SEIBERG-WITTEN-FLOER HOMOLOGY OF A SURFACE TIMES A CIRCLE FOR NON-TORSION SPIN$^{\mathbb{C}}$ STRUCTURES

VICENTE MUÑOZ AND BAI-LING WANG

Abstract. We determine the Seiberg-Witten-Floer homology groups of the 3-manifold $\Sigma \times \mathbb{S}^1$, where Σ is a surface of genus $g \geq 2$, together with its ring structure, for a Spin$^{\mathbb{C}}$ structure with non-vanishing first Chern class. We give applications to computing Seiberg-Witten invariants of 4-manifolds which are connected sums along surfaces and also we reprove the higher type adjunction inequalities obtained by Oszváth and Szabó.

1. Introduction

In this paper we study the gluing theory for Seiberg-Witten invariants of 4-manifolds split along the 3-manifold $Y = \Sigma \times \mathbb{S}^1$, where Σ is a surface of genus $g \geq 2$. This produces applications to the determination of the Seiberg-Witten invariants of 4-manifolds which are constructed as connected sums of other 4-manifolds along embedded surfaces, and to obtain constrains for the Seiberg-Witten invariants of 4-manifolds containing an embedded surface of genus g and non-negative self-intersection. The seminal work in this direction is provided by [12] leading to a proof of the generalized Thom conjecture. Analysis of this kind on non-trivial circle bundles over surfaces appears in [14] [22].

Before stating the results, we set up some notation. Let X be a closed, connected, oriented smooth 4-manifold with $b^+ > 0$ and a fixed homology orientation (i.e. an orientation of $H^1(X;\mathbb{R}) \oplus H^{2+}(X;\mathbb{R})$). For a Spin$^{\mathbb{C}}$ structure $\mathfrak{s}$, the Seiberg-Witten invariant [27] [23] [13] [25] [26] is a linear functional

$$SW_{X,\mathfrak{s}} : \mathbb{A}(X) \to \mathbb{Z},$$

where $\mathbb{A}(X) = \mathrm{Sym}^* H_0(X) \otimes \Lambda^* H_1(X)$, the free graded algebra generated by the class of the point $x \in H_0(X)$ and the 1-cycles $\gamma \in H_1(X)$ (we understand rational coefficients). We grade $\mathbb{A}(X)$ by declaring the degree of x to be 2 and the degree of the elements in $H_1(X)$ to be 1. The invariants are constructed by endowing X with a metric g and studying the moduli space $\mathcal{M}_X(\mathfrak{s})$ of solutions (A, Φ) modulo gauge to the Seiberg-Witten equations

$$(1) \qquad \begin{cases} \rho((F_A - \sqrt{-1}\,\xi)^+) = (\Phi\Phi^*)_0 \\ D_A \Phi = 0, \end{cases}$$

Date: April, 1999. Revised May, 2002.
Key words: 4-manifolds, Seiberg-Witten invariants, Seiberg-Witten-Floer homology.
Mathematics Subject Classification. Primary: 57R57. Secondary: 57N13.

where Φ is a section of the positive spin bundle W^+ of $\mathfrak{s}$, A is a connection on the determinant line bundle $L = \det W^{\pm}$, $D_A : \Gamma(W^+) \to \Gamma(W^-)$ is the Dirac operator twisted with the connection A, ρ denotes Clifford multiplication, $(\Phi\Phi^*)_0$ is the trace-free part of $(\Phi\Phi^*)$ interpreted as an endomorphism of W^+, ξ is a (small) closed real two-form introduced as a perturbation.

Note that the invariants are zero on elements whose degree is not $d(\mathfrak{s})$ where

$$d(\mathfrak{s}) = \frac{c_1(\mathfrak{s})^2 - (2\chi(X) + 3\sigma(X))}{4}$$

is the dimension of $\mathcal{M}_X(\mathfrak{s})$. When $b^+ > 1$ the Seiberg-Witten invariants are independent of metrics and perturbations. When $b^+ = 1$ the Seiberg-Witten invariants depend on a chamber structure. Fix a component $\mathcal{K}_0$ of the positive cone $\mathcal{K}(X) = \{x \in H^2(X;\mathbb{R}) - \{0\}/x^2 \geq 0\}$. Then we say that the Seiberg-Witten invariants are computed in $\mathcal{K}_0$ when the metric g and perturbation ξ satisfy $(c_1(\mathfrak{s}) + \frac{1}{2\pi}[\xi]) \cdot \omega_g < 0$, where $\omega_g \in H^2(X;\mathbb{Z})$ is a period point for the metric g lying in $\mathcal{K}_0$.

A basic class for X is a $\mathrm{Spin}^{\mathbb{C}}$ structure with non-zero Seiberg-Witten invariant. By analogy with the definitions of simple type in the context of Donaldson invariants [18, introduction], we give the following

Definition 1.1. Let X be a 4-manifold with $b^+ > 1$. We say that

- X is of *simple type* if $SW_{X,\mathfrak{s}}(z) = 0$ for any z in the ideal generated by x in $\mathbb{A}(X)$, for any $\mathfrak{s}$.
- X is of *H_1-simple type* if $SW_{X,\mathfrak{s}}(z) = 0$ for any z in the ideal of $\mathbb{A}(X)$ generated by $H_1(X)$, for any $\mathfrak{s}$.
- X is of *strong simple type* if it is both of simple type and of H_1-simple type, i.e. $SW_{X,\mathfrak{s}}(z) = 0$ whenever $\deg(z) > 0$, for any $\mathfrak{s}$.

Note that when X has $b_1 = 0$ it is automatically of H_1-simple type. There are manifolds not of H_1-simple type (for instance any manifold which is a connected sum $X \# \mathbb{S}^1 \times \mathbb{S}^3$, where X has $b^+ > 1$, see [21, proposition 2.2]), but it is an open question whether all 4-manifolds with $b^+ > 1$ are of simple type.

Now we are ready to state our main result. On the one hand, we have applications to computing the Seiberg-Witten invariants of connected sums along surfaces (see [15]). We prove the following results in section 5.

Theorem 1.2. *Let $\bar{X}_1$ and $\bar{X}_2$ be 4-manifolds with embedded surfaces $\Sigma \hookrightarrow \bar{X}_i$, $i = 1, 2$, of the same genus $g \geq 2$, self-intersection zero and representing non-torsion homology classes, and let $X = \bar{X}_1 \#_\Sigma \bar{X}_2$ be their connected sum along Σ. Suppose that $\bar{X}_1$, $\bar{X}_2$ are both of strong simple type, and let $\mathfrak{s}_1$, $\mathfrak{s}_2$ be $\mathrm{Spin}^{\mathbb{C}}$ structures on $\bar{X}_1$, $\bar{X}_2$ respectively, such that $c_1(\mathfrak{s}_1) \cdot \Sigma = c_1(\mathfrak{s}_2) \cdot \Sigma \neq 0$ and $d(\mathfrak{s}_1) = d(\mathfrak{s}_2) = 0$. Let $\mathfrak{s}_o$ be a $\mathrm{Spin}^{\mathbb{C}}$ structure on X obtained* *by gluing* $\mathfrak{s}_1$ *and* $\mathfrak{s}_2$ *and let* $z \in \mathbb{A}(X)$ *such that* $d(\mathfrak{s}_o) = \deg z$. *Then*

$$\sum_{h\in\mathcal{R}im} SW_{X,\mathfrak{s}_o+h}(z) = \begin{cases} SW_{\bar{X}_1,\mathfrak{s}_1}(1)\cdot SW_{\bar{X}_2,\mathfrak{s}_2}(1) & z=1,\ |c_1(\mathfrak{s})\cdot\Sigma| = 2g-2 \\ 0 & z=1,\ |c_1(\mathfrak{s})\cdot\Sigma| < 2g-2 \\ 0 & \deg(z)>0 \end{cases}$$

where $\mathcal{R}im \subset H^2(X;\mathbb{Z})$ *is the subspace generated by the rim tori (cf.* [6]*). If any of the manifolds has* $b^+ = 1$*, then the Seiberg-Witten invariants are computed for the component of the positive cone containing* $\varepsilon P.D.[\Sigma]$*, where* $\varepsilon = 1$ *if* $c_1(\mathfrak{s}_o)\cdot\Sigma < 0$ *and* $\varepsilon = -1$ *if* $c_1(\mathfrak{s}_o)\cdot\Sigma > 0$.

Moreover if the connected sum is admissible (see definition 5.3), then there are no basic classes $\mathfrak{s}$ *of* X *such that* $0 < |c_1(\mathfrak{s})\cdot\Sigma| < 2g-2$*, and the basic classes of* X *with* $c_1(\mathfrak{s})\cdot\Sigma = \pm(2g-2)$ *are in bijection with pairs of basic classes* $(\mathfrak{s}_1,\mathfrak{s}_2)$ *of* X_1 *and* X_2 *respectively, such that* $c_1(\mathfrak{s}_1)\cdot\Sigma = c_1(\mathfrak{s}_2)\cdot\Sigma = \pm(2g-2)$.

This theorem is analogous to [15, corollary 13] and [15, corollary 15] in the case of Donaldson invariants. It is generalised to the following analogue of [18, theorem 9.5],

Theorem 1.3. *Let* $\bar{X}_1$ *and* $\bar{X}_2$ *be* 4*-manifolds with embedded surfaces* $\Sigma \hookrightarrow \bar{X}_i$, $i = 1, 2$*, of the same genus* $g \geq 2$*, self-intersection zero and representing non-torsion homology classes, and let* $X = \bar{X}_1 \#_\Sigma \bar{X}_2$ *be their connected sum along* Σ*. Suppose that* $\bar{X}_1$*,* $\bar{X}_2$ *are of* H_1*-simple type, and let* $\mathfrak{s}_1$*,* $\mathfrak{s}_2$ *be Spin*$^{\mathbb{C}}$ *structures on* $\bar{X}_1$*,* $\bar{X}_2$ *respectively, such that* $c_1(\mathfrak{s}_1)\cdot\Sigma = c_1(\mathfrak{s}_2)\cdot\Sigma \neq 0$*. Let* $\mathfrak{s}_o$ *be a Spin*$^{\mathbb{C}}$ *structure on* X *obtained by gluing* $\mathfrak{s}_1$ *and* $\mathfrak{s}_2$ *and let* $z \in \mathbb{A}(X)$ *such that* $d(\mathfrak{s}_o) = \deg z$*. Then*

$$\sum_{h\in\mathcal{R}im} SW_{X,\mathfrak{s}_o+h}(z) = 0$$

if $\deg(z) > 0$ *or if* $c_1(\mathfrak{s})\cdot\Sigma \not\equiv 2g-2 \pmod 4$*. If* X *has* $b^+ = 1$ *then the Seiberg-Witten invariants are computed for the component of the positive cone containing* $\varepsilon P.D.[\Sigma]$*, where* $\varepsilon = 1$ *if* $c_1(\mathfrak{s}_o)\cdot\Sigma < 0$ *and* $\varepsilon = -1$ *if* $c_1(\mathfrak{s}_o)\cdot\Sigma > 0$.

Moreover if the connected sum is admissible then the basic classes $\mathfrak{s}$ *of* X *with* $c_1(\mathfrak{s})\cdot\Sigma \neq 0$ *all satisfy* $c_1(\mathfrak{s})\cdot\Sigma \equiv 2g-2 \pmod 4$.

The restriction $c_1(\mathfrak{s})\cdot\Sigma \neq 0$ in theorems 1.2 and 1.3 is due to the fact that the gluing theory for Seiberg-Witten invariants used here only works for non-torsion Spin$^{\mathbb{C}}$ structures.

On the other hand, our analysis also leads to a different proof of the higher type adjunction inequalities first obtained by Oszváth and Szabó in [21]. Our method of proof is more transparent and parallels that of [19] for proving the higher type adjunction inequalities for Donaldson invariants. Section 6 is devoted to this issue.

Theorem 1.4. ([21, theorem 1.4]) *Let* X *be a* 4*-manifold and let* $\Sigma \subset X$ *be an embedded surface of genus* $g \geq 2$ *representing a non-torsion homology class with self-intersection*

