

**WALL-CROSSING FORMULAE FOR ALGEBRAIC SURFACES
WITH $q > 0$**

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ABSTRACT. We extend the ideas of Friedman and Qin [5] to find the wall-crossing formulae for the Donaldson invariants of algebraic surfaces with $p_g = 0$, $q > 0$ and anticanonical divisor $-K$ effective, for any wall ζ with $l_\zeta = \frac{1}{4}(\zeta^2 - p_1)$ being 0 or 1.

1. INTRODUCTION

The Donaldson invariants of a smooth oriented 4-manifold X depend by definition on a Riemannian metric g . In the case $b^+ > 1$ they however turn out to be independent of g . When $b^+ = 1$, they depend on g through a structure of walls and chambers, that we recall briefly here (we refer to [8] [9] for more details).

Fix $w \in H^2(X; \mathbb{Z})$. Then for any $p_1 \leq 0$ with $p_1 \equiv w^2 \pmod{4}$, we set $d = -p_1 - \frac{3}{2}(1 - b_1 + b^+)$, for half of the dimension of the moduli space $\mathcal{M}_{X,g}^{w,d}$ of g -antiselfdual connections on the $SO(3)$ -principle bundle over X with second Stiefel-Whitney class the reduction mod 2 of w , and first Pontrjagin number p_1 . The corresponding Donaldson invariant will be denoted $D_{X,g}^{w,d}$. This is a linear functional on the elements of degree $2d$ of $\mathbb{A}(X) = \text{Sym}^*(H_0(X) \oplus H_2(X)) \otimes \wedge^*(H_1(X) \oplus H_3(X))$, where the degree of elements in $H_i(X)$ is $4 - i$ ($H_i(X)$ will always denote homology with rational coefficients, and similarly for $H^i(X)$). This invariant is only defined in principle for generic metrics.

From now on let X be a compact smooth oriented 4-manifold with $b^+ = 1$. Let \mathbb{H} be the image of the positive cone $\{x \in H^2(X; \mathbb{R})/x^2 > 0\}$ in $\mathbb{P}(H^2(X; \mathbb{R}))$, which is a model of the hyperbolic disc of dimension b^- . The period point of g is the line $\omega_g \in \mathbb{H} \subset \mathbb{P}(H^2(X; \mathbb{R}))$ given by the selfdual harmonic forms for g . A **wall** of type (w, p_1) is a non-empty hyperplane $W_\zeta = \{x \in \mathbb{H}/x \cdot \zeta = 0\}$ in \mathbb{H} , with $\zeta \in H^2(X; \mathbb{Z})$, such that $\zeta \equiv w \pmod{2}$ and $p_1 \leq \zeta^2 < 0$. The connected components of the complement of the walls of type (w, p_1) in \mathbb{H} are the **chambers** of type (w, p_1) .

Let \mathfrak{M} denote the space of metrics of X . Then we have a map $\mathfrak{M} \rightarrow \mathbb{H}$ which sends every metric g to its period point ω_g . The connected components of the preimage of the chambers of \mathbb{H} are, by definition, the chambers of \mathfrak{M} . A wall W'_ζ for \mathfrak{M} is a non-empty preimage of a wall W_ζ for \mathbb{H} . When g moves in a chamber \mathcal{C}' of \mathfrak{M} the Donaldson invariants do not change. But when it crosses a wall they change. So for any chamber \mathcal{C}' of \mathfrak{M} , we have defined $D_{X,\mathcal{C}'}^{w,d}$, by choosing any generic metric $g \in \mathcal{C}'$, so that the moduli space is smooth, and computing the corresponding Donaldson invariants (to avoid flat connections we might have to use the trick in [12]). For a path of metrics $\{g_t\}_{t \in [-1,1]}$, with $g_{\pm 1} \in \mathcal{C}'_{\pm}$, we have the difference term $\delta_X^{w,d}(\mathcal{C}'_-, \mathcal{C}'_+) = D_{X,\mathcal{C}'_+}^{w,d} - D_{X,\mathcal{C}'_-}^{w,d}$.

When $b_1 = 0$, Kotschick and Morgan [9] prove that the invariants only depend on the chamber \mathcal{C} of \mathbb{H} in which the period point of the metric lies. For this, they find that the change in the Donaldson invariant when the metric crosses a wall W'_ζ depends only on the class ζ and not on the particular metric having the reducible antiselfdual connection (Leness [11] points out that their argument is not complete and checks that it is true at least for the case $l_\zeta = \frac{1}{4}(\zeta^2 - p_1) \leq 2$). In this case, the difference term is defined as

$$\delta_X^{w,d}(\mathcal{C}_-, \mathcal{C}_+) = D_{X,\mathcal{C}_+}^{w,d} - D_{X,\mathcal{C}_-}^{w,d},$$

for chambers \mathcal{C}_{\pm} of \mathbb{H} . Then $\delta_X^{w,d}(\mathcal{C}_-, \mathcal{C}_+) = \sum \delta_{S,\zeta}^{w,d}$, where the sum is taken over all ζ defining walls separating \mathcal{C}_- and \mathcal{C}_+ . Moreover $\delta_{S,\zeta}^{w,d} = \varepsilon(\zeta, w) \delta_{S,\zeta}^d$, with $\delta_{S,\zeta}^d$ not dependent on w , $\varepsilon(\zeta, w) = (-1)^{\binom{\zeta-w}{2}}$.

Now suppose S is a smooth algebraic surface (not necessarily with $b_1 = 0$), endowed with a Hodge metric h corresponding to a polarisation H . Let $\mathfrak{M}_H(c_1, c_2)$ be the Gieseker compactification of the moduli space of H -stable rank two bundles V on X with $c_1(V) = \mathcal{O}(L)$ (a fixed line bundle with topological first Chern class equal to w) and $c_2 = \frac{1}{4}(c_1^2 - p_1)$. The Donaldson invariants (for the metric h) can be computed using $\mathfrak{M}_H(c_1, c_2)$ (see [4]) whenever the moduli spaces $\mathfrak{M}_H(c_1, c_2)$ are generic (i.e. $H^0(\text{End}_0 E) = H^2(\text{End}_0 E) = 0$, for every stable bundle $E \in \mathfrak{M}_H(c_1, c_2)$). The period point of h is the line spanned by $H \in H^2(X; \mathbb{Z}) \subset H^2(X; \mathbb{R})$. Now let $C_S \subset \mathbb{H}$ be the image of the **ample cone** of S , i.e. the subcone of the positive cone generated by the ample classes (polarisations). We have walls and chambers in C_S in the same vein as before (actually they are the intersections of the walls and chambers of \mathbb{H} with C_S , whenever this intersection is non-empty). Now $\mathfrak{M}_H(c_1, c_2)$ is constant on the chambers of C_S (and so the invariant stays the same), and when H crosses a wall W_ζ , $\mathfrak{M}_H(c_1, c_2)$ changes (see [13]). From the point of view of the Donaldson invariants, this corresponds to restricting our attention from the positive cone of S to its ample cone.

When the irregularity q of S is zero, the wall-crossing terms have been found

out in [5] [3] [2]. In [5] Friedman and Qin obtain some wall-crossing formulae for algebraic surfaces S with $-K$ being effective ($K = K_S$ the canonical divisor) and the irregularity $q = 0$ (equivalently, $b_1 = 0$). We want to adapt their results to the case $q > 0$ modifying their arguments where necessary. If $-K$ is effective then the change of $\mathfrak{M}_H(c_1, c_2)$ when H crosses a wall W can be described by a number of flips. We shall write the change of the Donaldson invariant as a sum of contributions $\delta_{S, \zeta}^{w, d}$, for the different ζ defining W .

Remark 1. The condition of $-K$ being effective can be relaxed for the case $q = 0$ to the following two conditions: K is not effective, $\pm\zeta + K$ are not effective for any ζ defining the given wall (we call such a wall a **good wall**, see [2] [3]). Probably the same is true for the case $q > 0$, since these two conditions ensure that the change in $\mathfrak{M}_H(c_1, c_2)$ when crossing a wall is described by flips. Nonetheless we will suppose $-K$ effective, which allows us to define the Donaldson invariants for any polarisation. Note that when $-K$ is effective, all walls are good.

The paper is organised as follows. In section 2 we extend the arguments of [5] to the case $q > 0$. In sections 3 and 4 we compute the wall crossing formulae for any wall with $l_\zeta = \frac{1}{4}(\zeta^2 - p_1)$ being 0 and 1 respectively. Then in section 5, we give the two leading terms of the wall crossing difference for any wall ζ . As a consequence of our results, we propose a conjecture on the shape of the wall crossing terms. In the appendix we give, for the convenience of the reader, a list of all the algebraic surfaces with $p_g = 0$ and $-K$ effective, i.e. the surfaces to which the results from this paper apply.

