\otimes k}$, whose existence is guaranteed by theorem 3.5.

Now fix a holomorphic foliation $\alpha \in H^0(T^*\mathbb{CP}^d \otimes \mathcal{O}(N))$ in $\mathbb{CP}^d$, such that the singular set S_α is a union of smooth complex submanifolds intersecting transversely. There are many examples of such foliations [3]. We want to study the restriction $\alpha_k = \phi_k^* \alpha$ of the foliation α to the sequences of embeddings ϕ_k and to prove that for suitable choice of embeddings we get asymptotically holomorphic foliations which are (γ, ε)-regular.

Proposition 5.1. *Let ϕ_k be an asymptotically sequence of embeddings of M into $\mathbb{CP}^d$ and let α be a foliation as above in the projective space. Then there is a C^2-close sequence of embeddings ψ_k which is γ-transverse to every submanifold of the singular set S_α. Then the induced foliation $\alpha_k = \psi_k^* \alpha$ in M is an asymptotically holomorphic foliation and satisfies conditions (i) and (ii) of definition 4.2 with $K_{\alpha_k} = \psi_k^{-1}(\psi_k(M) \cap S_\alpha)$.*

Proof. Let ϕ_k be an asymptotically sequence of embeddings of M into $\mathbb{CP}^d$. Write $S_\alpha = \cup S_i$, where $S_i \subset \mathbb{CP}^d$ are smooth complex submanifolds of $\mathbb{CP}^d$. We may apply theorem 3.7 to perturb ϕ_k to a C^2-close sequence of embeddings ψ_k which is γ-transverse to every submanifold S_i, for some uniform $\gamma > 0$. This implies that the submanifold $\psi_k(M)$ intersects S_i along an asymptotically holomorphic submanifold by [12, proposition 3.10].

Let $\alpha_k = \psi_k^*\alpha$ be the induced foliation in M. Then α_k is asymptotically holomorphic using the asymptotically holomorphic bounds of ψ_k and the holomorphicity of α. Note that $(\alpha_k)_{1,0} = \alpha \circ \partial\psi_k$ and $(\alpha_k)_{0,1} = \alpha \circ \bar{\partial}\psi_k$.

Now $K_{\alpha_k} = \psi_k^{-1}(\psi_k(M) \bigcap S_\alpha)$ is a finite union of asymptotically holomorphic submanifolds and it is included in S_{α_k}. The γ-transversality to S_i implies that for the points in a neighborhood of radius γ of S_i, the angle between the tangent space of $\psi_k(M)$ and the distribution D_{S_i} determined by S_i is bigger than γ. Now the regular leaves of the foliation α around S_i contain S_i in its closure, so that one may assume that $D_{S_i}(x) \subset \ker \alpha(x)$ in a neighborhood of S_i. We use a linear algebra result [12, proposition 3.5] that says that for U, V, W subspaces of a finite dimensional euclidean vector space with $V \subset W$ it is satisfied that $\angle_m(U, V) \leq \angle_m(U, W)$. Therefore

$$\angle_m(\ker \alpha(x), T_x\phi_k(M)) \geq \angle_m(D_{S_i}(x), T_x\psi_k(M)) \geq \gamma,$$

for any $x \in \psi^{-1}(B_\gamma(S_\alpha))$. This implies that the leaves are asymptotically holomorphic in some $B_{c_o\gamma}$, for a constant $c_o > 0$. This gives the sought property (maybe after multiplying γ by a suitable uniform constant). □

Theorem 5.2. *Let ϕ_k be a sequence of asymptotically holomorphic embeddings of M into $\mathbb{CP}^d$. Fix a holomorphic foliation $\alpha \in H^0(T^*\mathbb{CP}^d \otimes \mathcal{O}(N))$ in $\mathbb{CP}^d$ as above. Then there exists an arbitrarily C^2-close sequence of embeddings ψ_k such that $\alpha_k = \psi_k^*\alpha$ is an asymptotically holomorphic sequence of foliations of M with normal bundle $L^{\otimes Nk}$, which is (γ, ε)-regular for uniform $\gamma, \varepsilon > 0$.*

Moreover any two such embeddings ψ_k^i, $i = 0, 1$, induce isotopic foliations α_k^i, for k large enough.