$\Sigma^2 \geq 0$. Let $a \in \mathbb{A}(X)$ and $b \in \mathbb{A}(\Sigma)$. If X has $b^+ > 1$ and $\mathfrak{s}$ is a Spin$^{\mathbb{C}}$ structure with $SW_{X,\mathfrak{s}}(a\,b) \neq 0$ and $|c_1(\mathfrak{s}) \cdot \Sigma| + \Sigma^2 > 0$ then we have

$$|c_1(\mathfrak{s}) \cdot \Sigma| + \Sigma^2 + \deg(b) \leq 2g - 2.$$

Furthermore, when $b^+ = 1$ then for each Spin$^{\mathbb{C}}$ structure $\mathfrak{s}$ on X with $-c_1(\mathfrak{s}) \cdot \Sigma + \Sigma^2 > 0$, for which $SW_{X,\mathfrak{s}}(ab) \neq 0$, when calculated in the component of $\mathcal{K}(X)$ containing P.D.$[\Sigma]$, we have an inequality

$$-c_1(\mathfrak{s}) \cdot \Sigma + \Sigma^2 + \deg(b) \leq 2g - 2.$$

Theorem 1.5. ([21, theorem 1.1]) *Let X be a 4-manifold of H_1-simple type (e.g. with $b_1 = 0$) and let $\Sigma \subset X$ be an embedded surface of genus $g \geq 2$ representing a non-torsion homology class with $\Sigma^2 \geq 0$. If X has $b^+ > 1$ then for each basic class $\mathfrak{s}$ for X with $|c_1(\mathfrak{s}) \cdot \Sigma| + \Sigma^2 > 0$ we have*

$$|c_1(\mathfrak{s}) \cdot \Sigma| + \Sigma^2 + 2d(\mathfrak{s}) \leq 2g - 2.$$

If $b^+ = 1$ then for each basic class $\mathfrak{s}$ for the Seiberg-Witten invariants of X calculated in the component of $\mathcal{K}(X)$ which contains P.D.$[\Sigma]$ such that $-c_1(\mathfrak{s}) \cdot \Sigma + \Sigma^2 > 0$, we have an inequality

$$-c_1(\mathfrak{s}) \cdot \Sigma + \Sigma^2 + 2d(\mathfrak{s}) \leq 2g - 2.$$

□

This is a particular case of the more general result

Theorem 1.6. ([21, theorem 1.3]) *Let X be a 4-manifold with an embedded surface $\imath : \Sigma \hookrightarrow X$ of genus $g \geq 2$ representing a non-torsion homology class with $\Sigma^2 \geq 0$. Let l be an integer so that there is a symplectic basis $\{\gamma_i\}_{i=1}^{2g}$ of $H_1(\Sigma)$ with $\gamma_i \cdot \gamma_{g+i} = 1$, $1 \leq i \leq g$, satisfying that $\imath_*(\gamma_i) = 0$ in $H_1(X)$ for $i = 1, \ldots, l$. Let $a \in \mathbb{A}(X)$ and $b \in \mathbb{A}(\Sigma)$ be an element of degree $\deg(b) \leq l + 1$. If X has $b^+ > 1$ and $\mathfrak{s}$ is a Spin$^{\mathbb{C}}$ structure such that $SW_{X,\mathfrak{s}}(a\,b) \neq 0$ and $|c_1(\mathfrak{s}) \cdot \Sigma| + \Sigma^2 > 0$ then we have*

$$|c_1(\mathfrak{s}) \cdot \Sigma| + \Sigma^2 + 2\deg(b) \leq 2g - 2.$$

Furthermore, when $b^+ = 1$ then for each Spin$^{\mathbb{C}}$ structure $\mathfrak{s}$ on X with $-c_1(\mathfrak{s}) \cdot \Sigma + \Sigma^2 > 0$, for which $SW_{X,\mathfrak{s}}(a\,b) \neq 0$, when calculated in the component of $\mathcal{K}(X)$ containing P.D.$[\Sigma]$, we have

$$-c_1(\mathfrak{s}) \cdot \Sigma + \Sigma^2 + 2\deg(b) \leq 2g - 2.$$

Again the restriction $|c_1(\mathfrak{s}) \cdot \Sigma| + \Sigma^2 > 0$ is due to the fact that we only use the Seiberg-Witten-Floer theory for non-torsion Spin$^{\mathbb{C}}$ structures (see section 2).

In order to prove these results, we determine completely the structure of the Seiberg-Witten-Floer (co)homology of the three manifold $Y = \Sigma \times \mathbb{S}^1$, where Σ is a closed surface of genus $g \geq 2$, for any Spin$^{\mathbb{C}}$ structure with non-zero first Chern class. We use the Seiberg-Witten-Floer groups as defined in [4], since they satisfy a gluing theorem for Seiberg-Witten invariants. We prove

Theorem 1.7. *Let $\mathfrak{s}_Y$ be a Spin$^{\mathbb{C}}$ structure on $Y = \Sigma \times \mathbb{S}^1$ with $c_1(\mathfrak{s}_Y) \neq 0$. If $c_1(\mathfrak{s}_Y) \neq 2rP.D.[\mathbb{S}^1]$, with $-(g-1) \leq r \leq g-1$ then $HFSW^*(Y, \mathfrak{s}_Y) = 0$. Let $\mathfrak{s}_r$ be the Spin$^{\mathbb{C}}$ structure on Y with $c_1(\mathfrak{s}_r) = 2rP.D.[\mathbb{S}^1]$, $-(g-1) \leq r \leq g-1$, $r \neq 0$, and set $d = g-1-|r|$. Then there is an isomorphism of vector spaces $HFSW^*(Y, \mathfrak{s}_r) \cong H^*(s^d\Sigma)$, where $s^d\Sigma$ is the d-th symmetric product of Σ.* □

Theorem 1.7 will follow from theorem 3.2 and proposition 3.4. The Seiberg-Witten-Floer homology of $Y = \Sigma \times \mathbb{S}^1$ has a natural ring structure coming from the cobordism which is a pair of pants times Σ (cf. [5]). This should be closely related (if not isomorphic) to the quantum cohomology of the symmetric products of Σ (see [1] for a partial computation of the latter), in the same way as the instanton Floer cohomology of $\Sigma \times \mathbb{S}^1$ is isomorphic to the quantum cohomology of the moduli space of rank 2 odd degree stable vector bundles over Σ (see [17]). We fully compute the ring structure of $HFSW^*(Y, \mathfrak{s}_r)$, which is a deformation of the ring structure on $H^*(s^{g-1-|r|}\Sigma)$.

Theorem 1.8. *Let $-(g-1) \leq r \leq g-1$, $r \geq 0$, and set $d = g-1-|r|$. Then there is a presentation*

$$HFSW^*(\Sigma \times \mathbb{S}^1, \mathfrak{s}_r) = \bigoplus_{k=0}^{d} \Lambda_0^k \otimes \frac{\mathbb{Q}[\eta, \theta]}{(\tilde{\mathcal{R}}_k^g, \theta\tilde{\mathcal{R}}_{k+1}^g, \eta^{d+1}, \theta^{d+1})},$$

where $\theta \in \Lambda^2 H_1(\Sigma)$ represents the intersection form in $H^1(\Sigma)$ and

$$\Lambda_0^k = \Lambda_0^k H_1(\Sigma) = \ker\left(\theta^{g-k+1} : \Lambda^k H_1(\Sigma) \to \Lambda^{2g-k+2} H_1(\Sigma)\right)$$

is the primitive part. The polynomials are defined as

$$\tilde{\mathcal{R}}_k^g = \sum_{i=0}^{\alpha} \frac{\binom{d-k-\alpha+1}{i}}{i!\binom{g-k}{i}} (-1)^i \eta^{\alpha-i}\theta^i - \sum_{i=0}^{\alpha+|r|} \frac{\binom{\alpha+|r|}{i}}{i!\binom{g-k}{i}} \eta^{\alpha+|r|-i}\theta^i,$$

where $\alpha = [\frac{d-k}{2}] + 1$, $0 \leq k \leq d$, and $\tilde{\mathcal{R}}_{d+1}^g = 1$. □

This theorem will follow from theorem 4.9.

2. Review of Seiberg-Witten-Floer homology for non-torsion Spin$^{\mathbb{C}}$ structures

Let Y be an oriented 3-manifold with first Betti number $b_1 > 0$ and a Spin$^{\mathbb{C}}$ structure $\mathfrak{s}_Y$ on Y with $c_1(\mathfrak{s}_Y)$ non-torsion. We are going to review the construction of the Seiberg-Witten-Floer (co)homology groups $HFSW_*(Y, \mathfrak{s}_Y)$ of Y from [4].

2.1. **Definition of $HFSW(Y, \mathfrak{s}_Y)$.** We endow Y with a metric g_Y and fix a base connection A_0 on the determinant line bundle $L_Y = \det W_Y$, where W_Y stands for the spin bundle. There is a Chern-Simons Seiberg-Witten functional (taking values in $\mathbb{R}/\mathbb{Z}$) defined on the configuration space of gauge classes $[A, \psi]$ of a connection A on L_Y and a section ψ of W_Y,

$$\mathcal{C}_\eta(A, \psi) = -\frac{1}{2}\left(\int_Y (A - A_0) \wedge (F_A + F_{A_0} - 2 *_Y \sqrt{-1}\eta) + \langle \psi, D_A\psi \rangle dvol_Y\right),$$

where D_A stands for the Dirac operator on W_Y coupled with A, η is a (perturbative) real coexact one-form on Y. The critical points of $\mathcal{C}_\eta$ correspond to translation invariant solutions to (1) on the tube $Z = Y \times \mathbb{R}$ for the perturbation $\xi = *\eta$. The equations we are led to are

$$\left\{ \begin{array}{l} *F_A = q(\psi) + \eta \\ D_A\psi = 0, \end{array} \right. \tag{2}$$

where $q(\psi)$ is the standard quadratic form. When $c_1(\mathfrak{s}_Y)$ is non-torsion we can choose a generic perturbation parameter η to get a finite collection of non-degenerate irreducible (oriented) points [4]. Denoted by $\mathcal{M}_Y(\mathfrak{s}_Y, \eta)$ be the set of solutions to the perturbed Seiberg-Witten equations (2) on $(Y, \mathfrak{s}_Y)$. There is also a well-defined relative index on $\mathcal{M}_Y(\mathfrak{s}_Y, \eta)$, taking values in $\mathbb{Z}/N\mathbb{Z}$, where

$$N = \text{GCD } \{\langle c_1(\mathfrak{s}_Y), \sigma\rangle \,|\, \sigma \in H_2(Y, \mathbb{Z})\}.$$

The Seiberg-Witten-Floer chain complex $CFSW_*(Y)$ is the vector space generated by the gauge classes of solutions to (2) with the relative grading.

The downward flow equation of $\mathcal{C}_\eta$ is the 4-dimensional Seiberg-Witten equation on $(Y \times \mathbb{R}, g_Y + dt^2)$ with the pull-back $\text{Spin}^{\mathbb{C}}$ structure under the temporal gauge:

$$\left\{ \begin{array}{l} \dfrac{\partial A(t)}{\partial t} = -*F_A + q(\psi) + \eta \\ \dfrac{\partial \psi(t)}{\partial t} = -D_A\psi. \end{array} \right. \tag{3}$$

In general, in order to achieve the transversality property for the moduli space of the Seiberg-Witten solutions on $Y \times \mathbb{R}$ to (3) without destroying the $\mathbb{R}$-translation action and obtaining a natural compactification, we have to choose a suitable perturbation of $\mathcal{C}_\eta$ supported away from the set of critical points. See [4] for a detailed discussion. (Similar kind of perturbations were first constructed in [7].)

Let a and b in $\mathcal{M}_Y(\mathfrak{s}_Y, \eta)$ be two gauge classes of irreducible solutions to (2). For generic perturbations as in [4], any connected component of the moduli space $\hat{\mathcal{M}}(a, b)$, the gauge classes of solutions to the perturbed Seiberg-Witten equations on the tube $Y \times \mathbb{R}$ with limits a and b respectively, is smooth, orientable and admits a free $\mathbb{R}$-action. We shall denote by $\mathcal{M}^D(a, b)$ the components of dimension D in the quotient space of $\hat{\mathcal{M}}(a, b)$ by the $\mathbb{R}$-action.

Note that $D \equiv \mathrm{ind}(a,b) - 1 \pmod N$. We define a boundary map

$$\begin{aligned} \partial : CFSW_i(Y) &\rightarrow CFSW_{i-1}(Y) \\ a &\mapsto \sum_{\substack{b \in \mathcal{M}(Y,\mathfrak{s}_Y) \\ \mathrm{ind}(a,b)=1 \pmod N}} \#\mathcal{M}^0(a,b)\, b. \end{aligned}$$

The compactifications of the moduli space of the trajectory flow lines ensure that ∂ is well-defined and $\partial^2 = 0$ [4] so we obtain the Seiberg-Witten-Floer cohomology, denoted by $HFSW^*(Y,\mathfrak{s}_Y)$, which is $\mathbb{Z}_N$-graded abelian group.

Notice that the first Chern class of the non-torsion $\mathrm{Spin}^{\mathbb{C}}$ structure $\mathfrak{s}_Y$ defines a homomorphism $c_1(\mathfrak{s}_Y) : H^1(Y,\mathbb{Z}) \to \mathbb{Z}$ by $c_1(\mathfrak{s}_Y)([u]) = \langle [u] \cup c_1(\mathfrak{s}_Y), [Y] \rangle$ for any element $[u] \in H^1(Y,\mathbb{Z})$. For any subgroup $K \subset \ker(c_1(\mathfrak{s}_Y))$, there is a subgroup $\mathcal{G}_Y^K$ of the full gauge transformation group $\mathcal{G}_Y$, whose elements lie in the connected components determined by K (since the group of connected components of $\mathcal{G}_Y$ is $H^1(Y,\mathbb{Z})$). Consider the Chern-Simons-Dirac functional on the configuration space $\mathcal{A}_Y$ modulo the gauge group $\mathcal{G}_Y^K$. The critical point set, denoted by $\mathcal{M}_{Y,K}(\mathfrak{s}_Y,\eta)$, is a covering space

$$\pi_K : \mathcal{M}_{Y,K}(\mathfrak{s}_Y,\eta) \longrightarrow \mathcal{M}_Y(\mathfrak{s}_Y,\eta),$$

whose fiber is an $H^1(Y,\mathbb{Z})/K$-homogeneous space.

There is a variant of the Seiberg-Witten-Floer chain complex whose generators are elements in $\mathcal{M}_{Y,K}(\mathfrak{s}_Y,\eta)$ with relative $\mathbb{Z}$-graded indices and boundary operator ∂^K given by counting the gradient flow lines of the perturted Chern-Simons-Dirac functional on $\mathcal{A}_Y/\mathcal{G}_Y^K$ connecting two critical points with relative index 1. We denote the Seiberg-Witten-Floer homology in this setting by $HFSW_{*,[K]}(Y,\mathfrak{s}_Y)$.

The following theorem was established in [4] regarding various properties of these Seiberg-Witten-Floer homologies of $(Y,\mathfrak{s}_Y)$.

Theorem 2.1. *(Theorem 1.1* [4]*) For any closed oriented 3-manifold Y with $b_1(Y) > 0$ and a $\mathrm{Spin}^{\mathbb{C}}$ structure $\mathfrak{s}_Y$ such that $c_1(\mathfrak{s}_Y) \neq 0$ in $H^2(Y,\mathbb{Z})/torsion$, let N be the divisibility of $c_1(\mathfrak{s}_Y)$ in $H^2(Y,\mathbb{Z})/torsion$, that is*

$$N = GCD\ \{\langle c_1(\mathfrak{s}_Y), \sigma \rangle \,|\, \sigma \in H_2(Y,\mathbb{Z})\}.$$

Then there exists a finitely generated Seiberg-Witten-Floer complex whose homology $HFSW_*(Y,\mathfrak{s}_Y)$ satisfies the following properties:

(a) *$HFSW_*(Y,\mathfrak{s}_Y)$ is a topological invariant of $(Y,\mathfrak{s}_Y)$ and it is a $\mathbb{Z}_N$-graded abelian group.*
(b) *There is an action of*

$$\mathbb{A}(Y) = Sym^*(H_0(Y,\mathbb{Z})) \otimes \Lambda^*\big(H_1(Y,\mathbb{Z})/torsion\big)$$

on $HFSW_*(Y,\mathfrak{s}_Y)$ with elements in $H_0(Y,\mathbb{Z})$ and $H_1(Y,\mathbb{Z})/torsion$ decreasing degree in $HFSW_*(Y,\mathfrak{s}_Y)$ by 2 and 1 respectively.