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Note: After the completion of this work, L. Göttsche provided me with a copy of [3]. The arguments for computing the wall-crossing terms in [3] can also be extended to the case $q > 0$, in a similar fashion to the work carried out in this paper.

2. WALL-CROSSING FORMULAE

From now on, S is a smooth algebraic manifold with irregularity $q \geq 0$ and $p_g = 0$ (equivalently $b^+ = 1$) and with anticanonical divisor $-K$ effective. Let $w \in H^2(S; \mathbb{Z})$, $p_1 \equiv w^2 \pmod{4}$. Put

$$d = -p_1 - \frac{3}{2}(1 - b_1 + b^+) = -p_1 - 3(1 - q)$$

and let ζ define a wall of type (w, p_1) . In every chamber \mathcal{C} of the ample cone, we have well-defined the Donaldson invariant $D_{S, \mathcal{C}}^{w, d}$ associated to polarisations in that chamber.

For two different chambers \mathcal{C}_+ and \mathcal{C}_- , there is a **wall-crossing difference term**

$$\delta_S^{w,d}(\mathcal{C}_-, \mathcal{C}_+) = D_{S, \mathcal{C}_+}^{w,d} - D_{S, \mathcal{C}_-}^{w,d},$$

which can be written as a sum

$$\delta_S^{w,d}(\mathcal{C}_-, \mathcal{C}_+) = \sum_{\zeta} \delta_{S, \zeta}^{w,d},$$

where ζ runs over all walls of type (w, p_1) with $\mathcal{C}_- \cdot \zeta < 0 < \mathcal{C}_+ \cdot \zeta$.

Suppose from now on that \mathcal{C}_- and \mathcal{C}_+ are two adjacent chambers separated by a single wall W_ζ of type (w, p_1) . For simplicity, we will assume that the wall W_ζ is only represented by the pair $\pm\zeta$ since in the general case we only need to add up the contributions for every pair representing the wall. Then the wall-crossing term is $\delta_{S, \zeta}^{w,d}$. Set

$$l_\zeta = (\zeta^2 - p_1)/4 \in \mathbb{Z}.$$

Let ζ define the wall separating \mathcal{C}_- from \mathcal{C}_+ and put, as in [5, section 2], $E_\zeta^{n_1, n_2}$ to be the set of all isomorphism classes of non-split extensions of the form

$$0 \rightarrow \mathcal{O}(F) \otimes I_{Z_1} \rightarrow V \rightarrow \mathcal{O}(L - F) \otimes I_{Z_2} \rightarrow 0,$$

where F is a divisor such that $2F - L$ is homologically equivalent to ζ , and Z_1 and Z_2 are two zero-dimensional subschemes of S with $l(Z_i) = n_i$ and such that $n_1 + n_2 = l_\zeta$. Let us construct $E_\zeta^{n_1, n_2}$ explicitly. Consider $H_i = \text{Hilb}_{n_i}(S)$ the Hilbert scheme of n_i points on S , $J = \text{Jac}^F(S)$ the Jacobian parametrising divisors homologically equivalent to F , $\mathcal{Z}_i \subset S \times H_i$ the universal codimension 2 scheme, and $\mathcal{F} \subset S \times J$ the universal divisor. Then we define $\mathcal{E}_\zeta^{n_1, n_2} \rightarrow J \times H_1 \times H_2$ to be

$$\mathcal{E} = \mathcal{E}_\zeta^{n_1, n_2} = \mathcal{E}\text{xt}_{\pi_2}^1(\mathcal{O}_{S \times (J \times H_1 \times H_2)}(\pi_1^* L - \mathcal{F}) \otimes I_{\mathcal{Z}_2}, \mathcal{O}_{S \times (J \times H_1 \times H_2)}(\mathcal{F}) \otimes I_{\mathcal{Z}_1}),$$

for $\pi_1 : S \times (J \times H_1 \times H_2) \rightarrow S$ and $\pi_2 : S \times (J \times H_1 \times H_2) \rightarrow J \times H_1 \times H_2$, the projections (we do not denote all pull-backs of sheaves explicitly). This is a vector bundle over $J \times H_1 \times H_2$ of rank

$$\text{rk}(\mathcal{E}) = l_\zeta + h^1(\mathcal{O}_S(2F - L)) = l_\zeta + h(\zeta) + q,$$

where

$$h(\zeta) = \frac{\zeta \cdot K_S}{2} - \frac{\zeta^2}{2} - 1,$$

by Riemann-Roch [5, lemma 2.6]. Note that $l_\zeta \geq 0$ and $h(\zeta) + q \geq 0$. Put $N_\zeta = \text{rk}(\mathcal{E}) - 1$. Then $E_\zeta^{n_1, n_2} = \mathbb{P}((\mathcal{E}_\zeta^{n_1, n_2})^\vee)$ (we follow the convention $\mathbb{P}(\mathcal{E}) = \text{Proj}(\oplus_i S^i(\mathcal{E}))$), which is of dimension $q + 2l_\zeta + (l_\zeta + h(\zeta) + q)$. Also $N_\zeta + N_{-\zeta} + q + 2l_\zeta = d - 1$. We will have to treat the case $\text{rk}(\mathcal{E}) = 0$ (i.e. $l_\zeta = 0$ and $h(\zeta) + q = 0$) separately.

We can modify the arguments in sections 3 and 4 of [5] to get intermediate moduli spaces $\mathfrak{M}_0^{(k)}$ together with embeddings $E_\zeta^{l_\zeta-k,k} \hookrightarrow \mathfrak{M}_0^{(k)}$ and $E_{-\zeta}^{k,l_\zeta-k} \hookrightarrow \mathfrak{M}_0^{(k-1)}$, fitting in the following diagram

$$\begin{array}{ccccccc}
 & & \widetilde{\mathfrak{M}}_0^{(l_\zeta)} & & \dots & & \widetilde{\mathfrak{M}}_0^{(0)} & & \\
 & \swarrow & & \searrow & & \swarrow & & \searrow & \\
 \mathfrak{M}_0^{(l_\zeta)} & & & & \mathfrak{M}_0^{(l_\zeta-1)} & & & & \mathfrak{M}_0^{(0)} & & & & & \mathfrak{M}_0^{(-1)} \\
 \parallel & & & & & & & & & & & & \parallel & \\
 \mathfrak{M}_- & & & & & & & & & & & & \mathfrak{M}_+ &
 \end{array}$$

where $\widetilde{\mathfrak{M}}_0^{(k)} \rightarrow \mathfrak{M}_0^{(k)}$ is the blow-up of $\mathfrak{M}_0^{(k)}$ at $E_\zeta^{l_\zeta-k,k}$ and $\widetilde{\mathfrak{M}}_0^{(k)} \rightarrow \mathfrak{M}_0^{(k-1)}$ is the blow-up of $\mathfrak{M}_0^{(k-1)}$ at $E_{-\zeta}^{k,l_\zeta-k}$. This is what is called a flip. Basically, the space $E_\zeta = \sqcup E_\zeta^{l_\zeta-k,k}$ parametrises H_- -stable sheaves which are H_+ -unstable. Analogously, $E_{-\zeta} = \sqcup E_{-\zeta}^{k,l_\zeta-k}$ parametrises H_+ -stable sheaves which are H_- -unstable. Hence one could say that \mathfrak{M}_+ is obtained from \mathfrak{M}_- by removing E_ζ and then attaching $E_{-\zeta}$. The picture above is a nice description of this fact and allows us to find the universal sheaf for \mathfrak{M}_+ out of the universal sheaf for \mathfrak{M}_- by a sequence of elementary transforms.

The point is that whenever $-K_S$ is effective, we have an embedding $E_\zeta^{0,l_\zeta} \rightarrow \mathfrak{M}_-$ (the part of E_ζ consisting of bundles) and rational maps $E_\zeta^{k,l_\zeta-k} \dashrightarrow \mathfrak{M}_-$, $k > 0$, but if we blow-up \mathfrak{M}_- at E_ζ^{0,l_ζ} , we have already an embedding from $E_\zeta^{1,l_\zeta-1}$ to this latter space. Now we can proceed inductively for $k = 0, \dots, l_\zeta$. Analogously, we can have started from \mathfrak{M}_+ blowing-up $E_{-\zeta}^{k,l_\zeta-k}$ one by one. The diagram above says that we can perform these blow-ups and blow-downs alternatively, instead of first blowing-up $l_\zeta + 1$ times and then blowing-down $l_\zeta + 1$ times. We see that the exceptional divisor in $\widetilde{\mathfrak{M}}_0^{(k)}$ is a $\mathbb{P}^{N_\zeta} \times \mathbb{P}^{N_\zeta-k}$ -bundle over $J \times H_{l_\zeta-k} \times H_k$.