Proof. Recall that $\phi_k = \mathbb{P}(s_k)$ for a γ-asymptotically holomorphic sequence of sections s_k of $L^{\otimes k} \otimes \mathbb{C}^{d+1}$ which is γ-projectizable. The property of ϕ_k being γ-asymptotically holomorphic is open in C^1-sense, so any small perturbation will still be $\gamma/2$-asymptotically holomorphic. Using proposition 5.1 we may assume that ϕ_k is already γ-transverse to (every submanifold in) S_α (reducing γ if necessary). The property of an asymptotically holomorphic embedding being γ-transverse to S_α is open in C^1-sense [12, definition 3.11], so any small perturbation will still be $\gamma/2$-transverse. Denote by B_γ^k the tubular neighborhood of radius γ of $K_{\alpha_k} = \phi_k^{-1}(\phi_k(M) \bigcap S_\alpha)$ in M. We need to perturb ϕ_k to a sequence of embeddings such that $(\alpha_k)_{1,0}$ is ε-transverse to zero in $M - B_{\gamma/2}^k$.

We define the following property for sequences of sections s_k which are $\gamma/2$-projectizable and such that $\phi_k = \mathbb{P}(s_k)$ is $\gamma/2$-asymptotically holomorphic and $\gamma/2$-transverse to S_α: s_k satisfies the property $\mathcal{P}(\varepsilon, x)$ if $(\phi_k^*\alpha)_{1,0}$ is ε-transverse as a section of $T^*M \otimes L^{\otimes Nk}$ at the point x or else $x \in B_{\gamma/2}^k$. This property is local and open in C^2-sense (for ε small).

We want to use the globalization lemma in [2, proposition 3] which states the following: Let s_k be asymptotically holomorphic sections of $E_k = L^{\otimes k} \otimes \mathbb{C}^{d+1}$. If we can obtain for any point $x \in M$ and any $\delta > 0$ an asymptotically holomorphic sequence of sections $\tau_{k,x}$ with Gaussian decay away from x in C^r-norm and $|\tau_{k,x}|_{C^r, g_k} < \delta$ such that $s_k + \tau_{k,x}$ satisfies the property $\mathcal{P}(\sigma, y)$ for all y in a ball of uniform radius $B_{g_k}(x, c)$, with $\sigma = c'\delta(\log(\delta^{-1}))^{-p}$, with c, c', p independent of k, then, given any $\delta > 0$, there exist, for all large enough k, asymptotically holomorphic sections σ_k of E_k such that $|s_k - \sigma_k|_{C^r, g_k} < \delta$ and the sections σ_k satisfy $\mathcal{P}(\eta, x)$ for all $x \in M$ with $\eta > 0$ independent of k.

The transversality of ϕ_k to S_α implies that $\phi_k(M - B^k_{\gamma/2}) \subset \mathbb{CP}^d - B_{c_o\gamma}(S_\alpha)$, for some uniform constant $c_o > 0$. Now fix a finite covering U_j of $\mathbb{CP}^d - B_{c_o\gamma}(S_\alpha)$ such that in each of the sets U_j one may write $\alpha = h_j\, df_j$ where h_j is a (holomorphic) integrating factor and f_j is a first integral.

Let $x \in M - B^k_{\gamma/2}$. We may choose c small enough so that $\phi_k(B_{g_k}(x, c)) \subset U_j$ for some j (since $|\nabla \phi_k|_{g_k} \leq C$). Also any small perturbation will still be inside the same open set. Define $f_k^j = f_j \circ \phi_k$ and $h_k^j = h_j \circ \phi_k$. Both are asymptotically holomorphic in the ball. Moreover $\alpha_k = h_k^j\, df_k^j$. The functions h_k^j are bounded above and below by fixed constants. Therefore checking transversality for $(\alpha_k)_{1,0}$ is equivalent to checking transversality for ∂f_k^j.

With a transformation of $U(d+1)$ in $\mathbb{C}^{d+1}$ we may suppose that $s_k(x) = (s_k^0(x), 0, \ldots, 0)$. As s_k is γ-projectizable and asymptotically holomorphic, we suppose that $|s_k^0| \geq \gamma/2$ on $B_{g_k}(x, c)$ (maybe reducing $c > 0$). By [2, lemma 2] there are asymptotically holomorphic sections $s_{k,x}^{\mathrm{ref}}$ of $L^{\otimes k}$ with Gaussian decay away from x and with $|s_{k,x}^{\mathrm{ref}}| \geq c_1$ on $B_{g_k}(x, c)$, for some uniform $c_1 > 0$.