(c) *For $(-Y,-\mathfrak{s}_Y)$, where $-Y$ is Y with the reversed orientation and $-\mathfrak{s}_Y$ is the induced Spin$^{\mathbb{C}}$ structure, the corresponding Seiberg-Witten-Floer complex $C_*(-Y,-\mathfrak{s}_Y)$ is the dual complex of $C_*(Y,\mathfrak{s}_Y)$. There is a natural pairing*

$$\langle\ ,\ \rangle : HFSW_*(Y,\mathfrak{s}_Y)\times HFSW_{-*}(-Y,-\mathfrak{s}_Y)\longrightarrow \mathbb{Z}$$

such that $< z\cdot\Xi_1,\Xi_2 >=< \Xi_1, z\cdot\Xi_2 >$ for any $z\in\mathbb{A}(Y)\cong\mathbb{A}(-Y)$ and any cycles $\Xi_1\in HFSW_*(Y,\mathfrak{s}_Y)$ and $\Xi_2\in HFSW_{-*}(-Y,-\mathfrak{s}_Y)$ respectively.

(d) *For any subgroup $K\subset\ker(c_1(\mathfrak{s}_Y))\subset H^1(Y,\mathbb{Z})$, there is a variant of Seiberg-Witten-Floer homology denoted by $HFSW_{*,[K]}(Y,\mathfrak{s}_Y)$, which is a topological invariant and a $\mathbb{Z}$-graded $\mathbb{A}(Y)$ module. For any $[u]\in H^1(Y,\mathbb{Z})/K$, there is an action of $[u]$ on $HFSW_{*,[K]}(Y,\mathfrak{s}_Y)$ decreasing degrees by $\langle [u]\wedge c_1(\mathfrak{s}_Y),[Y]\rangle$. There is natural pairing*

$$\langle\ ,\ \rangle : HFSW_{*,[K]}(Y,\mathfrak{s}_Y)\times HFSW_{-*,[K]}(-Y,-\mathfrak{s}_Y)\longrightarrow \mathbb{Z}$$

satisfying $< z\cdot\Xi_1,\Xi_2 >=< \Xi_1, z\cdot\Xi_2 >$ for any $z\in\mathbb{A}(Y)\cong\mathbb{A}(-Y)$ and any cycles $\Xi_1\in HFSW_{*,[K]}(Y,\mathfrak{s}_Y)$ and $\Xi_2\in HFSW_{-*,[K]}(-Y,-\mathfrak{s}_Y)$ respectively. There is a $\mathbb{A}(Y)$-equivariant homomorphism:

$$\pi_K : HFSW_{*,[K]}(Y,\mathfrak{s}_Y)\longrightarrow HFSW_*(Y,\mathfrak{s}_Y).$$

If $K_1\subset K_2$ are two subgroups in $\ker(c_1(\mathfrak{s}_Y))$, there is a $\mathbb{A}(Y)$-equivariant homomorphism $HFSW_{*,[K_1]}(Y,\mathfrak{s}_Y)\to HFSW_{*,[K_2]}(Y,\mathfrak{s}_Y)$. Moreover, for any $m\in\mathbb{Z}$,

$$\pi_{\ker(c_1(\mathfrak{s}_Y))} : HFSW_{m,[\ker(c_1(\mathfrak{s}_Y))]}(Y,\mathfrak{s}_Y)\cong HFSW_{m \pmod N}(Y,\mathfrak{s}_Y).$$

2.2. **Relative Seiberg-Witten invariants and gluing formula.** Let X_1 be an oriented, connected 4-manifold furnished with a cylindrical end of the form $Y\times[0,\infty)$. Suppose we have a Spin$^{\mathbb{C}}$ structure $\mathfrak{s}_1$ over X_1 whose restriction to Y is a non-torsion Spin$^{\mathbb{C}}$ structure $\mathfrak{s}_Y$. Consider finite energy solutions to the Seiberg-Witten equations on X_1 with finite variations of the perturbed Chern-Simons-Dirac functional on the end, there is an associated boundary asymptotic value map

$$\partial_\infty : \mathcal{M}_{X_1}(\mathfrak{s}_1)\to\mathcal{M}_{Y,X_1}(\mathfrak{s}_Y,\eta)$$

where $\mathcal{M}_{Y,X_1}(\mathfrak{s}_Y,\eta)$ is the quotient of solutions to the perturbed Seiberg-Witten equations on $(Y,\mathfrak{s}_Y)$ by the action of those gauge transformations which can be extended to X_1. In fact, $\pi_1 : \mathcal{M}_{Y,X_1}(\mathfrak{s}_Y,\eta)\to\mathcal{M}_Y(\mathfrak{s}_Y,\eta)$ is a covering map with fiber an $H^1(Y,\mathbb{Z})/\mathrm{im}(i_1^*)$-homogeneous space. Here $\mathrm{im}(i_1^*)\subset\ker(c_1(\mathfrak{s}_Y))$ is the image of the map $i_1^* : H^1(X_1,\mathbb{Z})\to H^1(Y,\mathbb{Z})$ induced from the boundary embedding map i_1. Generically, the fiber of ∂_∞ is an oriented, smooth manifold of dimension given by Atiyah-Patodi-Singer index theorem, and it can be compactified to a smooth manifold with corners. See [4] for the detailed discussion.

The relative Seiberg-Witten invariant of $(X_1,\mathfrak{s}_1)$, as defined in [4], takes values in the Seiberg-Witten-Floer homology $HFSW_{*,[\mathrm{im}(i_1^*)]}(Y,\mathfrak{s}_Y)$, and defines an $\mathbb{A}(Y)$-equivariant linear map

$$\phi^{SW}_{X_1}(\mathfrak{s}_1,\cdot) : \mathbb{A}(X_1)\longrightarrow HFSW_{*,[\mathrm{im}(i_1^*)]}(Y,\mathfrak{s}_Y).$$

Here the $\mathbb{A}(Y)$-action on $\mathbb{A}(X_1)$ is induced from the homomorphism $(i_1)_* : \mathbb{A}(Y) \to \mathbb{A}(X_1)$. For any $z_1 \in \mathbb{A}(X_1)$ of degree d, $\phi^{SW}_{X_1}(\mathfrak{s}_1, z_1)$ can be expressed in terms of the Seiberg-Witten invariant from the components of dimension d in $\mathcal{M}_{X_1}(\mathfrak{s}_1)$.

For a second 4-manifold X_2 with a cylindrical end $(-Y) \times [0, \infty)$, we construct $X = X_1 \cup_Y X_2$ by cutting the ends and gluing along the common boundary Y. The resulting manifold may depend on the isotopy class of the diffeomorphism identifying the boundaries, but we shall not make the dependence explicit. If there is a $\mathrm{Spin}^{\mathbb{C}}$ structure $\mathfrak{s}_2$ on X_2 with $\mathfrak{s}_2|_Y \cong \mathfrak{s}_Y$, then we can glue $\mathfrak{s}_1$ and $\mathfrak{s}_2$. The indeterminacy for the gluing is parametrised by $\mathrm{coker}(H^1(X_1;\mathbb{Z}) \oplus H^1(X_2;\mathbb{Z}) \to H^1(Y;\mathbb{Z}))$. The following gluing formula is taken from [4].

Theorem 2.2 (Theorem 1.2 [4]). *Let X be a closed manifold with $b^+ \geq 1$ which is written as $X = X_1 \cup_Y X_2$, where X_1 and X_2 are 4-manifolds with boundary and $\partial X_1 = -\partial X_2 = Y$. Suppose that we have $Spin^{\mathbb{C}}$ structures $\mathfrak{s}_1$ and $\mathfrak{s}_2$ on X_1 and X_2 respectively such that $\mathfrak{s}_1|_Y \cong \mathfrak{s}_2|_Y \cong \mathfrak{s}_Y$ is a non-torsion $Spin^{\mathbb{C}}$ structure on Y. Then for any $Spin^{\mathbb{C}}$ structure $\mathfrak{s} = \mathfrak{s}_1 \#_{[u]} \mathfrak{s}_2$ obtained by gluing $\mathfrak{s}_1$ and $\mathfrak{s}_2$ along Y using an isomorphism $u \in Map(Y, U(1))$ representing $[u] \in coker(H^1(X_1;\mathbb{Z}) \oplus H^1(X_2;\mathbb{Z}) \to H^1(Y;\mathbb{Z}))$, we have the following gluing formula for $z_i \in \mathbb{A}(X_i)$, $i = 1, 2$,*

$$SW_{X,\mathfrak{s}}(z_1 z_2) = \langle [u](\pi_1(\phi^{SW}_{X_1}(\mathfrak{s}_1, z_1))), \pi_2(\phi^{SW}_{X_2}(\mathfrak{s}_2, z_2)) \rangle.$$

Here $[u]$ acts on $HFSW_{*,[im(i_1^*)+im(i_2^*)]}(Y, \mathfrak{s}_Y)$, π_1 and π_2 are the $\mathbb{A}(Y)$-equivariant homomorphisms induced from the inclusion maps $im(i_1^*) \subset (im(i_1^*) + im(i_2^*))$ and $im(i_2^*) \subset (im(i_1^*) + im(i_2^*))$ respectively, and the pairing on the right hand side is the natural pairing

$$HFSW_{*,[im(i_1^*)+im(i_2^*)]}(Y, \mathfrak{s}_Y) \times HFSW_{*,[im(i_1^*)+im(i_2^*)]}(-Y, -\mathfrak{s}_Y),$$

with the degrees in $HFSW_{*,[im(i_1^*)+im(i_2^*)]}(-Y, -\mathfrak{s}_Y)$ shifted by $\deg(z_1) + \deg(z_2)$. When $b^+ = 1$, the Seiberg-Witten invariants correspond to a metric giving a long neck. In particular, let $\mathcal{S}$ be the set of $Spin^{\mathbb{C}}$ structures on X with $\mathfrak{s}|_{X_i} = \mathfrak{s}_i$, $i = 1, 2$ and $\frac{1}{4}(c_1(\mathfrak{s})^2 - 2(\chi(X) + \sigma(X))) = \deg(z_1) + \deg(z_2)$, then

$$\sum_{\mathfrak{s} \in \mathcal{S}} SW_{X,\mathfrak{s}}(z_1 z_2) = \langle \pi(\phi^{SW}_{X_1}(\mathfrak{s}_1, z_1)), \pi(\phi^{SW}_{X_2}(\mathfrak{s}_2, z_2)) \rangle.$$

Here $\pi(\phi^{SW}_{X_i}(\mathfrak{s}_i, z_i))$ are elements in $HFSW_*(\pm Y, \pm \mathfrak{s}_Y)$ under the maps

$$\pi : HFSW_{*,[im(i_1^*)+im(i_2^*)]}(\pm Y, \pm \mathfrak{s}_Y) \to HFSW_*(\pm Y, \pm \mathfrak{s}_Y)$$

and the pairing on the right hand side is the pairing $HFSW_*(Y, \mathfrak{s}_Y) \times HFSW_*(-Y, -\mathfrak{s}_Y) \to \mathbb{Z}$.

3. Seiberg-Witten-Floer homology of $\Sigma \times \mathbb{S}^1$

From now on we shall consider the three-manifold $Y = \Sigma \times \mathbb{S}^1$, which is the central object of our study. As $H^2(Y;\mathbb{Z})$ has no 2-torsion, the $\mathrm{Spin}^{\mathbb{C}}$ structures $\mathfrak{s}_Y$ on Y are determined by the determinant line bundle $L_Y = c_1(\mathfrak{s}_Y)$. As $c_1(\mathfrak{s}_Y)$ reduces to $w_2(Y) = 0$ modulo 2, it has to be an even class in $H^2(Y;\mathbb{Z})$.

Proposition 3.1. *Let $\mathfrak{s}_Y$ be a Spin$^{\mathbb{C}}$ structure on Y. Let $\mathcal{M}$ be the moduli space of solutions to* (2) *with zero perturbation. Then $\mathcal{M}$ is empty unless $c_1(\mathfrak{s}_Y) = 2rP.D.[\mathbb{S}^1]$, with $-(g-1) \leq r \leq g-1$. For $c_1(\mathfrak{s}_Y) = 2rP.D.[\mathbb{S}^1]$, with $-(g-1) \leq r \leq g-1$ and $r \neq 0$, $\mathcal{M}$ is Morse-Bott irreducible and isomorphic to $s^d\Sigma$ with $d = g-1-|r|$.*

Proof. We choose a rotation invariant metric for Y of the form $g_\Sigma + d\theta \otimes d\theta$, where g_Σ is a metric on Σ with unit area and scalar curvature $-4\pi(2g-2)$, and θ is the coordinate on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. Think of $\Sigma \times \mathbb{S}^1$ as $\Sigma \times [0,1]$ with the boundaries $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$ identified by the identity. The line bundle L_Y is constructed out of the pull back under the projection $\Sigma \times [0,1] \to \Sigma$ of a line bundle L_Σ on Σ by gluing along the boundaries with an isomorphism $\sigma \in \mathcal{G}_\Sigma = \mathrm{Map}(\Sigma, \mathbb{S}^1)$. Then $c_1(L_Y) = c_1(L_\Sigma) + [\sigma] \otimes [\mathbb{S}^1]$, where $[\sigma]$ is the class of σ in $[\Sigma; \mathbb{S}^1] \cong H^1(\Sigma; \mathbb{Z})$. The Spin$^{\mathbb{C}}$ structure $\mathfrak{s}_Y$ induces a Spin$^{\mathbb{C}}$ structure $\mathfrak{s}_\Sigma$ on Σ with determinant line bundle L_Σ. The spin bundle of $\mathfrak{s}_\Sigma$ is of the form $W_\Sigma = (\Lambda^0 \oplus \Lambda^{0,1}) \otimes \mu$, for a line bundle μ such that $L_\Sigma = K_\Sigma^{-1} \otimes \mu^2$, where K_Σ stands for the canonical bundle of Σ.