When adapting the arguments of [5, sections 3 and 4], the only place requiring serious changes is proposition 3.7 in order to prove proposition 3.6.

Proposition 2. ([5, proposition 3.6]) *The map $E_\zeta^{l_\zeta-k,k} \rightarrow \mathfrak{M}_0^{(\zeta,k)}$ is an immersion. The normal bundle $\mathcal{N}_\zeta^{l_\zeta-k,k}$ to $E_\zeta^{l_\zeta-k,k}$ in $\mathfrak{M}_0^{(\zeta,k)}$ is exactly $\rho^* \mathcal{E}_{-\zeta}^{k,l_\zeta-k} \otimes \mathcal{O}_{E_\zeta^{l_\zeta-k,k}}(-1)$, where $\rho : E_\zeta^{l_\zeta-k,k} \rightarrow J \times H_{l_\zeta-k} \times H_k$ is the projection. Here we have defined $\mathcal{E}_{-\zeta}^{k,l_\zeta-k} = \mathcal{E}xt_{\pi_2}^1(\mathcal{O}_{S \times (J \times H_1 \times H_2)}(\mathcal{F}) \otimes I_{Z_1}, \mathcal{O}_{S \times (J \times H_1 \times H_2)}(\pi_1^* L - \mathcal{F}) \otimes I_{Z_2})$.*

Proposition 2 is proved as [5, proposition 3.6] making use of the following analogue of [5, proposition 3.7]

Proposition 3. *For all nonzero $\xi \in \text{Ext}^1 = \text{Ext}^1(\mathcal{O}(L - F) \otimes I_{Z_2}, \mathcal{O}(F) \otimes I_{Z_1})$, the natural map from a neighbourhood of ξ in $E_\zeta^{l_\zeta-k,k}$ to $\mathfrak{M}_0^{(\zeta,k)}$ is an immersion at ξ .*

The image of $T_\xi E_\zeta^{l_\zeta-k, k}$ in $\text{Ext}_0^1(V, V)$ (the tangent space to $\mathfrak{M}_0^{(\zeta, k)}$ at ξ , where V is the sheaf corresponding to ξ) is exactly the kernel of the natural map $\text{Ext}_0^1(V, V) \rightarrow \text{Ext}^1(\mathcal{O}(F) \otimes I_{Z_1}, \mathcal{O}(L-F) \otimes I_{Z_2})$, and the normal space to $E_\zeta^{l_\zeta-k, k}$ at ξ in $\mathfrak{M}_0^{(\zeta, k)}$ may be canonically identified with $\text{Ext}^1(\mathcal{O}(F) \otimes I_{Z_1}, \mathcal{O}(L-F) \otimes I_{Z_2})$.

Proof. We have that $\text{Ext}^1(I_Z, I_Z)$ parametrises infinitesimal deformations of I_Z as a sheaf. The deformations of I_Z are of the form $I_{Z'} \otimes \mathcal{O}(D)$ for $D \equiv 0$. The universal space parametrising these sheaves is $\text{Hilb}_r(S) \times \text{Jac}^0(S)$, where r is the length of Z . There is an exact sequence

$$0 \rightarrow H^0(\mathcal{E}\text{xt}^1(I_Z, I_Z)) \rightarrow \text{Ext}^1(I_Z, I_Z) \rightarrow H^1(\mathcal{H}\text{om}(I_Z, I_Z)) \rightarrow 0,$$

where $H^0(\mathcal{E}\text{xt}^1(I_Z, I_Z)) = H^0(\mathcal{H}\text{om}(I_Z, \mathcal{O}_Z)) = \text{Hom}(I_Z, \mathcal{O}_Z)$ is the tangent space to $\text{Hilb}_r(S)$ and $H^1(\mathcal{H}\text{om}(I_Z, I_Z)) = H^1(\mathcal{O})$ is the tangent space to the Jacobian. Analogously, $\text{Ext}^1(V, V)$ is the space of infinitesimal deformations of V (but the determinant is not preserved). The infinitesimal deformations preserving the determinant are given by the kernel $\text{Ext}_0^1(V, V)$ of a map $\text{Ext}^1(V, V) \rightarrow H^1(\mathcal{H}\text{om}(V, V)) \rightarrow H^1(\mathcal{O})$. Now $E = E_\zeta^{l_\zeta-k, k}$ sits inside the bigger space $\tilde{E} = \tilde{E}_\zeta^{l_\zeta-k, k}$ given as

$$\mathbb{P}(\mathcal{E}\text{xt}_{\pi_2}^1(\mathcal{O}_{S \times (J_1 \times H_1 \times J_2 \times H_2)}(\pi_1^* L - \mathcal{F}_2) \otimes I_{Z_2}, \mathcal{O}_{S \times (J_1 \times H_1 \times J_2 \times H_2)}(\mathcal{F}_1) \otimes I_{Z_1})^\vee),$$

for $J_1 = J_2 = J$, $\mathcal{F}_i \subset S \times J_i$ the universal divisor, and H_i the Hilbert scheme parametrising Z_i . The arguments in [5, proposition 3.7] go through to prove that for every non-zero $\xi \in \text{Ext}^1 = \text{Ext}^1(\mathcal{O}(L-F) \otimes I_{Z_2}, \mathcal{O}(F) \otimes I_{Z_1})$ we have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccc} T_\xi E & \longrightarrow & \text{Ext}_0^1(V, V) & \longrightarrow & \text{Ext}^1(\mathcal{O}(F) \otimes I_{Z_1}, \mathcal{O}(L-F) \otimes I_{Z_2}) \\ \downarrow & & \downarrow & & \parallel \\ T_\xi \tilde{E} & \longrightarrow & \text{Ext}^1(V, V) & \longrightarrow & \text{Ext}^1(\mathcal{O}(F) \otimes I_{Z_1}, \mathcal{O}(L-F) \otimes I_{Z_2}) \\ \downarrow & & \downarrow & & \\ H^1(\mathcal{O}) & \longequal{\quad} & H^1(\mathcal{O}) & & \end{array}$$

So the natural map from a neighbourhood of ξ in E to $\mathfrak{M}_0^{(\zeta, k)}$ is an immersion at ξ and the normal space may be canonically identified with $\text{Ext}^1(\mathcal{O}(F) \otimes I_{Z_1}, \mathcal{O}(L-F) \otimes I_{Z_2})$. \square

Therefore proposition 2 is true for $q > 0$. The set up is now in all ways analogous to that of [5]. We fix some notations [5, section 5]:

Notation 4. Let ζ define a wall of type (w, p_1) .

- λ_k is the tautological line bundle over $E_\zeta^{l_\zeta-k,k} = \mathbb{P}((\mathcal{E}_\zeta^{l_\zeta-k,k})^\vee)$. λ_k will also be used to denote its first Chern class.
- $\rho_k : S \times E_\zeta^{l_\zeta-k,k} \rightarrow S \times (J \times H_{l_\zeta-k} \times H_k)$ is the natural projection.
- $p_k : \widetilde{\mathfrak{M}}_0^{(k)} \rightarrow \mathfrak{M}_0^{(k)}$ is the blow-up of $\mathfrak{M}_0^{(k)}$ at $E_\zeta^{l_\zeta-k,k}$.
- $q_{k-1} : \widetilde{\mathfrak{M}}_0^{(k)} \rightarrow \mathfrak{M}_0^{(k-1)}$ is the contraction of $\widetilde{\mathfrak{M}}_0^{(k)}$ to $\mathfrak{M}_0^{(k-1)}$.
- The normal bundle of $E_\zeta^{l_\zeta-k,k}$ in $\mathfrak{M}_0^{(k)}$ is $\mathcal{N}_k = \rho_k^* \mathcal{E}_{-\zeta}^{k,l_\zeta-k} \otimes \lambda_k^{-1}$, where $\mathcal{E}_{-\zeta}^{k,l_\zeta-k} = \mathcal{E}xt_{\pi_2}^1(\mathcal{O}_{S \times (J \times H_1 \times H_2)}(\mathcal{F}) \otimes I_{\mathcal{Z}_1}, \mathcal{O}_{S \times (J \times H_1 \times H_2)}(\pi_1^* L - \mathcal{F}) \otimes I_{\mathcal{Z}_2})$.
- $D_k = \mathbb{P}(\mathcal{N}_k^\vee)$ is the exceptional divisor in $\widetilde{\mathfrak{M}}_0^{(k)}$.
- $\xi_k = \mathcal{O}_{\widetilde{\mathfrak{M}}_0^{(k)}}(-D_k)|_{D_k}$ is the tautological line bundle on D_k .
- $\mu^{(k)}(\alpha) = -\frac{1}{4}p_1(\mathfrak{g}_{\mathcal{U}^{(k)}})/\alpha$, for $\alpha \in H_2(S; \mathbb{Z})$ and $\mathcal{U}^{(k)}$ a universal sheaf over $S \times \mathfrak{M}_0^{(k)}$. Let $\mu^{(l_\zeta)}(\alpha) = \mu_-(\alpha)$ and $\mu^{(-1)}(\alpha) = \mu_+(\alpha)$.
- Let $z = x^r \alpha^s \gamma_1 \cdots \gamma_a A_1 \cdots A_b$ be any element in $\mathbb{A}(S)$, where $x \in H_0(S; \mathbb{Z})$ is the generator of the 0-homology, $\gamma_i \in H_1(S; \mathbb{Z})$, $\alpha \in H_2(S; \mathbb{Z})$, $A_i \in H_3(S; \mathbb{Z})$. Then we define $\mu^{(k)}(z)$ as $\mu^{(k)}(x)^r \mu^{(k)}(\alpha)^s \mu^{(k)}(\gamma_1) \cdots \mu^{(k)}(\gamma_a) \mu^{(k)}(A_1) \cdots \mu^{(k)}(A_b)$.