We use the standard chart Ψ_0 in $\mathbb{CP}^d$ around $p = \phi_k(x) = [1 : 0 \cdots : 0]$. With respect to this trivialization the map ϕ_k is given locally as

$$\begin{aligned} \Psi_0 \circ \phi_k : B_{g_k}(x, c) &\to \mathbb{C}^d \\ y &\to \left(\frac{s_k^1(y)}{s_k^0(y)}, \ldots, \frac{s_k^d(y)}{s_k^0(y)} \right). \end{aligned}$$

Now we can suppose that $|\partial f_j(0)| > c_2$, for a universal constant c_2 since we are well away from S_α. Also we may suppose that $\partial f_j(0) = (0, 0, \ldots, 0, \frac{\partial f_j}{\partial w_d}(0))$. Therefore $\frac{\partial f_j}{\partial w_d}$ is big enough in a small neighborhood.

We trivialize M at a neighborhood of x by using the asymptotically holomorphic charts $\Phi_k : B_{\mathbb{C}^n}(0, 1) \to B_{g_k}(x, c)$ provided in [2, lemma 3]. We denote by f_k^j and h_k^j again the corresponding functions defined in a ball of $\mathbb{C}^n$, which are asymptotically holomorphic. We define the "approximately orthogonal basis" as in [2]

$$\mu_k^i = \partial \left(z_i \frac{s_{k,x}^{\mathrm{ref}}}{s_k^0} \right). \tag{3}$$

At x it is an orthogonal basis and all the forms are asymptotically holomorphic. We can use (3) to locally trivialize the cotangent bundle. In particular we may write

$$\frac{\partial f_k^j}{(\partial f_j / \partial w_d) \circ \phi_k} = t_1 \mu_k^1 + \cdots t_n \mu_k^n.$$

It is easy to check that $t : B_{\mathbb{C}^n}(0, 1) \to \mathbb{C}^n$ defined by $t = (t_1, \ldots, t_n)$ is asymptotically holomorphic. This is because $\{\mu_k^1, \cdots, \mu_k^n\}$ is close to be an orthogonal matrix, and so all the eigenvalues are bounded below and above by positive uniform constants. Moreover the amount of transversality of t and of ∂f_k^j are related by non-zero uniform constants. So we need only get transversality for t.

The main local result is Donaldson's theorem 12 in [9] stating that there exists $w = (w_1, \dots, w_n) \in \mathbb{C}^n$ with $|w| < \delta$ such that $t - w$ is σ-transverse to 0 over the ball $B_{\mathbb{C}^n}(0, \frac{9}{10})$, with $\sigma = \delta(\log(\delta^{-1}))^{-p}$, for a universal $p > 0$ and k large enough.

Now define the perturbation

$$\tau_{k,x} = (0, \dots, 0, -\sum w_i z_i s_{k,x}^{\mathrm{ref}}).$$

This is asymptotically holomorphic, with Gaussian decay away from x and norm less than δ. The asymptotically holomorphic sequence $\hat{s}_k = s_k + \tau_{k,x}$ has the corresponding $\hat{t} = t - w$, so the induced $\hat{\alpha}_k = \hat{\phi}_k^* \alpha$ satisfies that $(\hat{\alpha}_k)_{1,0}$ is $C'\sigma$-transverse over $B_{g_k}(x, c)$, where C' is again another uniform constant. This concludes the proof.

For the one-parameter case, let $\psi_k^i : M \to \mathbb{CP}^d$, $i = 0, 1$, be two asymptotically holomorphic sequences of embeddings with respect to two compatible almost complex structures J_i, which are (γ, ε)-regular, for some $\gamma, \varepsilon > 0$. Consider a one-parameter family of compatible almost complex structures J_t, $t \in [0, 1]$, interpolating between J_0 and J_1 and let s_k^t be J_t-asymptotically holomorphic sections of $L^{\otimes k} \otimes \mathbb{C}^{d+1}$ such that $\psi_k^0 = \mathbb{P}(s_k^0)$ and $\psi_k^1 = \mathbb{P}(s_k^1)$.