Now consider any solution (A, ψ) to (2) with $\eta = 0$. In $\Sigma \times [0,1]$, we can kill, by a gauge transformation, the $d\theta$ component of A, i.e. we can suppose that we have a family A_θ, $\theta \in [0,1]$, of connections on L_Σ (up to a constant gauge) with the boundary condition $A_1 = \sigma^*(A_0)$, for some $\sigma \in \mathcal{G}_\Sigma$ in the homotopy class determined by L_Y. So (A, ψ) is interpreted as a path $(A_\theta, \alpha_\theta, \beta_\theta)$, $\theta \in [0,1]$, where $\alpha_\theta \in \Lambda^0 \otimes \mu$ and $\beta_\theta \in \Lambda^{0,1} \otimes \mu$. Let us rewrite equations (2) in this set-up. Clearly $*F_A = \Lambda F_{A_\theta} d\theta + *_\Sigma(\frac{\partial A}{\partial \theta})$, and the map q and the Dirac operator are as follows

$$q(\psi) = -* \frac{\left(\alpha\overline{\beta} + \overline{\alpha}\beta\right)}{2} + \frac{|\alpha|^2 - |\beta|^2}{2} d\theta,$$

$$D_A = \begin{pmatrix} -\sqrt{-1}\frac{\partial}{\partial\theta} & \sqrt{2}\bar{\partial}_A^* \\ \sqrt{2}\bar{\partial}_A & -\sqrt{-1}\frac{\partial}{\partial\theta} \end{pmatrix} : (\Lambda^0 \otimes \mu) \oplus (\Lambda^{0,1} \otimes \mu) \to (\Lambda^0 \otimes \mu) \oplus (\Lambda^{0,1} \otimes \mu).$$

So the solutions to (2) correspond to solutions to

$$\left\{ \begin{array}{l} \frac{\partial\alpha}{\partial\theta} = -\sqrt{-1}\sqrt{2}\bar{\partial}_{A_\theta}^*\beta \\ \frac{\partial\beta}{\partial\theta} = \sqrt{-1}\sqrt{2}\bar{\partial}_{A_\theta}\alpha \\ 2\frac{\partial A_\theta}{\partial\theta} = -\sqrt{-1}(\alpha\overline{\beta} + \beta\overline{\alpha}) \\ 2\sqrt{-1}\Lambda F_{A_\theta} = -|\alpha|^2 + |\beta|^2 \end{array} \right. \tag{4}$$

We can write $A_\theta = \partial_{A_\theta} + \bar{\partial}_{A_\theta}$, so the third line is $\frac{\partial}{\partial\theta}(\partial_{A_\theta}) = -\frac{\sqrt{-1}}{2}\alpha\overline{\beta}$, $\frac{\partial}{\partial\theta}(\bar{\partial}_{A_\theta}) = -\frac{\sqrt{-1}}{2}\overline{\alpha}\beta$. Now suppose we have a solution to (4). Then we work out the following expression (using $\sqrt{-1}\bar{\partial}^* = \Lambda\partial$ on $(0,1)$-forms and $|\beta|^2 = -\sqrt{-1}\Lambda\beta \wedge \overline{\beta}$)

$$\frac{\partial}{\partial\theta}(\bar{\partial}^*\beta) = -\frac{\partial}{\partial\theta}(\sqrt{-1}\Lambda\partial)\beta + \bar{\partial}^*(\frac{\partial\beta}{\partial\theta}),$$

with the given equalities to get

$$-\frac{1}{\sqrt{2}\sqrt{-1}}\frac{\partial^2\alpha}{\partial\theta^2} = \sqrt{-1}\Lambda\frac{\sqrt{-1}}{2}\alpha\overline{\beta}\beta - \bar{\partial}^*(-\sqrt{-1}\sqrt{2}\bar{\partial}\alpha),$$

$$-\frac{\partial^2\alpha}{\partial\theta^2}+\frac{\sqrt{2}}{2}\alpha|\beta|^2+2\bar{\partial}^*\bar{\partial}\alpha=0.$$

Take scalar product with α and integrate along Σ by parts to get

$$-\int_\Sigma\langle\frac{\partial^2\alpha}{\partial\theta^2},\alpha\rangle+\frac{\sqrt{2}}{2}\int_\Sigma|\alpha|^2|\beta|^2+2\int_\Sigma|\bar{\partial}\alpha|^2=0,$$

for every $\theta\in[0,1]$. This equation makes sense in $\mathbb{S}^1$, since the values for $\theta=0$ and $\theta=1$ coincide. Then we can integrate again by parts to get

$$||\frac{\partial}{\partial\theta}\alpha||^2+\frac{\sqrt{2}}{2}||\alpha\beta||^2+2||\bar{\partial}\alpha||^2=0.$$

So either $\alpha=0$ or $\beta=0$. In any case, A_θ, α_θ and β_θ are constant, i.e. if the line bundle L_Y admits solutions to (2) then it is pulled-back from Σ and any solution is invariant under rotations in the $\mathbb{S}^1$ factor.

Assume now that $c_1(L_Y)=2r\text{P.D.}[\mathbb{S}^1]$. For any solution to (4), either $\alpha=0$, $\bar{\partial}^*_{A_0}\beta=0$ or $\beta=0$, $\bar{\partial}_{A_0}\alpha=0$. Also $2r=c_1(L_\Sigma)=\frac{\sqrt{-1}}{2\pi}\int_\Sigma F_A=\frac{1}{4\pi}\int_\Sigma(|\beta|^2-|\alpha|^2)$. If $r<0$ then $\beta=0$ and the solutions to equations (4) are equivalent to the solutions to

$$\begin{cases}\bar{\partial}_A\alpha=0\\ 2\sqrt{-1}\Lambda F_A=-|\alpha|^2\end{cases}$$

on Σ. These are the typical vortex equations. The space of solutions is $s^d\Sigma$, where $d=g-1+r$. If $r<-(g-1)$ then there are no solutions. The case $r>0$ is analogous. $\square$

Theorem 3.2. *Let $\mathfrak{s}_Y$ be a Spin$^{\mathbb{C}}$ structure on Y with $c_1(\mathfrak{s}_Y)\neq 0$. Then $HFSW^*(Y,\mathfrak{s}_Y)=0$ unless $c_1(\mathfrak{s}_Y)=2rP.D.[\mathbb{S}^1]$, with $-(g-1)\leq r\leq g-1$. Let $\mathfrak{s}_r$ be the Spin$^{\mathbb{C}}$ structure on Y with $c_1(\mathfrak{s}_r)=2rP.D.[\mathbb{S}^1]$, $-(g-1)\leq r\leq g-1$, $r\neq 0$. Put $d=g-1-|r|\geq 0$, then* $\dim HFSW^*(Y,\mathfrak{s}_r)\leq\dim H^*(s^d\Sigma)$.

Proof. The first claim is a direct consequence of proposition 3.1. Also from proposition 3.1, we know that the unperturbed Chern-Simons Seiberg-Witten functional already has non-degenerate critical manifolds. As in [8] and [2, proposition 6], we can choose a perturbation modelled on the finite dimensional critical manifold $s^d\Sigma$. Choose a positive and perfect Morse function f on Σ, i.e. f has one critical point of index 0, $2g$ critical points of index 1 and one critical point of index 2. For any point $(x_1,x_2,\cdots,x_d)\in\Sigma$, define $F(x_1,x_2,\cdots,x_d)=\prod_{i=1}^d f(x_i)$, then it is easy to check that F is a Morse function on $s^d\Sigma$. The critical points of F consist of those $(x_1,x_2,\cdots,x_d)$ where x_i is a critical point of f, and the Morse index of $(x_1,x_2,\cdots,x_d)$ is the sum of the Morse indices of the x_i's. Therefore, the number of the critical points of F with Morse index i is given by

$$\binom{2g}{i}+\binom{2g}{i-2}+\cdots+\binom{2g}{i-2[i/2]},$$

which is exactly the i-th Betti number of $s^d\Sigma$ (see [9]). Hence, $F(x_1,x_2,\cdots,x_d)$ is a perfect Morse function on $s^d\Sigma$. Then we can perturb the Chern-Simons Seiberg-Witten functional such that there exists a one-to-one correspondence between the perturbed Seiberg-Witten monopoles on $\Sigma \times \mathbb{S}^1$ and the critical points of F on $s^d\Sigma$. Both sets of critical points are non-degenerate and have the same relative indices modulo $2|r|$. This implies that $\dim HFSW^*(Y, \mathfrak{s}_r) \leq \dim H^*(s^d\Sigma)$. □

To shorten the notation, we shall write from now on

$$V_r = HFSW^*(Y, \mathfrak{s}_r), \tag{5}$$

for $-(g-1) \leq r \leq g-1$, $r \neq 0$. In this section we will study the finite dimensional vector spaces V_r for $r \neq 0$. They have a natural $\mathbb{Z}/2|r|\mathbb{Z}$-grading. The only tools we shall use are the bound on the dimension provided by theorem 3.2 and the gluing theorem 2.2. First, it is easily seen that the diffeomorphism $f \times c : \Sigma \times \mathbb{S}^1 \to \Sigma \times \mathbb{S}^1$, where $f : \Sigma \to \Sigma$ is an orientation reversing diffeomorphism and $c : \mathbb{S}^1 \to \mathbb{S}^1$ is the conjugation, induces an isomorphism $V_r \cong V_{-r}$. Henceforth we shall suppose $r > 0$ in (5).

Let $A = \Sigma \times D^2$ be the 4-manifold given as the product of Σ times a 2-dimensional disc, so that $\partial A = \Sigma \times \mathbb{S}^1$. Let $\Delta = \text{pt} \times D^2 \subset A$. The $\text{Spin}^{\mathbb{C}}$ structures on A are parametrised by $H^2(A;\mathbb{Z}) = H^2(\Sigma;\mathbb{Z}) \cong \mathbb{Z}$. We write $\mathfrak{s}_r$ for the $\text{Spin}^{\mathbb{C}}$ structure on A with $c_1(\mathfrak{s}_r) = 2r\text{P.D.}[\Delta]$ (we use the same name $\mathfrak{s}_r$ for $\text{Spin}^{\mathbb{C}}$ structures on Y and on A. No confusion should arise from this, as they are compatible in the sense that $\mathfrak{s}_r|_Y = \mathfrak{s}_r$). Note that $\ker(c_1(\mathfrak{s}_r)) = \operatorname{im}(H^1(A;\mathbb{Z}) \to H^1(Y;\mathbb{Z}))$ and $HFSW^*(Y, \mathfrak{s}_r) \cong HFSW^*_{[\ker(c_1(\mathfrak{s}_r))]}(Y, \mathfrak{s}_r)$, the relative Seiberg-Witten invariants of A give a map

$$\begin{array}{ccc} \mathbb{A}(\Sigma) & \to & V_r = HFSW^*(Y, \mathfrak{s}_r) \\ z & \mapsto & \phi_A^{SW}(\mathfrak{s}_r, z). \end{array} \tag{6}$$

As $S = A \cup_Y A = \Sigma \times \mathbb{S}^2$, the gluing theorem 2.2 yields

$$\sum_{n \in \mathbb{Z}} SW_{S, \mathfrak{s}_r + n[\Sigma]}(z_1 z_2) = \langle \phi_A^{SW}(\mathfrak{s}_r, z_1), \phi_A^{SW}(\mathfrak{s}_r, z_2) \rangle, \tag{7}$$

for any $z_1, z_2 \in \mathbb{A}(\Sigma)$, where the $\text{Spin}^{\mathbb{C}}$ structure $\mathfrak{s}_r$ on S is the one with $c_1(\mathfrak{s}_r) = 2r\text{P.D.}[\mathbb{S}^2]$. The metric that we must use for the Seiberg-Witten invariants in the left hand side of (7) is one giving a long neck, i.e. with period point ω_g close to $[\Sigma]$ in $\mathcal{K}_0 = \{a[\mathbb{S}^2] + b[\Sigma]/a, b > 0\}$. This implies that $c_1(\mathfrak{s}_r + n[\Sigma]) \cdot \omega_g > 0$ as $r > 0$, so the invariants are calculated in the component $-\mathcal{K}_0$ of the positive cone.

As $r \neq 0$, there is at most one $n \in \mathbb{Z}$ that contributes to the left hand side in (7), since $c_1(\mathfrak{s}_r + n[\Sigma]) = 2r[\mathbb{S}^2] + 2n[\Sigma]$ and

$$d(\mathfrak{s}_r + n[\Sigma]) = 2rn + 2(g-1).$$

It is thus important to know the Seiberg-Witten invariants of $S = \Sigma \times \mathbb{S}^2$ for the component $-\mathcal{K}_0$, which we describe now. We fix the homology orientation given by the usual orientation of $H^1(S) = H^1(\Sigma)$ and the orientation of $H^2_+(S) = \mathbb{R}\omega_g$ determined by $-\omega_g$.