Although $\mathcal{U}^{(k)}$ might not exist, there is always a well-defined element $p_1(\mathfrak{g}_{\mathcal{U}^{(k)}})$. As in [5], we are using the natural complex orientations for the moduli spaces. These differ from the natural ones used in the definition of the Donaldson invariants by a factor $\epsilon_S(w) = (-1)^{\frac{K \cdot w + w^2}{2}}$. The analogues of lemma 5.2 and lemma 5.3 of [5] are

Lemma 5. *Let $\gamma \in H_1(S; \mathbb{Z})$, $\alpha \in H_2(S; \mathbb{Z})$, $A \in H_3(S; \mathbb{Z})$. Put $a = (\zeta \cdot \alpha)/2$. Then*

$$\begin{cases} p_k^* \mu^{(k)}(\alpha)|_{D_k} = (p_k|_{D_k})^* \left([\mathcal{Z}_{l_\zeta-k}]/\alpha + [\mathcal{Z}_k]/\alpha - a\lambda_k - c_1(\mathcal{F})^2/\alpha \right) \\ p_k^* \mu^{(k)}(\gamma)|_{D_k} = (p_k|_{D_k})^* \left([\mathcal{Z}_{l_\zeta-k}]/\gamma + [\mathcal{Z}_k]/\gamma - \lambda_k(c_1(\mathcal{F})/\gamma) \right) \\ p_k^* \mu^{(k)}(A)|_{D_k} = (p_k|_{D_k})^* \left([\mathcal{Z}_{l_\zeta-k}]/A + [\mathcal{Z}_k]/A - (\zeta c_1(\mathcal{F}))/A \right) \\ p_k^* \mu^{(k)}(x)|_{D_k} = (p_k|_{D_k})^* \left([\mathcal{Z}_{l_\zeta-k}]/x + [\mathcal{Z}_k]/x - \frac{1}{4}\lambda_k^2 \right) \end{cases}$$

Lemma 6. *Let $\gamma \in H_1(S; \mathbb{Z})$, $\alpha \in H_2(S; \mathbb{Z})$, $A \in H_3(S; \mathbb{Z})$. Put $a = (\zeta \cdot \alpha)/2$. Then*

$$\begin{cases} q_{k-1}^* \mu^{(k-1)}(\alpha) = p_k^* \mu^{(k)}(\alpha) - aD_k \\ q_{k-1}^* \mu^{(k-1)}(\gamma) = p_k^* \mu^{(k)}(\gamma) - (c_1(\mathcal{F})/\gamma)D_k \\ q_{k-1}^* \mu^{(k-1)}(A) = p_k^* \mu^{(k)}(A) \\ q_{k-1}^* \mu^{(k-1)}(x) = p_k^* \mu^{(k)}(x) - \frac{1}{4}(D_k^2 + 2\lambda_k D_k) \end{cases}$$

We immediately see that it is important to understand the cohomology classes $e_\alpha = c_1(\mathcal{F})^2/\alpha$, $e_\gamma = c_1(\mathcal{F})/\gamma$, and $e_S = c_1(\mathcal{F})^4/[S]$. We write $c_1(\mathcal{F}) = c_1(F) + \sum \beta_i \otimes \beta_i^\#$, the Künneth decomposition of $c_1(\mathcal{F}) \in H^2(S \times J)$, where $\{\beta_i\}$ is a basis for $H^1(S)$ and

$\{\beta_i^\#\}$ is the dual basis for $H^1(J) \cong H^1(S)^*$. Now we have more explicit expressions

$$(1) \quad \begin{cases} e_\alpha = -2 \sum_{i < j} \langle \beta_i \wedge \beta_j, \alpha \rangle \beta_i^\# \wedge \beta_j^\# \in H^2(J) \\ e_\gamma = \sum \langle \gamma, \beta_i \rangle \beta_i^\# \in H^1(J) \\ e_{\zeta A} = (\zeta c_1(\mathcal{F}))/A = \sum \langle \text{P.D.}[A] \wedge \beta_i, \zeta \rangle \beta_i^\# \in H^1(J) \\ e_S = \sum_{i,j,k,l} \langle \beta_i \wedge \beta_j \wedge \beta_k \wedge \beta_l, [S] \rangle \beta_i^\# \wedge \beta_j^\# \wedge \beta_k^\# \wedge \beta_l^\# \in H^4(J) \end{cases}$$

Theorem 7. *Let ζ define a wall of type (w, p_1) and $d = -p_1 - 3(1 - q)$. Suppose $l_\zeta + h(\zeta) + q > 0$. For $\alpha \in H_2(S; \mathbb{Z})$, put $a = (\zeta \cdot \alpha)/2$. Let $z = x^r \alpha^s \gamma_1 \cdots \gamma_a A_1 \cdots A_b \in \mathbb{A}(S)$ be of degree $2d$. Then $\delta_{S, \zeta}^{w, d}(\alpha)$ is $\epsilon_S(w)$ times*

$$\sum_{0 \leq k \leq l_\zeta} ([\mathcal{Z}_{l_\zeta - k}]/x + [\mathcal{Z}_k]/x - \frac{1}{4} X^2)^r ([\mathcal{Z}_{l_\zeta - k}]/\alpha + [\mathcal{Z}_k]/\alpha - e_\alpha + aX)^s ([\mathcal{Z}_{l_\zeta - k}]/\gamma_1 + [\mathcal{Z}_k]/\gamma_1 + e_{\gamma_1} X) \cdots ([\mathcal{Z}_{l_\zeta - k}]/A_b + [\mathcal{Z}_k]/A_b - e_{\zeta A_b})$$

where $X^N = (-1)^{N - N_\zeta} s_{N-1-N_\zeta-N_{-\zeta}}(\mathcal{E}_\zeta^{l_\zeta - k, k} \oplus (\mathcal{E}_{-\zeta}^{k, l_\zeta - k})^\vee)$, $s_i(\cdot)$ standing for the Segre class.

Proof. By lemma 6, $\mu^{(k-1)}(z)$ is equal to (we omit the pull-backs)

$$(\mu^{(k)}(x) - \frac{1}{4}(D_k^2 + 2\lambda_k D_k))^r (\mu^{(k)}(\alpha) - aD_k)^s (\mu^{(k)}(\gamma_1) - e_{\gamma_1} D_k) \cdots \mu^{(k)}(A_b)$$

which is $\mu^{(k)}(z)$ plus things containing at least one D_k . So $\mu^{(k-1)}(z) = \mu^{(k)}(z) + D_k \cdot s$, where s is formally (recall $\xi_k = -D_k|_{D_k}$)

$$\frac{1}{-\xi_k} \left((\mu^{(k)}(x)|_{D_k} - \frac{1}{4}(\xi_k^2 - 2\lambda_k \xi_k))^r (\mu^{(k)}(\alpha)|_{D_k} + a\xi_k)^s (\mu^{(k)}(\gamma_1)|_{D_k} + e_{\gamma_1} \xi_k) \cdots \cdots (\mu^{(k)}(\gamma_a)|_{D_k} + e_{\gamma_1} \xi_k) \mu^{(k)}(A_1)|_{D_k} \cdots \mu^{(k)}(A_b)|_{D_k} \right)_0$$

where the subindex 0 means “forgetting anything not containing at least one ξ_k ”. So s is (we drop the subindices)

$$-\frac{1}{\xi} \left(([\mathcal{Z}]/x + [\mathcal{Z}]/x - \frac{1}{4}(\xi - \lambda)^2)^r ([\mathcal{Z}]/\alpha + [\mathcal{Z}]/\alpha - e_\alpha + a(\xi - \lambda))^s ([\mathcal{Z}]/\gamma_1 + [\mathcal{Z}]/\gamma_1 + e_{\gamma_1}(\xi - \lambda)) \cdots ([\mathcal{Z}]/A_b + [\mathcal{Z}]/A_b - e_{\zeta A_b}) \right)_0$$