We initially perturb s_k^t using theorem 3.7 so that all $\psi_k^t = \mathbb{P}(s_k^t)$ are asymptotically holomorphic embeddings which are γ-transverse to S_α (reducing $\gamma > 0$ if necessary). Then the argument above works for one-parameter families of sections depending on $t \in [0, 1]$ since all the ingredients do (see [2, 9, 12]). This means that for a given $\delta > 0$, there exists, for large enough k, J_t-asymptotically holomorphic sections σ_k^t of $L^{\otimes k} \otimes \mathbb{C}^{d+1}$ such that $|s_k^t - \sigma_k^t| < \delta$ and $\mathbb{P}(\sigma_k^t)$ are $(\gamma/2, \eta)$-regular asymptotically holomorphic embeddings in $\mathbb{CP}^d$ for some uniform $\eta > 0$. Taking $\delta > 0$ very small, the linear segment $u s_k^0 + (1 - u)\sigma_k^0$, $u \in [0, 1]$, consists of sections inducing $(\gamma/2, \varepsilon/2)$-regular maps. This provides an isotopy ${s'}_k^t$ between s_k^0 and s_k^1. The foliations $\alpha_k^t = ({\psi'}_k^t)^* \alpha$, ${\psi'}_k^t = \mathbb{P}({s'}_k^t)$, $t \in [0, 1]$, provide an isotopy between the initial ones, as required. □

Remark 5.3. *The perturbation of the foliation carried out in section 4 can be done in a one-parameter family α_k^t, as long as we start with a one-parameter family of asymptotically holomorphic functions f_k^t and h_k^t to start with. Therefore for a family of (γ, ε)-regular asymptotically holomorphic foliations α_k^t, there exists a family of symplectic foliations $\hat{\alpha}_k^t$ interpolating between the perturbations $\hat{\alpha}_k^0$, $\hat{\alpha}_k^1$ of α_k^0, α_k^1 carried out in proposition 4.3. So the construction of the symplectic foliations is unique up to symplectic isotopy, for k large enough.*

Remark 5.4. *Suppose that (M, ω) is a symplectic manifold with $[\omega]/2\pi$ not an integer cohomology class in $H^2(M; \mathbb{R})$. Choose a compatible almost complex structure J. We may take a small perturbation ω' of ω which is still symplectic and compatible with J, such that $[\omega']/2\pi$ is a rational cohomology class. Therefore there is a positive integer M such that $M[\omega']/2\pi$ is an integer cohomology class.*

Applying the theorem above for $M\omega'$ we get asymptotically holomorphic foliations, and therefore symplectic foliations for (M, ω), with hermitian normal bundles $L^{\otimes Nk}$ where $c_1(L) = M[\omega']/2\pi$.

6. Examples

We can apply all the precedent constructions to any fixed foliation in the projective space. There is a large number of examples [3, 7]. We are going to compute explicitly some classical cases.

6.1. **Application to Lefschetz pencils.** First, we can recover Donaldson's result [9] on the existence of Lefschetz pencils. We need the following definition.

Definition 6.1. *A branched (p, q) Lefschetz pencil, with $p, q > 0$ relatively prime, over an oriented closed manifold M consists of the following set of data:*

i. *A codimension four smooth submanifold B.*
ii. *A map $f : M - B \to \mathbb{CP}^1$ which is a submersion outside a finite set of points Δ.*

Also the data fit in the following models

- *Given a point $x \in B$, there exists a neighborhood of x with oriented coordinates $(z_1, \dots, z_n)$ of M where the map f can be written as $f(z_1, \dots, z_n) = z_2^q/z_1^p$.*
- *Given a point $x \in \Delta$, there exists a neighborhood of x with oriented coordinates $(z_1, \dots, z_n)$ of M where the map f can be written as $f(z_1, \dots, z_n) = z_1^2 + \cdots + z_n^2$.*

A branched $(1, 1)$ Lefschetz pencil is called a simple Lefschetz pencil (or a Lefschetz pencil, for brevity). The main result in [9] is

Theorem 6.2. *Let (M, ω) be a symplectic manifold of integer class. There exists an integer $k_0 > 0$ such that for any $k > k_0$, M admits a Lefschetz pencil structure (f_k, B_k, Δ_k) where all the fibers of the map f_k are symplectic and Poincare dual to $k[\omega]/2\pi$ and B_k is also symplectic.*