Fix a symplectic basis $\{\gamma_1, \dots, \gamma_{2g}\}$ of $H_1(\Sigma;\mathbb{Z})$ with $\gamma_i \gamma_{i+g} = \text{pt}$, for $1 \leq i \leq g$. Then

$$\mathbb{A}(\Sigma) = \mathbb{Q}[x] \otimes \Lambda^*(\gamma_1, \dots, \gamma_{2g})$$

and there is an action of the mapping class group of Σ, $\pi_0(\text{Diff}(\Sigma))$, factoring through an action of the symplectic group $\text{Sp}\,(2g,\mathbb{Z})$, on both $\mathbb{A}(\Sigma)$ and V_r, making the map (6) equivariant. Define

$$\theta=\sum_{i=1}^{g}\gamma_i\gamma_{g+i}\in\Lambda^2H_1(\Sigma).$$

Then the invariant part $\mathbb{A}(\Sigma)_I$ of $\mathbb{A}(\Sigma)$ is generated by x and θ. We decompose $\mathbb{A}(\Sigma)$ in irreducible $\text{Sp}\,(2g,\mathbb{Z})$-representations as

$$\mathbb{A}(\Sigma)=\bigoplus_{k=0}^{g}\Lambda_0^k\otimes\frac{\mathbb{Q}[x,\theta]}{(\theta^{g+1-k})},$$

where

$$\Lambda_0^k=\Lambda_0^kH_1(\Sigma)=\ker(\theta^{g-k+1}:\Lambda^kH_1(\Sigma)\to\Lambda^{2g-k+2}H_1(\Sigma))$$

is the primitive component of $\Lambda^kH_1(\Sigma)$, for $0\le k\le g$. Then, as the Seiberg-Witten invariant $SW_{S,\mathfrak{s}}(z)$ is invariant under the action of $\text{Diff}(\Sigma)$, $SW_{S,\mathfrak{s}}(z)=0$ for any $z\in\bigoplus_{k=1}^{g}\Lambda_0^k\otimes\mathbb{Q}[x,\theta]/(\theta^{g+1-k})$, and it only matters to compute $SW_{S,\mathfrak{s}}(z)$ for $z=x^a\theta^b$.

Lemma 3.3. *Fix $0<r\le g-1$ and $n\in\mathbb{Z}$. Set $d=g-1-r$. Then $SW_{S,\mathfrak{s}_r+n[\Sigma]}$ is zero unless $n\le -1$ and $D=rn+g-1\ge 0$ (there is only a finite number of such n). In that case $SW_{S,\mathfrak{s}_r+n[\Sigma]}(x^a\theta^b)=\frac{g!}{(g-b)!}(-n)^{g-b}$, for $a+b=D$, $0\le b\le g$. Note that $D\le d$ and $D\equiv d \pmod r$. As a consequence, for $n=-1$ (i.e. $D=d$) we have $SW_{S,\mathfrak{s}_r-[\Sigma]}(z)=\langle z,[s^d\Sigma]\rangle$, for any $z\in\mathbb{A}(\Sigma)$ of degree $2d$.*

Proof. Let L be the determinant bundle of $\mathfrak{s}_r+n[\Sigma]$, so that $c_1(L)=2r[\mathbb{S}^2]+2n[\Sigma]$. Let $H=\Sigma+\epsilon\mathbb{S}^2$ be a polarisation close to $[\Sigma]$, i.e. $\epsilon>0$ small. Then $\deg_H L=2r+2n\epsilon>0$, so by [3, proposition 27] the non-perturbed Seiberg-Witten moduli space on S is $\mathbb{P}(H^0(K\otimes\mathcal{L}^\vee)^*)$, where $-K+2\mathcal{L}=L$, so $K-\mathcal{L}=\frac{K-L}{2}\equiv(g-1-r)[\mathbb{S}^2]+(-1-n)[\Sigma]$. For $n\ge 0$ this is empty and hence $SW_{S,\mathfrak{s}_r+n[\Sigma]}=0$.

For $n\le -1$, $d(\mathfrak{s}_r+n[\Sigma])=2(rn+g-1)$. Let $H_0=\epsilon\Sigma+\mathbb{S}^2$ be a polarisation close to $[\mathbb{S}^2]$, i.e. $\epsilon>0$ small. Then $\deg_{H_0}L=2r\epsilon+2n<0$, so by [3, proposition 27] the non-perturbed Seiberg-Witten moduli space on S is $\mathbb{P}(H^0(\mathcal{L})^*)$, where $\mathcal{L}=\frac{K+L}{2}\equiv(g-1+r)[\mathbb{S}^2]+(-1+n)[\Sigma]$. Hence the moduli space is empty and the Seiberg-Witten invariant for this polarisation, is zero. The Seiberg-Witten invariant $SW_{S,\mathfrak{s}_r+n[\Sigma]}$ is obtained via wall-crossing from [20]. With the notations therein, $u_c\in\Lambda^2H_1(S;\mathbb{Z})$ is given by $u_c(\gamma_i\wedge\gamma_j)=\frac{1}{2}\langle\gamma_i\cup\gamma_j,c_1(L)\rangle$, i.e. $u_c=n\theta$, and

$$SW_{S,\mathfrak{s}_r+n[\Sigma]}(x^a\theta^b)=\langle\theta^b\frac{(-u_c)^{g-b}}{(g-b)!},[\text{Jac}\,S]\rangle=\frac{g!}{(g-b)!}(-n)^{g-b}.$$

The sign is as stated as there is a minus sign coming in as we compute the invariants in the component $-\mathcal{K}_0$ and another minus sign because we orient H^2_+ with $-\omega_g$.

The last statement follows from [9]. □

Proposition 3.4. *Fix $0 < r \leq g-1$ and put $d = g-1-r$. Let $z_i \in \mathbb{A}(\Sigma)$, $i \in I$, be homogeneous elements such that $\{z_i\}_{i\in I}$ is a basis for $H^*(s^d\Sigma)$, under the epimorphism (8). Consider for each $i \in I$ the element $e_i = \phi^{SW}_A(\mathfrak{s}_r, z_i) \in V_r = HFSW^*(Y, \mathfrak{s}_r)$. Then $\{e_i\}_{i\in I}$ is a basis for V_r. Therefore $H^*(s^d\Sigma) \to V_r$, $z_i \mapsto e_i$, is a ($Sp(2g,\mathbb{Z})$-equivariant, non-canonical) isomorphism of vector spaces.*

Proof. Without loss of generality, we may suppose that $\{z_i\}_{i\in I}$ is a basis formed by homogeneous elements with non-decreasing degrees. The intersection matrix $(\langle z_i, z_j\rangle)$ is then of the form

$$\begin{pmatrix} 0 & \cdots & 0 & A_0 \\ 0 & \cdots & A_1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ A_{2d} & \cdots & 0 & 0 \end{pmatrix},$$

where A_i are the sub-matrices corresponding to the pairing $H^i(s^d\Sigma)\otimes H^{2d-i}(s^d\Sigma) \to \mathbb{Q}$. So $\det A_i \neq 0$, for $0 \leq i \leq 2d$. By the formula (7) and lemma 3.3, $\langle e_i, e_j\rangle = 0$ if $\deg z_i + \deg z_j > 2d$ and $\langle e_i, e_j\rangle = \langle z_i, z_j\rangle$ if $\deg z_i + \deg z_j = 2d$. Therefore the intersection matrix $(\langle e_i, e_j\rangle)$ is of the form

$$\begin{pmatrix} * & \cdots & * & A_0 \\ * & \cdots & A_1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ A_{2d} & \cdots & 0 & 0 \end{pmatrix},$$

which is invertible. This implies in particular that $\dim V_r \geq \dim H^*(s^d\Sigma)$. As we already have the opposite inequality from theorem 3.2, it must be $\dim V_r = \dim H^*(s^d\Sigma)$ and $\{e_i\}_{i\in I}$ is a basis for V_r. □

The proof of this proposition shows that the map (6) is surjective. We have the following

Criterium 3.5. Let $z \in \mathbb{A}(\Sigma)$ and $0 < |r| \leq g-1$. Then the following are equivalent:

- $\phi^{SW}_A(\mathfrak{s}_r, z) = 0$.
- $SW_{S,\mathfrak{s}_r+n[\Sigma]}(zz_i) = 0$ for all $i \in I$ and integer n.
- $SW_{S,\mathfrak{s}_r+n[\Sigma]}(zz') = 0$ for all $z' \in \mathbb{A}(\Sigma)$ and integer n.

4. Ring structure of $HFSW^*(\Sigma\times\mathbb{S}^1, \mathfrak{s}_r)$

Recall our basic set up. We have the three manifold $Y = \Sigma\times\mathbb{S}^1$ together with the Spin$^{\mathbb{C}}$ structure $\mathfrak{s}_r$ with $c_1(\mathfrak{s}_r) = 2r\text{P.D.}[\mathbb{S}^1] \in H^2(Y;\mathbb{Z})$, $0 < |r| \leq g-1$, and put $d = g-1-|r|$. We can define a product on $V_r = HFSW^*(Y, \mathfrak{s}_r)$ as follows. By criterium 3.5,

$$\mathcal{I}_g = \{z \in \mathbb{A}(\Sigma) \mid \phi^{SW}_A(\mathfrak{s}_r, z) = 0\}$$

is an ideal of $\mathbb{A}(\Sigma)$. So we define an associative and graded commutative ring structure on V_r by

$$\phi^{SW}_A(\mathfrak{s}_r, z_1)\cdot\phi^{SW}_A(\mathfrak{s}_r, z_2) = \phi^{SW}_A(\mathfrak{s}_r, z_1z_2),$$

for any $z_1, z_2 \in \mathbb{A}(\Sigma)$. Therefore $V_r = \mathbb{A}(\Sigma)/\mathcal{I}_g$. This makes the map (6) an epimorphism of rings.

Lemma 4.1. *Let X_1 be a 4-manifold with boundary $\partial X_1 = Y$ and let $\mathfrak{s}$ be a Spin$^{\mathbb{C}}$ structure such that $\mathfrak{s}|_Y = \mathfrak{s}_r$. Then for any $z_1 \in \mathbb{A}(X_1)$ and $z_2 \in \mathbb{A}(\Sigma)$ we have*

$$\phi_A^{SW}(\mathfrak{s}_r, z_2) \cdot \phi_{X_1}^{SW}(\mathfrak{s}, z_1) = \phi_{X_1}^{SW}(\mathfrak{s}, z_2 z_1).$$

Proof. First, for any $\phi \in V_r$ we have $\langle \phi \cdot \phi_A^{SW}(\mathfrak{s}_r, z_2), \phi_A^{SW}(\mathfrak{s}_r, z)\rangle = \langle \phi, \phi_A^{SW}(\mathfrak{s}_r, z_2 z)\rangle$. This is true since by the very definition of the product it holds for the elements $\phi = \phi_A^{SW}(\mathfrak{s}_r, z')$, which generate V_r.

Now for any $z \in \mathbb{A}(\Sigma)$ we have

$$\begin{aligned}&\langle \phi_{X_1}^{SW}(\mathfrak{s}, z_1) \cdot \phi_A^{SW}(\mathfrak{s}_r, z_2), \phi_A^{SW}(\mathfrak{s}_r, z)\rangle = \langle \phi_{X_1}^{SW}(\mathfrak{s}, z_1), \phi_A^{SW}(\mathfrak{s}_r, z_2 z)\rangle = \\ &= SW_{X,\mathfrak{s}_r+n[\Sigma]}(z_1 z_2 z) = \langle \phi_{X_1}^{SW}(\mathfrak{s}, z_1 z_2), \phi_A^{SW}(\mathfrak{s}_r, z)\rangle,\end{aligned}$$

where $X = X_1 \cup_Y A$ and n is a suitable integer. By criterium 3.5 we have the result. □

Note that the isomorphism $V_r \cong V_{-r}$ intertwines the ring structures, so we may restrict to the case $r > 0$. Recall that $d = g - 1 - r$.

Theorem 4.2. *Denote by $\cdot$ the product induced in $H^*(s^d\Sigma)$ by the product in V_r under the isomorphism of proposition 3.4. Then $\cdot$ is a deformation of the cup product graded modulo $2r = 2(g - 1 - d)$, i.e. for $f_1 \in H^i(s^d\Sigma)$, $f_2 \in H^j(s^d\Sigma)$, it is $f_1 \cdot f_2 = \sum_{m\geq 0} \Phi_m(f_1, f_2)$, where $\Phi_m \in H^{i+j+2mr}(s^d\Sigma)$ and $\Phi_0 = f_1 \cup f_2$.*

Proof. By lemma 3.3, for any $i, j \in I$, $\langle e_i, e_j\rangle$ is zero unless $\deg z_i + \deg z_j = 2d - 2mr$, with $m \geq 0$. Moreover, when $\deg z_i + \deg z_j = 2d$, it is $\langle e_i, e_j\rangle = \langle z_i, z_j\rangle$. Now the same argument as in [16, theorem 5] accomplishes the result, the only difference being that, in our present case, the deformation produces terms of increasing degrees. □

Corollary 4.3. *Let $f \in \mathbb{A}(\Sigma)$ be an homogeneous element of degree strictly bigger than $2d$. Then f is zero in V_r.* □

The last ingredient that we need in order to describe V_r is the cohomology $H^*(s^d\Sigma)$ of the d-th symmetric product $s^d\Sigma$ of the surface Σ. Here $d = g - 1 - r$, so d is in the range $0 \leq d < g - 1$. This cohomology ring was initially described in [9] and revisited in [19, section 4] where it was described as $\mathrm{Sp}\,(2g, \mathbb{Z})$-representation, which is the form well suited for our purposes. Put an auxiliary complex structure on Σ and interpret $s^d\Sigma$ as the moduli space of degree d effective divisors on Σ. Let $D \subset s^d\Sigma \times \Sigma$ be the universal divisor. Then

$$\begin{cases} \eta = c_1(D)/x \in H^2(s^d\Sigma) & \\ \psi_i = c_1(D)/\gamma_i \in H^1(s^d\Sigma), & 1 \leq i \leq 2g \end{cases}$$

are generators of the ring $H^*(s^d\Sigma)$, i.e. there is a graded $\mathrm{Sp}\,(2g, \mathbb{Z})$-equivariant epimorphism

$$\mathbb{A}(\Sigma) \cong \mathbb{Q}[\eta] \otimes \Lambda^*(\psi_1, \ldots, \psi_{2g}) \twoheadrightarrow H^*(s^d\Sigma). \tag{8}$$

We set $\theta = \sum_{i=1}^{g} \psi_i \psi_{g+i} \in H^2(s^d\Sigma)$, abusing a little bit notation since it correspond to the element θ under (8). Also we identify $\Lambda_0^k = \Lambda_0^k(\psi_1, \ldots, \psi_{2g})$ under the same map. Clearly η and θ generate the invariant part $H^*(s^d\Sigma)_I$. The description of $H^*(s^d\Sigma)$ as $\mathrm{Sp}\,(2g,\mathbb{Z})$-representation is given in the following