We need the easy formula (which can be proved by induction)

$$\frac{1}{\xi} \left((\xi - \lambda)^N \right)_0 = \frac{(\xi - \lambda)^N - (-\lambda)^N}{\xi} = \sum_{i=0}^{N-1} (-\lambda)^i (\xi - \lambda)^{N-i-1}$$

As $\xi - \lambda$ is the tautological bundle corresponding to $\mathcal{E}_{-\zeta}^{k, l_\zeta - k}$ (see items 5 to 7 in notation 4), we have

$$\begin{cases} \lambda^u = s_{u-N_\zeta}(\mathcal{E}_\zeta^{l_\zeta - k, k}) \cdot \lambda^{N_\zeta} + O(\lambda^{N_\zeta - 1}) \\ (\xi - \lambda)^u = s_{u-N_\zeta}(\mathcal{E}_{-\zeta}^{k, l_\zeta - k}) \cdot (\xi - \lambda)^{N_\zeta} + O((\xi - \lambda)^{N_\zeta - 1}) \end{cases}$$

Evaluating (and doing the sum from $k = 0$ to $k = l_\zeta$) we get the statement of the theorem where

$$\begin{aligned} X^N &= - \sum (-1)^i s_{i-N_\zeta}(\mathcal{E}_\zeta^{l_\zeta - k, k}) \cdot s_{N-i-1-N_\zeta}(\mathcal{E}_{-\zeta}^{k, l_\zeta - k}) = \\ &= \sum (-1)^{N-N_\zeta} s_{i-N_\zeta}(\mathcal{E}_\zeta^{l_\zeta - k, k}) \cdot s_{N-i-1-N_\zeta}((\mathcal{E}_{-\zeta}^{k, l_\zeta - k})^\vee) = \\ &= (-1)^{N-N_\zeta} s_{N-1-N_\zeta-N_\zeta}(\mathcal{E}_\zeta^{l_\zeta - k, k} \oplus (\mathcal{E}_{-\zeta}^{k, l_\zeta - k})^\vee). \end{aligned}$$

□

An immediate corollary which generalises [5, theorem 5.4] is

Corollary 8. *Let ζ define a wall of type (w, p_1) and $d = -p_1 - 3(1 - q)$. Suppose $l_\zeta + h(\zeta) + q > 0$. For $\alpha \in H_2(S; \mathbb{Z})$, put $a = (\zeta \cdot \alpha)/2$. Then $\mu_+(\alpha^d) - \mu_-(\alpha^d)$ is equal to*

$$\sum (-1)^{h(\zeta) + l_\zeta + j} \frac{d!}{j!b!(d-j-b)!} a^{d-j-b} ([\mathcal{Z}_{l_\zeta - k}]/\alpha + [\mathcal{Z}_k]/\alpha)^j \cdot e_\alpha^b \cdot s_{2l_\zeta - j + q - b}(\mathcal{E}_\zeta^{l_\zeta - k, k} \oplus (\mathcal{E}_{-\zeta}^{k, l_\zeta - k})^\vee),$$

where the sum runs through $0 \leq j \leq 2l_\zeta$, $0 \leq b \leq q$, $0 \leq k \leq l_\zeta$. As $\mu_+(\alpha^d) - \mu_-(\alpha^d)$ is computed using the complex orientation, we have that $\delta_{S, \zeta}^{w, d}(\alpha^d) = \epsilon_S(w)(\mu_+(\alpha^d) - \mu_-(\alpha^d))$.

Remark 9. In Kotschick notation [8], $\varepsilon(\zeta, w) = (-1)^{\binom{\zeta - w}{2}}$. So $\epsilon_S(w)(-1)^{h(\zeta)} = (-1)^{d+q} \varepsilon(\zeta, w)$.

Theorem 10. *Let ζ define a wall of type (w, p_1) and $d = -p_1 - 3(1 - q)$. Suppose $l_\zeta + h(\zeta) + q = 0$ i.e. $l_\zeta = 0$ and $h(\zeta) + q = 0$. For $\alpha \in H_2(S; \mathbb{Z})$, put $a = (\zeta \cdot \alpha)/2$. Let $z = x^r \alpha^s \gamma_1 \cdots \gamma_a A_1 \cdots A_b \in \mathbb{A}(S)$ be of degree $2d$. Then $\delta_{S, \zeta}^{w, d}(\alpha)$ is $\epsilon_S(w)$ times*

$$\left(-\frac{1}{4}X^2\right)^r (-e_\alpha + aX)^s (e_{\gamma_1} X) \cdots (-e_{\zeta A_b})$$

where $X^N = s_{N-N_\zeta}(\mathcal{E}_{-\zeta}^{0,0}) = (-1)^{N-N_\zeta} s_{N-1-N_\zeta-N_\zeta}(\mathcal{E}_\zeta^{0,0} \oplus (\mathcal{E}_{-\zeta}^{0,0})^\vee)$.

Proof. Now \mathfrak{M}_+ is \mathfrak{M}_- with an additional connected component $D = E_{-\zeta}^{0,0}$ which is a \mathbb{P}^{d-q} -bundle over J , since $E_\zeta^{0,0} = \emptyset$. The universal bundle over $E_{-\zeta}^{0,0}$ is given by an extension

$$0 \rightarrow \pi^* \mathcal{O}_{S \times J}(\pi_1^* L - \mathcal{F}) \otimes p^* \lambda \rightarrow \mathcal{U} \rightarrow \pi^* \mathcal{O}_{S \times J}(\mathcal{F}) \rightarrow 0,$$

where $\pi : S \times E_{-\zeta}^{0,0} \rightarrow S \times J$ and $p : S \times E_{-\zeta}^{0,0} \rightarrow E_{-\zeta}^{0,0}$ are projections and λ is the tautological line bundle. From this

$$\begin{cases} \mu(\alpha)|_D = a\lambda - e_\alpha \\ \mu(\gamma)|_D = \lambda e_\gamma \\ \mu(A)|_D = -e_{\zeta A} \\ \mu(x)|_D = -\frac{1}{4}\lambda^2 \end{cases}$$

with notations as in theorem 7. As in the proof of theorem 7, $\lambda^u = s_{u-N_{-\zeta}}(\mathcal{E}_{-\zeta}^{0,0}) \cdot \lambda^{-N_{\zeta}} + O(\lambda^{-N_{\zeta}-1})$, so the expression of the statement of the theorem follows with $X^N = s_{N-N_{-\zeta}}(\mathcal{E}_{-\zeta}^{0,0})$. \square

The next step is to find more handy expressions for the set of classes given by (1).

Lemma 11. *Let S be a manifold with $b^+ = 1$. Then there is a (rational) cohomology class $\Sigma \in H^2(S)$ such that the image of $\wedge : H^1(S) \otimes H^1(S) \rightarrow H^2(S)$ is $\mathbb{Q}[\Sigma]$. Also $e_S = 0$.*

Proof. Let $\beta_1, \beta_2, \beta_3, \beta_4 \in H^1(S)$. If $\beta_1 \wedge \beta_2 \wedge \beta_3 \wedge \beta_4 \neq 0$ then the image of $\wedge : H^1(S) \otimes H^1(S) \rightarrow H^2(S)$ contains the subspace V generated by $\beta_i \wedge \beta_j$, which has dimension 6, with $b^+ = 3$ and $b^- = 3$. This is absurd, so $\beta_1 \wedge \beta_2 \wedge \beta_3 \wedge \beta_4 = 0$. Then $e_S = 0$.

Now let $\Sigma_1 = \beta_1 \wedge \beta_2, \Sigma_2 = \beta_3 \wedge \beta_4 \in H^2(S)$. Then $\Sigma_1^2 = \Sigma_2^2 = 0$ together with the fact that $b^+ = 1$ imply that $\Sigma_1 \cdot \Sigma_2 \neq 0$ unless Σ_1 and Σ_2 are proportional. Since $\Sigma_1 \cdot \Sigma_2 = 0$ by the above, it must be the case that Σ_1 and Σ_2 are proportional. \square

Remark 12. If $S \rightarrow C_g$ is a ruled surface with $g > 0$ and fiber class f , then $\Sigma = f$. Note also that the class Σ does not change under blow-ups.