Donaldson constructs two asymptotically holomorphic sections $s_k^1, s_k^2 \in L^{\otimes k}$ satisfying certain transversality properties. The map f_k is defined by $f_k = s_k^2/s_k^1$, and for k large, it satisfies the required properties. Moreover, the form $\alpha_k = s_k^1\, ds_k^2 - s_k^2\, ds_k^1$ is an asymptotically holomorphic symplectic foliation. This follows from the fact that α_k is just a rescaling of the differential of f_k which is obviously defining a symplectic foliation whenever it is well defined. The rescaling is performed in order to α_k be defined all over the manifold. Moreover the foliation is as well symplectic in B_k, and so it is symplectic all over M. We also can prove

Theorem 6.3. *Let (M, ω) be a symplectic manifold of integer class. There exists an integer $k_0 > 0$ such that for any $k > k_0$, M admits a branched (p, q) Lefschetz pencil structure (f_k, B_k, Δ_k) where all the fibers of the map f_k are symplectic and Poincare dual to $k(p + q)[\omega]/2\pi$, and B_k is also symplectic.*

Proof. Fix sections $s_1 \in H^0(T^*\mathbb{CP}^d \otimes H^{\otimes q})$ and $s_2 \in H^0(T^*\mathbb{CP}^d \otimes H^{\otimes p})$, where H is the hyperplane line bundle over $\mathbb{CP}^d$. Moreover, suppose that these sections are transverse to zero and $Z(s_1) \cap Z(s_2)$ is a transverse intersection. Therefore

$$f = \frac{s_2^{\otimes q}}{s_1^{\otimes p}}$$

defines a branched (p, q) Lefschetz pencil over $\mathbb{CP}^d$. Moreover

$$\alpha = q s_1\, ds_2 - p s_2\, ds_1 \in H^0(T^*\mathbb{CP}^d \otimes H^{\otimes(p+q)})$$

is a holomorphic foliation on $\mathbb{CP}^d$ satisfying the hypothesis required in section 5. Then by theorem 5.2, there exists an embedding ϕ_k of M in $\mathbb{CP}^d$ such that $\alpha_k = \phi_k^*\alpha$ is an asymptotically holomorphic and (γ, ε)-regular foliation, for uniform $\gamma, \varepsilon > 0$. The perturbation performed in proposition 4.3 takes place well away from the singular locus and changes $f_k = \phi_k^* f$ into an integrating function with a suitable form around the critical points. Therefore this perturbation may be done by perturbing either $s_k^1 = \phi_k^* s_1$ or $s_k^2 = \phi_k^* s_2$ (since one of them is non-zero). We obtain a symplectic foliation and the map $f_k = (s_k^2)^{\otimes q}/(s_k^1)^{\otimes p}$ defines a branched (p, q) Lefschetz pencil. □

6.2. **Deformations of Lefschetz pencils with non-trivial holonomy.** We may deform the Lefschetz pencils in the presence of fundamental group as in the algebraic case. We say that a symplectic foliation α on a symplectic manifold (M, ω) is a deformed Lefschetz pencil if there is a connected smooth codimension four symplectic submanifold $B \subset M$ such that

- i. Given a point $x \in B$, there are adapted coordinates $(z_1, \dots, z_n)$ around x where the leaves of the foliation are of the form $z_2/z_1 = \lambda$.
- ii. There is a finite set of critical points $x_j \in M - B$ such that at any x_j there are adapted coordinates $(z_1, \dots, z_n)$ where the leaves of the foliation are of the form $z_1^2 + \cdots + z_n^2 = \lambda$.

Suppose α is a deformed Lefschetz pencil with base locus B. The holonomy $H : \pi_1(B) \to \mathrm{PU}(2)$ is defined as follows. Fix $p_0 \in B$ and consider a small transversal 2-dimensional disk Δ to B. Identify $\mathbb{P}(\Delta) = \mathbb{CP}^1$. For any loop ς and any $\lambda \in \mathbb{CP}^1$, lift the path ς to a path in a tubular neighborhood of B inside the leaf of α corresponding to the value λ. The endpoint is defined to be $H(\varsigma)(\lambda) \in \mathbb{CP}^1$.