Proposition 4.4. ([19, proposition 3.5]) *For $0 \leq d \leq g-1$ there is a presentation*

$$H^*(s^d\Sigma) = \bigoplus_{k=0}^{d} \Lambda_0^k \otimes \frac{\mathbb{Q}[\eta,\theta]}{J_k^g},$$

where $J_k^g = (R_k^g, \theta R_{k+1}^g, \theta^2 R_{k+2}^g, \ldots, \theta^{d+1-k})$, $0 \leq k \leq d$, and

$$R_k^g = \sum_{i=0}^{\alpha} \frac{\binom{(d-k)-\alpha+1}{i}}{\binom{g-k}{i}} \frac{(-\theta)^i}{i!} \eta^{\alpha-i},$$

for $0 \leq k \leq d$, with $\alpha = [\frac{d-k}{2}] + 1$ (for consistency, $R_{d+1}^g = 1$). Actually $J_k^g = (R_k^g, \theta R_{k+1}^g)$. A basis for $\mathbb{Q}[\eta,\theta]/J_k^g$ as vector space is given by $\eta^a\theta^b$, with $2a + b \leq d - k$. □

For the space V_r we set

$$\begin{cases} \eta = \phi_A^{SW}(\mathfrak{s}_r, x) \in V_r \\ \psi_i = \phi_A^{SW}(\mathfrak{s}_r, \gamma_i) \in V_r, \qquad 1 \leq i \leq 2g \end{cases}$$

where η has degree 2 and ψ_i degree 1. (We name with the same letters elements in V_r and in $H^*(s^d\Sigma)$ as they are obviously related.) These elements are generators of V_r as algebra. This means that (6) is a $\mathrm{Sp}\,(2g,\mathbb{Z})$-equivariant epimorphism

$$\mathbb{A}(\Sigma) \cong \mathbb{Q}[\eta] \otimes \Lambda^*(\psi_1, \ldots, \psi_{2g}) \twoheadrightarrow V_r.$$

Clearly η and $\theta = \sum_{i=1}^{g} \psi_i \psi_{g+i}$ generate the invariant part of V_r. Now we are going to relate the ring structure of $H^*(s^d\Sigma)$ with that of V_r.

Proposition 4.5. *Let $0 < r \leq g-1$ and set $d = g-1-r$. Then there is a presentation*

$$V_r = \bigoplus_{k=0}^{d} \Lambda_0^k \otimes \frac{\mathbb{Q}[\eta,\theta]}{I_k^g},$$

where $I_k^g = (\mathcal{R}_k^g, \theta\mathcal{R}_{k+1}^g) \subset \mathbb{Q}[\eta,\theta]$ are ideals (dependent on g, k and r) such that

$$\mathcal{R}_k^g = R_k^g + \sum_{\substack{i=2\alpha+2mr-(d-k) \\ m>0}}^{\alpha+mr} \frac{a_{im}}{i!\binom{g-k}{i}} \eta^{\alpha+mr-i}\theta^i, \tag{9}$$

where $\alpha = [\frac{d-k}{2}] + 1$, R_k^g are given in proposition 4.4 and a_{im} are some complex numbers (dependent on g, k, r). A basis for $\mathbb{Q}[\eta,\theta]/I_k^g$ is given by $\eta^a\theta^b$, with $2a + b \leq d - k$.

Proof. Let $\{z_i^{(k)}\}$ be a basis for Λ_0^k. Then by proposition 4.4, $z_i^{(k)} x^a \theta^b$, $2a + b + k \leq d$, form a basis for $H^*(s^d\Sigma)$. We use this basis in proposition 3.4 to construct a ($\mathrm{Sp}\,(2g,\mathbb{Z})$-equivariant) isomorphism $H^*(s^d\Sigma) \cong V_r$. The fact that $R_k^g \in J_k^g$ means that $z_i^{(k)} R_k^g = 0$ in

$H^*(s^d\Sigma)$. Fix $z_0^{(k)} = \psi_1 \cdots \psi_k \in \Lambda_0^k$, then $\Lambda_0^k = \text{Span} < \text{Sp}\,(2g,\mathbb{Z}) z_0^{(k)} >$. Rewriting $z_0^{(k)} R_k^g$ in terms of the product $\cdot$ of theorem 4.2, and using the arguments of [24, section 2] (and the fact that the action of $\text{Sp}\,(2g,\mathbb{Z})$ is compatible with the ring structure on V_r), we get that

$$z_0^{(k)} R_k^g + \sum_{m>0} z_i^{(k)} R_{kim}^g = 0 \tag{10}$$

in V_r, where $\deg R_{kim}^g = \deg R_k^g + mr = \alpha + mr$, and R_{kim}^g is expressible in terms of the chosen basis, i.e. as a linear combination of the monomials $\eta^{\alpha+mr-j}\theta^j$, for $2\alpha + 2mr - (d-k) \leq j \leq \alpha + mr$. As in the proof of [17, proposition 16], we have that the only nonvanishing R_{kim}^g in (10) correspond to $z_0^{(k)}$ (otherwise one can find an element of $\text{Sp}\,(2g,\mathbb{Z})$ only fixing $z_0^{(k)}$, which would produce a relation between the elements of the basis of V_r, which is impossible), so (10) reduces to $z_0^{(k)}(R_k^g + \sum_{m>0} R_{k0m}^g) = 0$ in V_r. This produces the relation $\mathcal{R}_k^g = R_k^g + \sum_{m>0} R_{k0m}^g$ as stated in (9).

Also $\theta \mathcal{R}_{k+1}^g \in I_k^g$ since $\theta I_{k+1}^g \subset I_k^g$. Now I_k^g is generated by these two elements since J_k^g is generated by R_k^g and θR_{k+1}^g (see [24, section 2]). □

Remark 4.6. Note that for d odd, $\mathcal{R}_k^g$ is the relation uniquely determined by expressing $\eta^\alpha \in \mathbb{Q}[\eta,\theta]/I_k^g$, $\alpha = [\frac{d-k}{2}] + 1$, in terms of the monomials of the basis $\eta^a\theta^b$, $2a + b \leq d - k$. For d even, $\mathcal{R}_k^g$ is the relation uniquely determined by expressing $\eta^\alpha - \frac{(d-k)-\alpha+1}{g-k}\eta^{\alpha-1}\theta$ in terms of the monomials of the basis.

Corollary 4.7. *There is an isomorphism of associated graded rings*

$$Gr_\theta V_r \cong Gr_\theta H^*(s^d\Sigma),$$

where $d = g - 1 - |r|$. *Let* $HF^*(\Sigma \times \mathbb{S}^1)$ *be the instanton Floer homology of* $Y = \Sigma \times \mathbb{S}^1$ *with* $SO(3)$*-bundle with* $w_2 = P.D.[\mathbb{S}^1]$, *which was computed in* [16]. *This can be decomposed* [16, proposition 20] *according to the eigenvalues of* $\alpha = 2\mu(\Sigma)$ *as* $HF^*(\Sigma \times \mathbb{S}^1) = \oplus_{r=-(g-1)}^{g-1} H_r$, *where* α *has eigenvalue* $4r$ *(if* r *is odd) or* $4\sqrt{-1}r$ *(if* r *is even) on* H_r, $-(g-1) \leq r \leq g-1$. *Then* [19, corollary 3.7] *gives an isomorphism*

$$Gr_\theta V_r \cong Gr_\gamma H_r,$$

where $\gamma = -2\sum \mu(\gamma_i)\mu(\gamma_{g+i})$. □

Lemma 4.8. *Fix* $r > 0$. *Then the ideals of proposition 4.5 satisfy the recursion* $I_k^g = I_{k-1}^{g-1}$, *for* $k > 0$ *and* $r \leq g - 2$. *Equivalently,* $\mathcal{R}_k^g = \mathcal{R}_0^{g-k}$.

Proof. By the computation of the Seiberg-Witten invariants of S in lemma 3.3 and the invariance under the action of $\mathrm{Sp}\,(2g,\mathbb{Z})$, we have

$$\begin{aligned} SW_{S,\mathfrak{s}_r+n[\Sigma]}(\gamma_1\cdots\gamma_k\gamma_{g+1}\cdots\gamma_{g+k}x^a\theta^b) &= SW_{S,\mathfrak{s}_r+n[\Sigma]}\left(\frac{1}{k!}\binom{g}{k}^{-1}\theta^k x^a\theta^b\right) \\ &= \frac{(g-k)!}{(g-k-b)!}(-n)^{g-k-b}, \end{aligned}$$

for $a+b=g-k-1-rn$. Therefore for any $R\in\mathbb{A}(\Sigma)_I$, $z\in\mathbb{A}(\Sigma)$,

$$\langle\psi_1\cdots\psi_{k-1}\psi_g R,\psi_{2g}z\rangle_g=\langle\psi_1\cdots\psi_{k-1}R,z\rangle_{g-1},$$

where the subindex means the genus of the surface Σ. This implies the statement by criterium 3.5. The last part follows from remark 4.6. $\square$

Now we aim to compute the coefficients a_{im} of $\mathcal{R}_0=\mathcal{R}_0^g$ in (9). Let $m>0$. We collect the coefficients together in a polynomial

$$p_m(x)=\sum a_{im}x^{g-i}, \tag{11}$$

where we consider $a_{im}=0$ for $i\notin[2\alpha+2mr-d,\alpha+mr]\cap\mathbb{Z}$, $\alpha=[\frac{d}{2}]+1$. Note that there are a finite number of non-zero polynomials. By analogy we consider

$$p_0(x)=(x-1)^{d-\alpha+1}x^{g-(d-\alpha-1)}, \tag{12}$$

so that $R_0=\sum\frac{a_{i0}}{i!\binom{g}{i}}\eta^{\alpha-i}\theta^i$, as given in proposition 4.4. By definition $\mathcal{R}_0=0\in V_r$, therefore we have $\langle\mathcal{R}_0,\eta^a\theta^b\rangle=0$, whenever $\alpha+a+b=d-kr$, $k\geq 0$. Now using the computation of the invariants of S in lemma 3.3, we get

$$\sum_{m=0}^k\frac{a_{im}}{i!\binom{g}{i}}\frac{g!}{(g-b-i)!}(k-m+1)^{g-b-i}=\sum_{m=0}^k a_{im}\frac{(g-i)!}{(g-b-i)!}(k-m+1)^{g-b-i}=0,$$

for all $k\geq 0$ and $0\leq b\leq d-\alpha-kr$. So

$$\sum_{m=0}^k\frac{d^b}{dx^b}p_m(x)\Big|_{x=k-m+1}=0.$$

for all $k\geq 0$ and $0\leq b\leq d-\alpha-kr$. By Taylor expansion, this is equivalent to saying that

$$p_k(x)\equiv-\left(p_0(x+k)+p_1(x+k-1)+\cdots+p_{k-1}(x+1)\right)\pmod{(x-1)^{d-\alpha-kr+1}}. \tag{13}$$

This condition, together with the fact that $p_k(x)$ has degree $g-(2\alpha+2kr-d)$ and it is divisible by $x^{g-(\alpha+kr)}$, uniquely determines $p_k(x)$ by recursion.

For instance, let us calculate explicitly $p_1(x)$. From (12) we have that

$$\begin{aligned} p_0(x+1) &= x^{d-\alpha+1}(x+1)^{g-(d-\alpha-1)}=x^{g-\alpha-r}(x+1)^{\alpha+r} \\ &= x^{g-\alpha-r}\sum_{k=0}^{\alpha+r}\binom{\alpha+r}{k}2^k(x-1)^{\alpha+r-k}, \end{aligned}$$

using that $d = g-1-r$. Now $p_1(x)$ is divisible by $x^{g-\alpha-r}$, has degree $(g-\alpha-r)+(d-\alpha-r)$ and $p_1(x) \equiv -p_0(x+1) \pmod{(x-1)^{d-\alpha-r+1}}$. Therefore

$$\begin{aligned} p_1(x) &= -x^{g-\alpha-r} \sum_{k=2\alpha+2r-d}^{\alpha+r} \binom{\alpha+r}{k} 2^k (x-1)^{\alpha+r-k} = \\ &= -x^{g-\alpha-r} \sum_{\substack{2\alpha+2r-d\leq k\leq\alpha+r \\ 0\leq j\leq \alpha+r-k}} 2^k \binom{\alpha+r}{k}\binom{\alpha+r-k}{j} (-1)^j x^{\alpha+r-k-j} = \\ &= \sum_{\substack{2\alpha+2r-d\leq k\leq\alpha+r \\ 0\leq j\leq \alpha+r-k}} (-1)^{j+1} \frac{(\alpha+r)!}{k!j!(\alpha+r-k-j)!} 2^k x^{g-k-j}. \end{aligned}$$

From this we may write the coefficients a_{i1} as

$$a_{i1} = \sum_{j=0}^{i-(2\alpha+2r-d)} (-1)^{j+1} \frac{(\alpha+r)!}{(i-j)!j!(\alpha+r-i)!} 2^{i-j},$$

for $2\alpha+2r-d \leq i \leq \alpha+r$.

We can compute the rest of the coefficients a_{im}, for $m > 1$, by recurrence using this method but the result is a collection of rather cumbersome formulae which do not shed light on the ring structure of V_r. This is to no surprise: the shape of the relations $\mathcal{R}_k^g$ depends on the basis of $\mathbb{Q}[\eta,\theta]/I_k^g$ that we have chosen in proposition 4.5, and this basis has been chosen rather arbitrarily. We shall present now a slightly modified version of the previous argument which computes explicitly (a full set of) relations for V_r, by just not fixing any basis for $\mathbb{Q}[\eta,\theta]/I_k^g$. This leads to a closed formula for generators of the ideals I_k^g.