Now write $\beta_1, \dots, \beta_{2q}$ for a basis of $H^1(S)$ and fix a generator Σ of the image of $\wedge : H^1(S) \otimes H^1(S) \rightarrow H^2(S)$. Let $\delta_1, \dots, \delta_{2q}$ be the dual basis for $H_1(S)$. Put $\beta_i \wedge \beta_j = a_{ij}\Sigma$. The Jacobian of S is $J = H^1(S; \mathbb{R})/H^1(S; \mathbb{Z})$, so naturally $H^1(J) \xrightarrow{\sim} H^1(S)^*$. Let $\mathcal{L} \rightarrow S \times J$ be the universal bundle parametrising divisors homologically equivalent to zero. Then $E = c_1(\mathcal{L}) = \sum \beta_i \otimes \beta_i^\#$, with $\beta_i^\#$ corresponding to δ_i under the isomorphism $H^1(J) \cong H_1(S)$. So

$$(2) \quad \begin{cases} e_\alpha = -2 \sum_{i < j} a_{ij} (\Sigma \cdot \alpha) \beta_i^\# \wedge \beta_j^\# = -2(\Sigma \cdot \alpha)\omega \\ e_{\delta_i} = \beta_i^\# \\ e_{\zeta \beta_i} = \sum (\Sigma \cdot \zeta) a_{ij} \beta_j^\# = (\Sigma \cdot \zeta) i_{\beta_i} \omega \end{cases}$$

where we write $\omega = \sum_{i < j} a_{ij} (\beta_i^\# \wedge \beta_j^\#) \in H^2(J)$, which is an element independent of the chosen basis. We also have implicitly assumed $H_3(S) \cong H^1(S)$ through Poincaré

duality, in the third line. We define

$$F : \mathbb{A}(S) \rightarrow \Lambda^* H_1(S) \otimes \Lambda^* H_3(S) \rightarrow \mathbb{Q}$$

given by projection followed by the map sending $\gamma_1 \wedge \cdots \wedge \gamma_a \otimes A_1 \wedge \cdots \wedge A_b$ to zero when $a + b$ is odd and to

$$\int_J (\gamma_1 \wedge \cdots \wedge \gamma_a \wedge i_{A_1} \omega \wedge \cdots \wedge i_{A_b} \omega \wedge \omega^{q-(a+b)/2})$$

when $a + b$ is even. We note that we always can find a basis $\beta_1, \dots, \beta_{2q}$ with

$$\omega = a_1 \beta_1^\# \wedge \beta_2^\# + a_2 \beta_3^\# \wedge \beta_4^\# + \cdots + a_r \beta_{2r-1}^\# \wedge \beta_{2r}^\#,$$

where $a_i \neq 0$ are integers and $r \leq q$. So if ω is degenerate, $F(1) = \int_J \omega^q = 0$. In general, for a basis element $z = x^r \alpha^s \delta_{i_1} \cdots \delta_{i_a} \beta_{j_1} \cdots \beta_{j_b}$, $F(z)$ is zero unless z contains $\delta_{2r+1} \cdots \delta_{2q}$, and for every pair $(2i-1, 2i)$, $1 \leq i \leq r$, either $\delta_{2i-1} \delta_{2i}$, $\beta_{2i-1} \beta_{2i}$, $\delta_{2i-1} \beta_{2i-1}$, $\delta_{2i} \beta_{2i}$ or nothing. In any case, for subsequent use, we set

$$\text{vol} = \frac{1}{q!} \int_J \omega^q.$$

The number vol depends on the choice of Σ , as when Σ is changed to $r\Sigma$, vol is changed to $r^{-q} \text{vol}$. The final expressions we get for the wall-crossing terms are (as expected) independent of this choice. Also we are going to need the following

Proposition 13. *For any sheaf \mathcal{F} on any complex variety, the Segre classes of \mathcal{F} are given by $s_t(\mathcal{F}) = c_t(\mathcal{F})^{-1}$. For the relationship between the Chern classes of \mathcal{F} and its Chern character, write a_i for $i!$ times the i -th component of $\text{ch } \mathcal{F}$. Then*

$$c_n(\mathcal{F}) = \frac{1}{n!} \begin{vmatrix} a_1 & n-1 & 0 & \cdots & 0 \\ a_2 & a_1 & n-2 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 1 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{vmatrix}$$

and

$$s_n(\mathcal{F}) = \frac{1}{n!} \begin{vmatrix} -a_1 & -(n-1) & 0 & \cdots & 0 \\ a_2 & -a_1 & -(n-2) & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 1 \\ (-1)^n a_n & (-1)^{n-1} a_{n-1} & (-1)^{n-2} a_{n-2} & \cdots & -a_1 \end{vmatrix}$$

3. THE CASE $l_\zeta = 0$

In this section we are going to compute $\delta_{S,\zeta}^{w,d}$ in the case $l_\zeta = 0$, i.e. when $\zeta^2 = p_1$. We have the following theorem which extends [5, theorems 6.1 and 6.2] [8].

Theorem 14. *Let ζ be a wall with $l_\zeta = 0$. Then $\delta_{S,\zeta}^{w,d}(x^r \alpha^{d-2r})$ is equal to*

$$\varepsilon(\zeta, w) \sum_{0 \leq b \leq q} (-1)^{r+d} 2^{3q-b-d} \frac{q!}{(q-b)!} \binom{d-2r}{b} (\zeta \cdot \alpha)^{d-2r-b} (\Sigma \cdot \alpha)^b (\Sigma \cdot \zeta)^{q-b} \text{vol},$$

where terms with negative exponent are meant to be zero.

Proof. For simplicity of notation, let us do the case $r = 0$ (the other case is very similar). Recall that F is a divisor such that $2F - L$ is homologically equivalent to ζ and $J = \text{Jac}^F(S)$ is the Jacobian parametrising divisors homologically equivalent to F . Then $\mathcal{F} \subset S \times J$ denotes the universal divisor. Now $\mathcal{E}_\zeta = \mathcal{E}_\zeta^{0,0} = R^1 \pi_* (\mathcal{O}_{S \times J}(2\mathcal{F} - \pi_1^* L))$ (with $\pi : S \times J \rightarrow J$ the projection) is a vector bundle over J . We note that $H^0(\mathcal{O}_S(2F - L)) = 0$ and $H^0(\mathcal{O}_S(L - 2F) \otimes K) = 0$, as ζ is a good wall, so $R^0 \pi_*$ and $R^2 \pi_*$ vanish. Then

$$\begin{aligned} \text{ch } \mathcal{E}_\zeta &= -\text{ch } \pi_! (\mathcal{O}_{S \times J}(2\mathcal{F} - \pi_1^* L)) = -\pi_* (\text{ch } \mathcal{O}_{S \times J}(2\mathcal{F} - \pi_1^* L) \cdot \text{Todd } T_S) = \\ (3) \quad &= -\left(\frac{\zeta^2}{2} - \frac{\zeta \cdot K}{2} + 1 - q\right) + e_{K-2\zeta} - \frac{2}{3}e_S = \text{rk}(\mathcal{E}_\zeta) + e_{K-2\zeta}, \end{aligned}$$

since $e_S = 0$ (lemma 11). A fortiori $\text{ch } \mathcal{E}_{-\zeta}^\vee = -\left(\frac{\zeta^2}{2} + \frac{\zeta \cdot K}{2} + 1 - q\right) - e_{K+2\zeta}$ and

$$\text{ch}(\mathcal{E}_\zeta \oplus \mathcal{E}_{-\zeta}^\vee) = (-\zeta^2 + 2q - 2) - 4e_\zeta.$$

From proposition 13, $s_i(\mathcal{E}_\zeta \oplus \mathcal{E}_{-\zeta}^\vee) = \frac{4^i}{i!} e_\zeta^i$. This together with theorem 10 or theorem 7 (depending on whether $h(\zeta) + q$ is zero or not) gives

$$\begin{aligned} \delta_{S,\zeta}^{w,d}(\alpha^d) &= \epsilon_S(w) \sum_{0 \leq b \leq q} (-1)^{h(\zeta)} \binom{d}{b} a^{d-b} e_\alpha^b \cdot s_{q-b}(\mathcal{E}_\zeta \oplus \mathcal{E}_{-\zeta}^\vee) = \\ &= \epsilon_S(w) \sum_{0 \leq b \leq q} (-1)^{h(\zeta)} \binom{d}{b} a^{d-b} e_\alpha^b \cdot \frac{4^{q-b}}{(q-b)!} e_\zeta^{q-b} = \\ &= \epsilon_S(w) \sum_{0 \leq b \leq q} (-1)^{h(\zeta)+q} \frac{2^{3q-b-d}}{(q-b)!} \binom{d}{b} (\zeta \cdot \alpha)^{d-b} (\Sigma \cdot \alpha)^b (\Sigma \cdot \zeta)^{q-b} \omega^q, \end{aligned}$$

using (2). Now we substitute $\epsilon_S(w)(-1)^{h(\zeta)} = (-1)^{d+q} \varepsilon(\zeta, w)$ (remark 9), to get the desired result. \square