Theorem 6.4. *Let (M, ω) be a symplectic manifold of integer class such that $\dim M = 2n \geq 6$ and $\pi_1(M) \neq 1$. Let $L \to M$ be the complex line bundle with $c_1(L) = [\omega]/2\pi$. Then for k large enough there are deformed Lefschetz pencils $\alpha_k \in \mathcal{C}^\infty(T^*M \otimes L^{\otimes k})$ with non-trivial holonomy.*

Proof. Let $\rho : \pi_1(M) \to SU(2)$ be a representation and $E_\rho \to M$ the corresponding flat $\mathbb{C}^2$-bundle. Suppose that ρ is a small deformation of the trivial representation, so that E_ρ is a topologically trivial bundle. We may understand $E_\rho = M \times \mathbb{C}^2$ with a flat connection $\nabla_\rho = \nabla + \varpi$, where $\varpi \in \Omega^1(\mathfrak{su}(2))$. Now $|\nabla_\rho - \nabla|_{g_k} = |\varpi|_{g_k} = k^{-1/2}|\varpi|_g$, i.e., ∇_ρ is "asymptotically trivial" connection.

Endow M with a compatible almost complex structure and let $L \to M$ be the hermitian line bundle with connection with curvature $-i\omega$. As in the previous section, there are asymptotically holomorphic sections $s_k = (s_k^1, s_k^2)$ of $\mathbb{C}^2 \otimes L^{\otimes k}$ such that $f_k = s_k^2/s_k^1$ is a symplectic Lefschetz pencil for k large enough. Let α_k be the associated foliation with base locus $B_k = Z(s_k)$. Consider the morphism

$$M - B_k \xrightarrow{(1, f_k)} M \times \mathbb{CP}^1 \cong \mathbb{P}(E_\rho).$$

We pull back the flat distribution of $\mathbb{P}(E_\rho)$ under this map to get a foliation α_k' in M.

Let $K_{\alpha_k} = B_k$ be the Kupka set of α_k and let B_γ^k be the neighborhood of g_k-radius γ of K_{α_k}. Then in $M - B_\gamma^k$ the 1-form $(\alpha_k)_{1,0}$ is ε-transverse to zero. On the other hand, the horizontal distribution of $M \times \mathbb{CP}^1$ is given by $\alpha = d\lambda$, where λ is the coordinate in the $\mathbb{CP}^1$-direction, and the horizontal distribution for $\mathbb{P}(E_\rho)$ is given by $\alpha' = d\lambda + O(k^{-1/2})$.

Since $\alpha_k = (1, f_k)^*\alpha$ and $\alpha'_k = (1, f_k)^*\alpha'$ (up to a factor $(s^1_k)^{\otimes 2}$, which is uniformly bounded) we have that $|\alpha_k - \alpha'_k|_{C^1, g_k} = O(k^{-1/2})$. So α'_k is an asymptotically holomorphic foliation such that $(\alpha'_k)_{1,0}$ is $\varepsilon/2$-transverse to zero in $M - B^k_\gamma$. After the perturbation in proposition 4.3, it defines a symplectic foliation.

Now we look at B^k_γ, where f_k has transversal type $z_1 dz_2 - z_2 dz_1$. Let us see that this is stable for small perturbations. We construct E_ρ in a different way: take the universal covering space $\pi : \tilde{M} \to M$ and consider $\tilde{M} \times \mathbb{C}^2$ with the trivial connection. Identify (x, f) with $(h(x), \rho(h)f)$ for any deck transformation h, to get $E_\rho \to M$. Consider the identification of E_ρ with the trivial bundle as a (not connection preserving) isomorphism $\psi : \underline{\mathbb{C}}^2 \to E_\rho$. This lifts to $\tilde{\psi} : \tilde{M} \times \mathbb{C}^2 \to \tilde{M} \times \mathbb{C}^2$, where $\tilde{\psi}^*\nabla = \nabla + \tilde{\varpi}$, $\tilde{\varpi} = \pi^*\varpi$. Therefore $\tilde{\varpi} = \tilde{\psi}^{-1} d\tilde{\psi}$ and we may consider $\tilde{\psi}$ as a map $\tilde{M} \to \mathrm{SU}(2)$ satisfying $\tilde{\psi}(h(x)) = \rho(h)\tilde{\psi}(x)$. Look at the map

$$\tilde{M} - \tilde{B}_k \xrightarrow{(1, \tilde{f}_k)} \tilde{M} \times \mathbb{CP}^1 \overset{\tilde{\psi}}{\cong} \tilde{M} \times \mathbb{CP}^1,$$

where $\tilde{f}_k = f_k \circ \pi$ and $\tilde{B}_k = \pi^{-1}(B_k)$. Fix a point $p \in B_k$ and $(z_1, z_2, \ldots, z_n)$ coordinates around p such that $K_{\alpha_k} = \{z_1 = z_2 = 0\}$ and $f_k = z_2/z_1$. Looking at any point in $\tilde{M}$ over p we see that the leaves of α'_k are the level sets of $\tilde{\psi} \circ \tilde{f}_k$. Denoting