Theorem 4.9. *Let $0 < r \leq g-1$ and set $d = g-1-r$. Then there is a presentation*

$$V_r = \bigoplus_{k=0}^{d} \Lambda_0^k \otimes \frac{\mathbb{Q}[\eta,\theta]}{(\tilde{\mathcal{R}}_k^g, \theta\tilde{\mathcal{R}}_{k+1}^g, \eta^{d+1}, \theta^{d+1})},$$

where

$$\tilde{\mathcal{R}}_k^g = \sum_{i=0}^{\alpha} \frac{\binom{d-k-\alpha+1}{i}}{i!\binom{g-k}{i}} (-1)^i \eta^{\alpha-i}\theta^i - \sum_{i=0}^{\alpha+r} \frac{\binom{\alpha+r}{i}}{i!\binom{g-k}{i}} \eta^{\alpha+r-i}\theta^i,$$

where $\alpha = [\frac{d-k}{2}]+1$, for $0 \leq k \leq d$, and $\tilde{\mathcal{R}}_{d+1}^g = 1$.

Proof. By lemma 4.8 it is enough to find a relation for $k = 0$,

$$\tilde{\mathcal{R}}_0 = R_0 + \sum_{\substack{m>0 \\ 0\leq i\leq \alpha+mr}} \frac{a_{im}}{i!\binom{g}{i}} \eta^{\alpha+mr-i}\theta^i, \tag{14}$$

This time we do not restrict the range for i. We only note that we can suppose $a_{im} = 0$ if $i > g$, since $\theta^{g+1} = 0$. As before, we collect the coefficients a_{im} of (14) in a polynomial $p_m(x) =$

$\sum a_{im} x^{g-i}$, where $a_{im}=0$ for $i \notin [0, \alpha+mr] \cap \mathbb{Z}$. Also $p_0(x)=(x-1)^{d-\alpha+1}x^{g-(d-\alpha-1)}$. The condition that $\mathcal{R}_0$ be a relation is translated into

$$(15) \quad p_k(x) \equiv -\left(p_0(x+k)+p_1(x+k-1)+\cdots+p_{k-1}(x+1)\right) \quad (\text{mod } (x-1)^{d-\alpha-kr+1}).$$

We want to find polynomials $p_k(x)$ of degree g solving (15). This time the $p_k(x)$ are not determined uniquely, but we only need to find one solution. Since $p_0(x+1)=x^{d-\alpha+1}(x+1)^{g-(d-\alpha-1)}=x^{g-\alpha-r}(x+1)^{\alpha+r}$, we may choose

$$p_1(x)=-x^{g-\alpha-r}(x+1)^{\alpha+r}$$

and $p_k(x)=0$ for $k \geq 2$. This gives $a_{i1}=-\binom{\alpha+r}{i}$, $0 \leq i \leq \alpha+r$, and $a_{im}=0$ for $m \geq 2$. In this way we have found $\tilde{\mathcal{R}}^g_k, \theta\tilde{\mathcal{R}}^g_{k+1} \in I^g_k$ as given in the statement. However they do not generate the whole ideal as may be seen by looking at the associated graded ring $\mathrm{Gr}_\theta\left(\mathbb{Q}[\eta,\theta]/(\tilde{\mathcal{R}}^g_k, \theta\tilde{\mathcal{R}}^g_{k+1})\right)$, so we need to add more relations. The nilpotence relations η^{d+1}, θ^{d+1} are always satisfied by corollary 4.3. To see that these relations suffice, write any $f \in I^g_k$ as $f=a_1\mathcal{R}^g_k+a_2\theta\mathcal{R}^g_{k+1}$, by proposition 4.5. Then $f-a_1\tilde{\mathcal{R}}^g_k-a_2\theta\tilde{\mathcal{R}}^g_{k+1} \in I^g_k$ and has higher degree than that of f. Proceed recursively until we get a polynomial in $(\eta^{d+1}, \theta^{d+1})$. □

5. Seiberg-Witten invariants of connected sums along surfaces

We want to show, as a first application, how the knowledge of the previous sections can be used to compute the Seiberg-Witten invariants of 4-manifolds which appear as connected sums along surfaces of other 4-manifolds. This was first dealt with in a particular case in [12] to get a proof of the symplectic Thom conjecture. In the context of Donaldson invariants it has been extensively treated in [15] [18].

The set up is as follows (see [15]). Let $\bar{X}_1$ and $\bar{X}_2$ be smooth oriented 4-manifolds and let Σ be a compact oriented surface of genus $g \geq 2$. Suppose that we have embeddings $\Sigma \hookrightarrow \bar{X}_i$ with image Σ_i representing a *non-torsion* element in homology whose self-intersection is zero. This implies that $b^+>0$. Now take small closed tubular neighbourhoods N_{Σ_i} of Σ_i which are isomorphic to $A=\Sigma \times D^2$. Let X_i be the closure of $\bar{X}_i - N_{\Sigma_i}$, $i=1,2$. Then X_i is a 4-manifold with boundary $\partial X_i = Y = \Sigma \times \mathbb{S}^1$ and $\bar{X}_i = X_i \cup_Y A$. Take an identification $\phi : \partial X_1 \to -\partial X_2$ (i.e. an orientation reversing bundle isomorphism). We define the connected sum of $\bar{X}_1$ and $\bar{X}_2$ along Σ as

$$X = X(\phi) = X_1 \cup_\phi X_2.$$

The resulting 4-manifold depends in general on the isotopy class of ϕ, but we shall drop ϕ from the notation when there is no danger of confusion, and write then $X=\bar{X}_1 \#_\Sigma \bar{X}_2$. Consider $\mathrm{Spin}^{\mathbb{C}}$ structures $\mathfrak{s}_i$ on X_i such that $\mathfrak{s}_1|_Y \cong -\mathfrak{s}_2|_Y \cong \mathfrak{s}_Y$, with $c_1(\mathfrak{s}_Y) \neq 0$, so that they can be glued together to get a $\mathrm{Spin}^{\mathbb{C}}$ structure $\mathfrak{s}_o$ on X. The $\mathrm{Spin}^{\mathbb{C}}$ structures $\mathfrak{s}$ such that $\mathfrak{s}|_{X_i}=\mathfrak{s}_i$, $i=1,2$, are those of the form $\mathfrak{s}_o+h$, where h is an element in the image of $H_2(Y;\mathbb{Z}) \to H_2(X;\mathbb{Z}) \cong H^2(X;\mathbb{Z})$, where the last map is Poincaré duality.

Let $\mathcal{R}im \subset H^2(X;\mathbb{Z})$ be the subspace generated by the rim tori [6], i.e. the image of $H_1(\Sigma;\mathbb{Z}) \otimes [\mathbb{S}^1] \subset H_2(Y;\mathbb{Z})$ in $H^2(X;\mathbb{Z})$. Then any Spin$^{\mathbb{C}}$ structure $\mathfrak{s}$ such that $\mathfrak{s}|_{X_i} = \mathfrak{s}_i$, $i = 1, 2$, is of the form $\mathfrak{s} = \mathfrak{s}_o + h + n\Sigma$, where $h \in \mathcal{R}im$, $n \in \mathbb{Z}$.

If $\mathfrak{s}_Y$ has $c_1(\mathfrak{s}_Y) \neq 2r\text{P.D.}[\mathbb{S}^1]$, for any $-(g-1) \leq r \leq g-1$, $r \neq 0$, then theorem 3.2 tells us that $SW_{X,\mathfrak{s}} = 0$. Now consider the case $\mathfrak{s}_Y = \mathfrak{s}_r$, with $-(g-1) \leq r \leq g-1$, $r \neq 0$. Set $d = g - 1 - |r|$ as usual.

Theorem 5.1. *Fix $z_i \in \mathbb{A}(\Sigma)$, $i \in I$, homogeneous elements such that $\{z_i\}_{i\in I}$ is a basis for $H^*(s^d\Sigma)$. Then there exists a universal matrix $(m_{ij})_{i,j\in I}$ such that for every connected sum $X = \bar{X}_1 \#_\Sigma \bar{X}_2$ along a surface of genus g, Spin$^{\mathbb{C}}$ structures $\bar{\mathfrak{s}}_i$ on $\bar{X}_i$ with $c_1(\bar{\mathfrak{s}}_i) \cdot \Sigma = 2r$ and Spin$^{\mathbb{C}}$ structure $\mathfrak{s}_o$ on X obtained by gluing $\bar{\mathfrak{s}}_1$ and $\bar{\mathfrak{s}}_2$, we have*

$$\sum_{h\in\mathcal{R}im} SW_{X,\mathfrak{s}_o+h}(z_1 z_2) = \sum_{\substack{n,m\in\mathbb{Z}\\ i,j\in I}} m_{ij} SW_{\bar{X}_1,\bar{\mathfrak{s}}_1+n\Sigma}(z_1 z_i) \cdot SW_{\bar{X}_2,\bar{\mathfrak{s}}_2+m\Sigma}(z_2 z_j),$$

for any $z_1 \in \mathbb{A}(\bar{X}_1)$ and $z_2 \in \mathbb{A}(\bar{X}_2)$ with $d(\mathfrak{s}_o) = \deg z_1 + \deg z_2$ (note that at most one n and one m appear in every summand of the right hand side). If any of the manifolds involved has $b^+ = 1$ then its Seiberg-Witten invariants are computed for the component of the positive cone containing $-rP.D.[\Sigma]$.

Proof. Let $\mathfrak{s}_i = \bar{\mathfrak{s}}_i|_{X_i}$, $i = 1, 2$. By proposition 3.4, the elements $e_i = \phi^{SW}_A(\mathfrak{s}_r, z_i)$, $i \in I$, form a basis for $V_r = HFSW^*(Y, \mathfrak{s}_r)$. Therefore $V_r \to \mathbb{R}^{|I|}$, given as $\phi \mapsto (\langle \phi, \phi^{SW}_A(\mathfrak{s}_r, z_i)\rangle)_{i\in I}$, is an isomorphism such that

$$\phi^{SW}_{X_1}(\mathfrak{s}_1, z_1) \mapsto \left(\sum_{n\in\mathbb{Z}} SW_{\bar{X}_1,\bar{\mathfrak{s}}_1+n\Sigma}(z_1 z_i) \right)_{i\in I}.$$

Theorem 2.2 says that

$$\sum_n SW_{\bar{X}_1,\bar{\mathfrak{s}}_1+n\Sigma}(z_1 z_i) = \langle \phi^{SW}_{X_1}(\mathfrak{s}_1, z_1), e_i\rangle, \quad \sum_m SW_{\bar{X}_2,\bar{\mathfrak{s}}_2+m\Sigma}(z_2 z_j) = \langle \phi^{SW}_{X_2}(\mathfrak{s}_2, z_2), e_j\rangle$$

and

$$\sum_{\{\mathfrak{s}/\mathfrak{s}|_{X_i}=\mathfrak{s}_i,\, i=1,2\}} SW_{X,\mathfrak{s}}(z_1 z_2) = \langle \phi^{SW}_{X_1}(\mathfrak{s}_1, z_1), \phi^{SW}_{X_2}(\mathfrak{s}_2, z_2)\rangle.$$

Only the Spin$^{\mathbb{C}}$ structures of the form $\mathfrak{s} = \mathfrak{s}_o + h$, $h \in \mathcal{R}im$, satisfy $d(\mathfrak{s}) = \deg z_1 + \deg z_2$. The result follows with (m_{ij}) being the inverse of the intersection matrix for the basis $\{e_i\}_{i\in I}$. Note that this matrix is explicitly computable, since by lemma 3.3 the products $\langle e_i, e_j\rangle$ are known. □

Corollary 5.2. *If either of $\bar{X}_i$ has simple type then $\sum_{h\in\mathcal{R}im} SW_{X,\mathfrak{s}_o+h}(xz) = 0$, for any $z \in \mathbb{A}(X)$ and Spin$^{\mathbb{C}}$ structure $\mathfrak{s}_o$ on X with $c_1(\mathfrak{s}_o) \cdot \Sigma \neq 0$. Analogously, if either of $\bar{X}_i$ has strong simple type then $\sum_{h\in\mathcal{R}im} SW_{X,\mathfrak{s}_o+h}(z) = 0$, for any $z \in \mathbb{A}(X)$ with $\deg(z) > 0$ and any Spin$^{\mathbb{C}}$ structure $\mathfrak{s}_o$ on X with $c_1(\mathfrak{s}_o) \cdot \Sigma \neq 0$.* □

In order to remove the summation over the subspace $\mathcal{R}im$ in corollary 5.2 we need an extra condition.

Definition 5.3. A connected sum $X = \bar{X}_1 \#_\Sigma \bar{X}_2$ is admissible if $\mathcal{R}im$ has no torsion and there exists a subspace $V \subset H^2(X;\mathbb{Z})$ such that $H^2(X;\mathbb{Z}) = V \oplus \mathcal{R}im$ and $c_1(\mathfrak{s}) \in V$ for every basic class $\mathfrak{s}$ of X.

Remark 5.4. Suppose that $\bar{X}_1$ and $\bar{X}_2$ are Kähler surfaces and $\Sigma_i \subset \bar{X}_i$ are smooth complex curves of genus g, isomorphic as complex curves, such that there is a deformation Kähler family $\mathcal{Z} \xrightarrow{\pi} D^2 \subset \mathbb{C}$ with fiber $Z_t = \pi^{-1}(t)$, $t \neq 0$, smooth and $Z_0 = \pi^{-1}(0) = \bar{X}_1 \cup_\Sigma \bar{X}_2$, the union of $\bar{X}_1$ and $\bar{X}_2$ along $\Sigma_1 = \Sigma_2$ with a normal crossing. Then the general fiber $X = Z_t$ is the connected sum $X = \bar{X}_1 \#_\Sigma \bar{X}_2$ with identification given by the isomorphism between the normal bundles of Σ_1 and Σ_2. If $H^2(X;\mathbb{Z})$ has no torsion then this identification is admissible, since for any basic class $\mathfrak{s}$ one has $c_1(\mathfrak{s}) \in H^{1,1}$ and this space is orthogonal to $\mathcal{R}im$, as for any $T \in \mathcal{R}im$, it is $T^2 = 0$ and $\omega \cdot T = 0$ (ω standing for the Kähler form). This implies that $T \notin H^{1,1}$ unless $T = 0$.