We can also generalise introducing classes of odd degree. If $z = x^r \alpha^s \gamma_1 \cdots \gamma_a A_1 \cdots A_b$ with $d = s + 2r + \frac{3}{2}a + \frac{1}{2}b$, then theorem 10 or theorem 7 gives

$$\begin{aligned} \delta_{S,\zeta}^{w,d}(z) &= \epsilon_S(w) \left(-\frac{1}{4}X^2\right)^r (-e_\alpha + aX)^s (e_{\gamma_1}X) \cdots (-e_{\zeta A_b}) \\ &= \epsilon_S(w) \sum_j \left(-\frac{1}{4}\right)^r (-1)^b \binom{s}{j} a^{s-j} 2^j (\Sigma \cdot \alpha)^j (\Sigma \cdot \zeta)^b \omega^j \gamma_1 \cdots \gamma_a i_{A_1} \omega \cdots i_{A_b} \omega \cdot \\ &\quad \cdot (-1)^{2r+a+s-j-N_{-\zeta}} s_{2r+a+s-j-1-N_{-\zeta}-N_{-\zeta}}(\mathcal{E}_\zeta \oplus \mathcal{E}_{-\zeta}^\vee). \end{aligned}$$

Now $2r + a + s - 1 - N_\zeta - N_{-\zeta} = q - (a + b)/2$, so

$$\begin{aligned} \delta_{S,\zeta}^{w,d}(z) &= \epsilon(\zeta, w) \sum_j (-1)^{r+d+b} 2^{3q-d-b-j} \binom{s}{j} \frac{F(z)}{(q - (a + b)/2 - j)!} \cdot \\ &\quad \cdot (\zeta \cdot \alpha)^{s-j} (\Sigma \cdot \alpha)^j (\Sigma \cdot \zeta)^{q+(b-a)/2-j} \text{vol}. \end{aligned}$$

4. THE CASE $l_\zeta = 1$

Now we want to compute $\delta_{S,\zeta}^{w,d}$ in the case $l_\zeta = 1$, i.e. when $\zeta^2 = p_1 + 4$. In this case, $J \times H_{l_\zeta-k} \times H_k \cong J \times S$, both for $k = 0$ and $k = 1$. The universal divisor $\mathcal{Z}_1 \subset S \times H_1 = S \times S$ is the diagonal Δ . Let again $\mathcal{L} \rightarrow S \times J$ be the universal bundle parametrising divisors homologically equivalent to zero, so $\mathcal{F} = \pi_1^* F + \mathcal{L}$. With this understood, we have the following easy extension of [5, lemma 5.11]

Lemma 15. *Let $\text{Hom} = \text{Hom}(I_{\mathcal{Z}_k}, I_{\mathcal{Z}_{l_\zeta-k}})$ and $\text{Ext}^1 = \text{Ext}^1(I_{\mathcal{Z}_k}, I_{\mathcal{Z}_{l_\zeta-k}})$, π_1 , p and π_2 be the projections from $S \times (J \times H_{l_\zeta-k} \times H_k)$ to S , $S \times J$ and $J \times H_{l_\zeta-k} \times H_k$, respectively. Let $E = c_1(\mathcal{L})$. Then we have the following exact sequences*

$$0 \rightarrow R^1 \pi_{2*}(p^*(\zeta + 2E) \otimes \text{Hom}) \rightarrow \mathcal{E}_\zeta^{l_\zeta-k,k} \rightarrow \pi_{2*}(p^*(\zeta + 2E) \otimes \text{Ext}^1) \rightarrow 0$$

$$0 \rightarrow \pi_{2*}(p^*(\zeta + 2E) \otimes \mathcal{O}_{\mathcal{Z}_{l_\zeta-k}}) \rightarrow R^1 \pi_{2*}(p^*(\zeta + 2E) \otimes \text{Hom}) \rightarrow R^1 \pi_{2*}(p^*(\zeta + 2E)) \rightarrow 0$$

where the last sheaf is $M_\zeta = R^1 \pi_{2*}(\mathcal{O}_{S \times J}(2\mathcal{F} - \pi_1^* L))$, which is a line bundle over J with $\text{ch} M_\zeta = rk M_\zeta + e_{K-2\zeta}$ (computed in equation (3)).

We apply this lemma to our case $l_\zeta = 1$. Then for $k = 0$, $\text{Hom} = \text{Hom}(\mathcal{O}, I_\Delta) = I_\Delta$, $\text{Ext}^1 = \text{Ext}^1(\mathcal{O}, I_\Delta) = 0$, and for $k = 1$, $\text{Hom} = \text{Hom}(I_\Delta, \mathcal{O}) = \mathcal{O}_{S \times S}$, $\text{Ext}^1 = \text{Ext}^1(I_\Delta, \mathcal{O}) = \mathcal{O}_\Delta(\Delta)$. Using lemma 15 and the fact $\pi_{2*}(\mathcal{O}_\Delta(\Delta)) = \mathcal{O}_S(-K)$, we get

$$\text{ch } \mathcal{E}_\zeta^{1,0} = \text{ch } M_\zeta + \text{ch } \zeta \text{ ch } 2E$$

$$\text{ch } \mathcal{E}_\zeta^{0,1} = \text{ch } M_\zeta + \text{ch } (\zeta - K) \text{ ch } 2E$$

We recall from notation 4,

$$\begin{aligned}\mathcal{E}_\zeta^{l_\zeta-k,k} &= \mathcal{E}\text{xt}_{\pi_2}^1(\mathcal{O}_{S \times (J \times H_1 \times H_2)}(\pi_1^*L - \mathcal{F}) \otimes I_{\mathcal{Z}_2}, \mathcal{O}_{S \times (J \times H_1 \times H_2)}(\mathcal{F}) \otimes I_{\mathcal{Z}_1}) \\ &= \mathcal{E}\text{xt}_{\pi_2}^1(I_{\mathcal{Z}_2}, \mathcal{O}_{S \times (J \times H_1 \times H_2)}(\zeta + 2E) \otimes I_{\mathcal{Z}_1}), \\ \mathcal{E}_{-\zeta}^{k,l_\zeta-k} &= \mathcal{E}\text{xt}_{\pi_2}^1(I_{\mathcal{Z}_1}, \mathcal{O}_{S \times (J \times H_1 \times H_2)}(-\zeta - 2E) \otimes I_{\mathcal{Z}_2}).\end{aligned}$$

Then we have $\text{ch } \mathcal{E}_{-\zeta}^{1,0} = \text{ch } M_{-\zeta} + \text{ch } (-\zeta) \text{ch } (-2E)$ and $\text{ch } \mathcal{E}_{-\zeta}^{0,1} = \text{ch } M_{-\zeta} + \text{ch } (-\zeta - K) \text{ch } (-2E)$. So

$$\text{ch } (\mathcal{E}_\zeta^{1,0} \oplus (\mathcal{E}_{-\zeta}^{0,1})^\vee) = (-\zeta^2 + 2q - 2) - 4e_\zeta + 2\text{ch } \zeta \text{ch } 2E + \frac{K^2}{2} + K\zeta + K(1 + 2E + 2E^2)$$

$$\text{ch } (\mathcal{E}_\zeta^{0,1} \oplus (\mathcal{E}_{-\zeta}^{1,0})^\vee) = (-\zeta^2 + 2q - 2) - 4e_\zeta + 2\text{ch } \zeta \text{ch } 2E + \frac{K^2}{2} - K\zeta - K(1 + 2E + 2E^2)$$

(4)

We shall compute $s_i = s_i(\mathcal{E}_\zeta^{1,0} \oplus (\mathcal{E}_{-\zeta}^{0,1})^\vee) + s_i(\mathcal{E}_\zeta^{0,1} \oplus (\mathcal{E}_{-\zeta}^{1,0})^\vee)$, as a class on $J \times S$. From proposition 13, s_i is an polynomial expression on $a_i^{(k)} = i! \text{ch}_i(\mathcal{E}_\zeta^{1-k,k} \oplus (\mathcal{E}_{-\zeta}^{k,1-k})^\vee)$, $k = 0, 1$. Furthermore, s_i is invariant under $K \mapsto -K$, and hence an even function of $K\zeta$, K , KE and KE^2 . Now the only non-zero even combinations of $K\zeta$, K , KE and KE^2 are 1 and $K \cdot K$. The first consequence is that we can ignore $K\zeta$, KE and KE^2 in $a_i^{(k)}$ for the purposes of computing s_i . So we can suppose

$$\begin{cases} a_1^{(k)} = -4e_\zeta + 2\zeta + 4E + (-1)^k K \\ a_2 = 2\zeta^2 + 8E^2 + K^2 + 8E\zeta \\ a_3 = 24E^2\zeta = 24e_\zeta[S] \end{cases}$$