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \tilde{\psi}(z_1, \ldots, z_n) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

we have new adapted coordinates $(w_1, w_2, z_3, \ldots, z_n)$ such that the leaves of α'_k are of the form $w_2/w_1 = \lambda$. Note that the change of coordinates is asymptotically holomorphic, since $|d\tilde{\psi}|_{C^1} = O(k^{-1/2})$.

We have a symplectic foliation α'_k with $K_{\alpha_k} = B_k$ a symplectic smooth submanifold of codimension 4. Let us see that this foliation has non-trivial holonomy (and therefore it is not a Lefschetz pencil). By the Lefschetz theorem in [1], B_k is connected and $\pi_1(B_k) \twoheadrightarrow \pi_1(M)$ for k large enough. Therefore ρ defines a non-trivial representation of $\pi_1(B_k)$ which determines $\mathbb{P}(E_\rho)|_{B_k}$. Fix $p \in B_k$ and Δ a small transversal 2-dimensional disk to B_k at p. Identify $\mathbb{P}(\Delta) = \mathbb{CP}^1$. Let $\varsigma \in \pi_1(B_k)$ be a loop and $\lambda \in \mathbb{CP}^1$. We move ς into a regular leaf starting at the leaf determined by λ at p. Then looking at the picture in $\tilde{M}$ we see that the end-point is $\rho(\varsigma)(\lambda)$. Note that if $\rho(\pi_1(M))\lambda \subset \mathbb{CP}^1$ is infinite then the leaf corresponding to λ is not compact. Moreover if ρ is not the identity in PU(2), then (the closures of) the leaves are not smooth at B_k and therefore α'_k does not define a Lefschetz pencil. □

6.3. **Application to asymptotically holomorphic logarithmic foliations.** Another useful example of foliations in the projective space are the logarithmic foliations as discussed in subsection 2.2.

Definition 6.5. *Let M be a closed manifold, $L_1, \ldots, L_p$ a family of complex line bundles over M and $\lambda_1, \ldots, \lambda_p \in \mathbb{C}$ complex numbers such that $\sum \lambda_i c_1(L_i) = 0 \in H^2(M; \mathbb{C})$. Choose sections $f_1, \ldots, f_p$ of the bundles $L_1, \ldots, L_p$. Then a logarithmic foliation with normal bundle $L = L_1 \otimes \cdots \otimes L_p$ is given by the twisted 1-form*

$$\alpha = f_1 \cdots f_p \sum_{i=1}^{p} \lambda_i \frac{df_i}{f_i} \in \mathcal{C}^\infty(M, T^*M \otimes L).$$

The condition above ensures that in the open subset $\{f_1 \neq 0, \dots, f_p \neq 0\}$ we have $\alpha = f_1 \cdots f_p \, d(\log F)$, where $F = f_1^{\lambda_1} \cdots f_p^{\lambda_p}$ is a function. It is easy to see that this gives a well defined L-valued 1-form all over M. Now consider the manifold and the sections to be holomorphic. We say that the foliation is generic if:

i. $p \geq 3$ and for any $i = 1, \dots, p$, the line bundle L_i is positive.
ii. For any $i = 1, \dots, p$, the hypersurface defined by the equation $\{f_i = 0\}$ is irreducible and $\{f_1 \cdots f_p = 0\}$ is a divisor with normal crossings.

In the particular case of the projective space we have that $f_i \in H^0(\mathbb{CP}^d, \mathcal{O}(n_i))$ is a homogeneous polynomial of degree n_i. The condition $\sum \lambda_i c_1(L_i) = 0$ translates into

$$\sum_{i=1}^{p} n_i \lambda_i = 0. \tag{4}$$

With this kind of foliations at hand we can prove

Theorem 6.6. *Let (M, ω) be a symplectic manifold of integer class and let $L \to M$ be a complex line bundle with $c_1(L) = [\omega]/2\pi$. Given $(n_1, \dots, n_p)$ positive integers and $(\lambda_1, \dots, \lambda_p)$ satisfying the condition (4), then for k large enough there exists an asymptotically holomorphic sequence of sections $(f_k^1, \dots, f_k^p)$ of $L^{\otimes kn_1} \oplus \cdots \oplus L^{\otimes kn_p}$ such that the associated logarithmic foliation*

$$\alpha_k = f_k^1 \cdots f_k^p \sum_{i=1}^{p} \lambda_i \frac{df_k^i}{f_k^i} \in \mathcal{C}^\infty(M, T^*M \otimes L^{\otimes kN}),$$

where $N = n_1 + \cdots + n_p$, is a symplectic foliation.