Remark 5.5. In [11, definition 4.1], Morgan and Szabó define admissible identification when there exists a collection of primitive embedded (-2)-spheres in X (obtained by pasting embedded (-1)-discs in X_1 and X_2) generating a subspace $V \subset H_2(X)$ such that $H_2(X) = \mathcal{H} \oplus V$, where $\mathcal{H} = \{D \in H_2(X)/D|_Y = k[\mathbb{S}^1], \text{some } k\}$. Then $c_1(\mathfrak{s}) \cdot V = 0$ for any basic class $\mathfrak{s}$, and this implies admissibility in the sense of definition 5.3 (assuming again that $H^2(X;\mathbb{Z})$ has no torsion).

Corollary 5.6. *Suppose that the connected sum $X = \bar{X}_1 \# \bar{X}_2$ is admissible. If either of $\bar{X}_i$ has simple type then $SW_{X,\mathfrak{s}}(xz) = 0$ for any $z \in \mathbb{A}(X)$ and any Spin$^{\mathbb{C}}$ structure $\mathfrak{s}$ with $c_1(\mathfrak{s}) \cdot \Sigma \neq 0$.*

If either of $\bar{X}_i$ has strong simple type then $SW_{X,\mathfrak{s}}(z) = 0$ for any $z \in \mathbb{A}(X)$ with $\deg(z) > 0$ and any Spin$^{\mathbb{C}}$ structure $\mathfrak{s}$ with $c_1(\mathfrak{s}) \cdot \Sigma \neq 0$. □

The formula in theorem 5.1 becomes simpler when both $\bar{X}_i$ have $b_1 = 0$. We have the following result

Theorem 5.7. *Let $X = \bar{X}_1 \#_\Sigma \bar{X}_2$ be a connected sum along a surface of genus g where both $\bar{X}_1$ and $\bar{X}_2$ are of H_1-simple type. Let $\bar{\mathfrak{s}}_i$ be Spin$^{\mathbb{C}}$ structures on $\bar{X}_i$ with $c_1(\bar{\mathfrak{s}}_i) \cdot \Sigma = 2r \neq 0$ and let $\mathfrak{s}_o$ be a Spin$^{\mathbb{C}}$ structure on X obtained by gluing $\bar{\mathfrak{s}}_1$ and $\bar{\mathfrak{s}}_2$. Suppose $d = g-1-|r| \geq 0$. Let $z \in \mathbb{A}(X)$ with $\deg(z) = d(\mathfrak{s}_o)$. Then $\sum_{h \in \mathcal{R}im} SW_{X,\mathfrak{s}_o+h}(z) =$*

$$\begin{cases} \sum\limits_{n,m\in\mathbb{Z}} (-1)^{d/2} \binom{g-1}{d/2} SW_{\bar{X}_1,\bar{\mathfrak{s}}_1+n\Sigma}(x^{d/2}) \cdot SW_{\bar{X}_2,\bar{\mathfrak{s}}_2+m\Sigma}(x^{d/2}), & z = 1,\ d \text{ even} \\ 0, & \text{otherwise.} \end{cases}$$

Note that at most one n and one m contribute to this formula.

Proof. By lemma 4.1, $\psi_j \phi^{SW}_{X_1}(\mathfrak{s}_1, z_1) = \phi^{SW}_{X_1}(\mathfrak{s}_1, \gamma_j z_1) = 0$, for $j = 1, \ldots, 2g$. Therefore

$$\phi^{SW}_{X_1}(\mathfrak{s}_1, z_1) \in K = \bigcap_{1\leq j\leq 2g} \ker \psi_j,$$

for any $z_1 \in \mathbb{A}(X_1)$. Since $K \subset V_r$ is invariant under the action of $\mathrm{Sp}\,(2g,\mathbb{Z})$, we have that $K \subset \mathbb{Q}[\eta,\theta]/I^g_0$ by proposition 4.5. Now $f \in K$ if and only if $\psi_j f = 0$, for $j = 1, \ldots, 2g$. This means that $f \in I^g_1$ in the notation of proposition 4.5. So $K = I^g_1/I^g_0$, where the generators of I^g_1 are $\mathcal{R}^g_1$ and $\theta\mathcal{R}^g_2$. The intersection pairing $\langle,\rangle : K \otimes K \to \mathbb{Q}$ is the restriction of the pairing of V_r. Now by lemma 3.3 $\langle e_i, e_j\rangle = 0$ if $\deg z_i + \deg z_j > 2d$. For d odd, all the homogeneous components of all the elements in I^g_1 have degree strictly bigger than d (note that the component R^g_1 of $\mathcal{R}^g_1$ has degree $2([\frac{d-1}{2}] + 1) = d + 1$ and it is the component of lowest degree). So $K \otimes K \to \mathbb{Q}$ is the zero map for d odd, which proves the second line.

For d even, all the homogeneous components of all the elements in I^g_1 have degree strictly bigger than d, except for R^g_1, which has degree d. So $K \otimes K \to \mathbb{Q}$ has rank 1. Hence, for $z_1 \in \mathbb{A}(\Sigma)$ and $z_2 \in \mathbb{A}(\Sigma)$, we have that

$$\sum_{h\in\mathcal{R}im} SW_{X,\mathfrak{s}_o+h}(z_1z_2) = \sum_{n,m\in\mathbb{Z}} c\, SW_{\bar{X}_1,\bar{\mathfrak{s}}_1+n\Sigma}(z_1x^{d/2}) SW_{\bar{X}_2,\bar{\mathfrak{s}}_2+m\Sigma}(z_2x^{d/2}),$$

where we have used $SW_{\bar{X}_i,\bar{\mathfrak{s}}_i+n\Sigma}(z_iR^g_1) = SW_{\bar{X}_i,\bar{\mathfrak{s}}_i+n\Sigma}(z_ix^{d/2})$, and with $c = \langle R^1_g, R^1_g\rangle^{-1}$. To compute c, note that $\theta R^1_g = 0$ so

$$\langle R^1_g, R^1_g\rangle = \langle R^1_g, \eta^{d/2}\rangle = \sum_{i=0}^{\alpha} \frac{\binom{d/2}{i}}{i!\binom{g-1}{i}}(-1)^i\frac{g!}{(g-i)!} = \sum_{i=0}^{\alpha}(-1)^i\binom{\alpha}{i}\frac{g}{g-i} = (-1)^\alpha\binom{g-1}{\alpha}^{-1},$$

with $\alpha = d/2$. Finally corollary 4.3 implies that $SW_{\bar{X}_i,\bar{\mathfrak{s}}_i+n\Sigma}(z_ix^{d/2}) = 0$ for any z_i with $\deg(z_i) > 0$. □

The following corollary is analogue to the result in [15, corollary 15] regarding the Kronheimer-Mrowka basic classes.

Corollary 5.8. *Suppose $\bar{X}_1$ is of strong simple type and $\bar{X}_2$ has H_1-simple type. Suppose that the connected sum $X = \bar{X}_1\#_\Sigma\bar{X}_2$ is admissible. Then there are no basic classes $\mathfrak{s}$ with $0 < |c_1(\mathfrak{s})\cdot\Sigma| < 2g-2$. The basic classes for X with $c_1(\mathfrak{s}) = \pm(2g-2)$ are indexed by pairs of basic classes $(\bar{\mathfrak{s}}_1, \bar{\mathfrak{s}}_2)$ for $\bar{X}_1$ and $\bar{X}_2$ respectively, such that $\bar{\mathfrak{s}}_1\cdot\Sigma = \bar{\mathfrak{s}}_2\cdot\Sigma = \pm(2g-2)$.* □

6. Higher type adjunction inequalities

In this section we shall reprove the higher type adjunction inequalities for non-simple type 4-manifolds obtained by Oszváth and Szabó in [21]. Our method of proof is considerable simpler and parallels the proof of the higher type adjunction inequalities in the context of Donaldson invariants given in [19]. On the weak side, we cannot deal with the case $\Sigma^2 = 0$, $c_1(\mathfrak{s})\cdot\Sigma = 0$, due to the fact that we have to restrict to non-torsion $\mathrm{Spin}^{\mathbb{C}}$ structures for our study of the Seiberg-Witten-Floer homology.

Proof of theorem 1.4. Without loss of generality, by reversing the orientation of Σ in the case $b^+ > 1$, we can suppose that $c_1(\mathfrak{s}) \cdot \Sigma \leq 0$. We reduce to the case of self-intersection zero by blowing-up. Let $N = \Sigma^2$ and consider the blow-up $\tilde{X} = X \# N\overline{\mathbb{CP}}^2$ with exceptional divisors $E_1, \ldots, E_N$. Let $\tilde{\Sigma} = \Sigma - E_1 - \cdots - E_N$ be the proper transform of Σ, which is an embedded surface of self-intersection zero and genus g, with $b \in \mathbb{A}(\tilde{\Sigma}) \cong \mathbb{A}(\Sigma)$. Consider the $\mathrm{Spin}^{\mathbb{C}}$ structure $\tilde{\mathfrak{s}}$ on $\tilde{X}$ with $c_1(\tilde{\mathfrak{s}}) = c_1(\mathfrak{s}) - E_1 - \cdots - E_N$. Then $d(\mathfrak{s}) = d(\tilde{\mathfrak{s}})$,

$$-c_1(\tilde{\mathfrak{s}}) \cdot \tilde{\Sigma} + \tilde{\Sigma}^2 = -c_1(\mathfrak{s}) \cdot \Sigma + \Sigma^2,$$

and $SW_{\tilde{X},\tilde{\mathfrak{s}}}(a\, b) = SW_{X,\mathfrak{s}}(a\, b) \neq 0$.

Therefore we can suppose that $\Sigma^2 = 0$ and $c_1(\mathfrak{s}) \cdot \Sigma = -2r$, with $0 < r \leq g-1$. Let $\{\gamma_i\}$ be a symplectic basis of $H_1(\Sigma)$ with $\gamma_i \cdot \gamma_{g+i} = 1$, $1 \leq i \leq g$. Without loss of generality we may also suppose that $b = x^p \gamma_{i_1} \cdots \gamma_{i_m}$, $\deg(b) = 2p + m$. Now let $A = \Sigma \times D^2$ be a small tubular neighbourhood of $\Sigma \subset X$ and consider the splitting $X = X_1 \cup_Y A$, where X_1 is the closure of the complement of A and $\partial X_1 = \partial A = Y = \Sigma \times \mathbb{S}^1$. In this case $\mathfrak{s}$ is the only $\mathrm{Spin}^{\mathbb{C}}$ structure appearing in the gluing formula in theorem 2.2, so

$$0 \neq SW_{X,\mathfrak{s}}(a\, b) = \langle \phi^{SW}_{X_1}(\mathfrak{s}, a), \phi^{SW}_A(\mathfrak{s}_{-r}, b) \rangle.$$

(In the case $b^+ = 1$, the metric giving a long neck for X has period point close to $[\Sigma]$. Therefore we are calculating the invariants in the component $\mathcal{K}(X)$ of the positive cone containing P.D.$[\Sigma] \in H^2(X;\mathbb{Z})$, since $c_1(\mathfrak{s}) \cdot \Sigma < 0$.) Then $\phi^{SW}_A(\mathfrak{s}_{-r}, b) \in HFSW^*(Y, \mathfrak{s}_{-r}) = V_{-r} \cong V_r$ is non-zero and therefore $\eta^p \psi_{i_1} \cdots \psi_{i_m} \neq 0 \in V_r$. By corollary 4.3, this implies $2p + m \leq 2d = 2(g - 1 - |r|)$. Therefore $2r + \deg(b) \leq 2g - 2$. □

Proof of theorem 1.6. Again we may suppose that $\Sigma^2 = 0$ and $c_1(\mathfrak{s}) \cdot \Sigma = -2r$, with $0 < r \leq g-1$. Suppose also that $b = x^p \gamma_{i_1} \cdots \gamma_{i_m}$, $\deg(b) = 2p + m$. Now let $A = \Sigma \times D^2$ be a small tubular neighbourhood of $\Sigma \subset X$ and consider the splitting $X = X_1 \cup_Y A$. Then

$$0 \neq SW_{X,\mathfrak{s}}(a\, b) = \langle \phi^{SW}_{X_1}(\mathfrak{s}, a), \phi^{SW}_A(\mathfrak{s}_{-r}, b) \rangle.$$

Here $\phi^{SW}_{X_1}(\mathfrak{s}, a) \in V_{-r}$ lives in the kernels of $\psi_1, \ldots, \psi_l$, since as $\imath_*(\gamma_j) = 0 \in H_1(X)$,

$$\psi_j \phi^{SW}_{X_1}(\mathfrak{s}, a) = \phi^{SW}_{X_1}(\mathfrak{s}, \gamma_j a) = 0, \qquad j = 1, \ldots, l.$$

Therefore it must be $\phi^{SW}_A(\mathfrak{s}_{-r}, b) = \eta^p \psi_{i_1} \cdots \psi_{i_m} \notin (\psi_1, \ldots, \psi_l)$ in V_{-r}. The argument in [19, proposition 4.5] (using proposition 4.5) shows that any element of degree bigger strictly bigger than $g - 1 - |r|$ must lie in the ideal $(\psi_1, \ldots, \psi_{g-1-|r|})$ of V_{-r}. So if $l \geq g - 1 - |r|$ then $2p + m \leq g - 1 - |r|$, i.e. $\deg(b) \leq g - 1 - |r|$ and $|2r| + 2\deg(b) \leq 2(g-1)$. On the other hand, if $l + 1 \leq g - 1 - |r|$ then obviously $|2r| + 2\deg(b) \leq 2(g-1)$ as $\deg(b) \leq l + 1$ by hypothesis. □

Acknowledgments: First author would like to express his gratitude to the Fakultät für Mathematik and to Prof. Stefan Bauer for their hospitality and support during his stay at Universität Bielefeld when part of this work was carried out. He is specially indebted to Rogier Brussee with whom he discussed many of the ideas that gave rise to this work. Also thanks to Zoltán Szabó and Cliff Taubes for helpful correspondence. Second author's research was supported by the Australian Research Council Fellowship.

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Departamento de Matemáticas, Facultad de Ciencias, Universidad Autónoma de Madrid, Ciudad Universitaria Cantoblanco, 28049 Madrid, Spain
vicente.munoz@uam.es

Max-Planck-Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn, Germany
bwang@mpim-bonn.mpg.de