Proof. Choose a generic logarithmic foliation α in the projective space with polynomials $(f_1, \dots, f_p)$ of degrees $(n_1, \dots, n_p)$ and complex numbers $(\lambda_1, \dots, \lambda_p)$ as in the statement. Then the foliation satisfies the conditions of theorem 5.2, so there is a family of asymptotically holomorphic embeddings in the projective space $\phi_k : M \to \mathbb{CP}^d$ such that the pull-back foliations $\phi_k^* \alpha$ give an asymptotically holomorphic sequence of foliations which are (γ, ε)-regular. The perturbation of proposition 4.3 moves the integrating factor $\log\left((f_k^1)^{\lambda_1} \cdots (f_k^p)^{\lambda_p}\right)$, where $f_k^i = \phi_k^* f_i$. When the critical point is well away from every $D_k^i = Z(f_k^i)$ this perturbation can be absorbed into a perturbation of some f_k^i. To avoid that the critical points get close to D_k^i, just take the embeddings ϕ_k to be transverse to every $D_i = \{f_i = 0\} \subset \mathbb{CP}^d$ by using theorem 3.7. This produces logarithmic symplectic foliations for k large enough. □

Remark that any element of this family of foliations is not equivalent to a Lefschetz pencil. In particular, they are not of Kupka type.

References

[1] D. Auroux. *Asymptotically holomorphic families of symplectic submanifolds*. Geom. Funct. Anal. **7**, 971–995 (1997).
[2] D. Auroux. *Symplectic 4-manifolds as branched coverings of* $\mathbb{CP}^2$. Invent. Math. **139**, 551–602 (2000).
[3] O. Calvo. *Irreducible components in the spaces of holomorpihc foliations*. Matematische Annalen **299**, 751–767 (1994).
[4] O. Calvo. *Deformation of the holonomy of a Lefschetz pencil*. Qual. Theory Dyn. Syst. **1**, 231–245 (2000).

[5] D. Cerveau, J. F. Mattei. *Formes intégrables holomorphes singulières.* Astérisque **97** (1982).
[6] D. Cerveau, A. Lins Neto. *Codimension one foliations in* $\mathbb{CP}^n$, $n \geq 3$, *with Kupka components.* Complex Analytic Methods in Dynamical Systems. Eds C. Camacho, A. Lins, R. Mossou, P. Sad. Astérisque **222**, 93–133 (1994).
[7] D. Cerveau, A. Lins Neto. *Irreducible components of the space of holomorphic foliations of degree two in* $\mathbb{CP}(n)$, $n \geq 3$. Ann. Math. **143**, 577–612 (1996).
[8] S. K. Donaldson. *Symplectic submanifolds and almost-complex geometry.* J. Diff. Geom. **44**, 666–705 (1996).
[9] S. K. Donaldson. *Lefschetz pencils on symplectic manifolds.* J. Diff. Geom. **53**, 205–236 (1999).
[10] X. Gómez-Mont X, A. Lins Neto. *Structural Stability of foliations with a meromorphic first integral.* Topology **30**, 315–334 (1991).
[11] D. McDuff, D. Salamon. *Introduction to symplectic topology.* Oxford Math. Monographs (1994).
[12] V. Muñoz, F. Presas, I. Sols. *Almost holomorphic embeddings in grassmanians with applications to singular symplectic submanifolds.* To appear in J. reine angew. Math.
[13] E. Paul. *Etude topologique des formes logarithmiques fermées.* Inventiones Math. **95**, 395–420 (1989).

Cimat: Ap. Postal 402, Guanajuato, Gto. 36000 México
E-mail address: omegar@fractal.cimat.mx

Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain
E-mail address: vicente.munoz@uam.es

Mathematics Department. Bldg. 380, Stanford University, Stanford, CA 90304, Spain
E-mail address: fpresas@math.stanford.edu