where $a_i = a_i^{(0)} = a_i^{(1)}$ for $i \geq 2$, and $a_i = 0$ for $i \geq 4$ (here we have used that $E^3 = 0$ and $E^4 = 0$ as a consequence of lemma 11). Put $a_1 = -4e_\zeta + 2\zeta + 4E$ and define

$$I_n = \begin{vmatrix} -a_1 & -(n-1) & \cdots & 0 \\ a_2 & -a_1 & \cdots & 0 \\ -a_3 & a_2 & \cdots & 0 \\ 0 & -a_3 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & -a_1 \end{vmatrix}$$

and $I_n^{(k)}$ defined similarly with $a_1^{(k)}$ in the place of a_1 . Then by proposition 13, $n! s_n = I_n^{(0)} + I_n^{(1)}$. Easily we have $n! s_n = 2I_n + 2\binom{n}{2} K^2 (4e_\zeta)^{n-2}$. Now we can

look for an inductive formula for I_n . For $n \geq 2$,

$$\begin{aligned}
I_n &= -a_1 I_{n-1} + (n-1)(2\zeta^2 + K^2 + 8E\zeta)(4e_\zeta - 4E)^{n-2} + (n-1)8E^2 I_{n-2} \\
&\quad - 6(n-1)(n-2)[S](4e_\zeta)^{n-2} = \\
&= -a_1 I_{n-1} + (n-1)(2\zeta^2 + K^2 + 8E\zeta)(4e_\zeta)^{n-2} - (n-1)8E\zeta(n-2)(4E)(4e_\zeta)^{n-3} \\
&\quad + (n-1)8E^2 I_{n-2} - 6(n-1)(n-2)[S](4e_\zeta)^{n-2} = \\
&= -a_1 I_{n-1} + (n-1)(2\zeta^2 + K^2 + 8E\zeta)(4e_\zeta)^{n-2} - 8(n-1)(n-2)[S](4e_\zeta)^{n-2} \\
&\quad + (n-1)8E^2 \left((4e_\zeta)^{n-2} - (n-2)(4e_\zeta)^{n-3} 2\zeta \right) - 6(n-1)(n-2)[S](4e_\zeta)^{n-2} = \\
&= -a_1 I_{n-1} + (n-1)(4e_\zeta)^{n-2} (2\zeta^2 + K^2 + 8E\zeta + 8E^2 - 18(n-2)[S]).
\end{aligned}$$

In the first equality we have used that $I_n P = (4e_\zeta - 4E)^n P$, for any $P \in H^i(S) \otimes H^j(J)$ with $i = 3, 4$. In the third equality we use that $I_n E^2 = (4e_\zeta - 2\zeta)^n E^2$. With this inductive formula for I_n , we get, for $n \geq 2$,

$$\begin{aligned}
I_n &= (4e_\zeta - 2\zeta - 4E)^n + \sum_{i=2}^n (4e_\zeta - 2\zeta - 4E)^{n-i} (i-1)(4e_\zeta)^{i-2} \left(2\zeta^2 + K^2 \right. \\
(5) \quad &\quad \left. + 8E\zeta + 8E^2 - 18(i-2)[S] \right).
\end{aligned}$$

Now for any $k \geq 0$ (we always understand $\binom{k}{i} = 0$ if either $i < 0$ or $i > k$),

$$\begin{aligned}
(4e_\zeta - 2\zeta - 4E)^k &= (4e_\zeta)^k + k(-2\zeta - 4E)(4e_\zeta)^{k-1} + \binom{k}{2}(-2\zeta - 4E)^2(4e_\zeta)^{k-2} \\
&\quad + \binom{k}{3}3(-2\zeta)(-4E)^2(4e_\zeta)^{k-3} = \\
&= (4e_\zeta)^k + k(-2\zeta - 4E)(4e_\zeta)^{k-1} \\
&\quad + \binom{k}{2}(4\zeta^2 + 16E\zeta + 16E^2)(4e_\zeta)^{k-2} - 24\binom{k}{3}[S](4e_\zeta)^{k-2}.
\end{aligned}$$

Substituting this into (5), we have

$$\begin{aligned}
I_n &= (4e_\zeta)^n + n(-2\zeta - 4E)(4e_\zeta)^{n-1} + \binom{n}{2}(4\zeta^2 + 16E\zeta + 16E^2)(4e_\zeta)^{n-2} \\
&\quad - 24\binom{n}{3}[S](4e_\zeta)^{n-2} + \sum_{i=2}^n \left((4e_\zeta)^{n-i}(i-1)(4e_\zeta)^{i-2}(2\zeta^2 + K^2 - 18(i-2)[S]) \right. \\
&\quad \left. + ((4e_\zeta)^{n-i} + (n-i)(-4E)(4e_\zeta)^{n-i-1})(i-1)(4e_\zeta)^{i-2}8E\zeta \right. \\
&\quad \left. + ((4e_\zeta)^{n-i} + (n-i)(-2\zeta)(4e_\zeta)^{n-i-1})(i-1)(4e_\zeta)^{i-2}8E^2 \right) = \\
&= (4e_\zeta)^n - n(2\zeta + 4E)(4e_\zeta)^{n-1} + (4e_\zeta)^{n-2} \left[\binom{n}{2}(4\zeta^2 + 16E\zeta + 16E^2) - 24\binom{n}{3}[S] \right. \\
&\quad \left. + \sum_{i=2}^n (i-1)(2\zeta^2 + K^2 - 18(i-2)[S] + 8E\zeta + 8E^2 - 8(n-i)[S] - 4(n-i)[S]) \right].
\end{aligned}$$

Putting this into the expression for s_n , we get

$$\begin{aligned}
s_n &= \frac{2}{n!} \left((4e_\zeta)^n - n(2\zeta + 4E)(4e_\zeta)^{n-1} + (4e_\zeta)^{n-2} \left[\binom{n}{2}(6\zeta^2 + 24E\zeta + 24E^2 + K^2) \right. \right. \\
&\quad \left. \left. - 24\binom{n}{3}[S] + \sum_{i=2}^n (i-1)(36 - 12n - 6i)[S] \right] \right) + \frac{2}{n!} \binom{n}{2} K^2 (4e_\zeta)^{n-2}.
\end{aligned}$$

The expression in the summatory adds up to $-48\binom{n}{3}$, so finally

$$s_n = 2 \frac{(4e_\zeta)^n}{n!} - (4\zeta + 8E) \frac{(4e_\zeta)^{n-1}}{(n-1)!} + (6\zeta^2 + 2K^2 + 24E\zeta + 24E^2) \frac{(4e_\zeta)^{n-2}}{(n-2)!}$$

$$(6) \quad -24[S] \frac{(4e_\zeta)^{n-2}}{(n-3)!},$$

for $n \geq 2$ (where the last summand is understood to be zero when $n = 2$). This expression is actually valid for $n \geq 0$ under the proviso that the terms with negative exponent are zero.

Theorem 16. *Let ζ be a wall with $l_\zeta = 1$. Then $\delta_{\zeta, \zeta}^{w, d}(x^r \alpha^{d-2r})$ is equal to $\varepsilon(\zeta, w)$ times*

$$\begin{aligned}
&\sum_{b=0}^q (-1)^{r+d+1} 2^{3q-b-d} \left[(\zeta \cdot \alpha)^{d-2r-b} \left(\binom{d-2r}{b} (6\zeta^2 + 2K^2 - 24q - 8r) + 8 \binom{d-2r}{b+1} \binom{b+1}{1} \right) + \right. \\
&\quad \left. + 8(\zeta \cdot \alpha)^{d-2r-b-2} \alpha^2 \binom{d-2r}{b+2} \binom{b+2}{2} \right] (\Sigma \cdot \alpha)^b (\Sigma \cdot \zeta)^{q-b} \frac{q!}{(q-b)!} \text{vol},
\end{aligned}$$

where terms with negative exponent are meant to be zero.

Proof. By theorem 7, $\delta_{S,\zeta}^{w,d}(x^r \alpha^{d-2r}) = \epsilon_S(w)([S] - \frac{1}{4}X^2)^r(\alpha - e_\alpha + aX)^{d-2r}$ evaluated on $J \times S$, where

$$\begin{aligned} X^N &= (-1)^{N-N-\zeta} \left(s_{N-1-N_\zeta-N_{-\zeta}}(\mathcal{E}_\zeta^{1,0} \oplus (\mathcal{E}_{-\zeta}^{0,1})^\vee) + s_{N-1-N_\zeta-N_{-\zeta}}(\mathcal{E}_\zeta^{0,1} \oplus (\mathcal{E}_{-\zeta}^{1,0})^\vee) \right) = \\ &= (-1)^{N-N-\zeta} s_{N-1-N_\zeta-N_{-\zeta}}. \end{aligned}$$

Hence

$$\begin{aligned} \delta_{S,\zeta}^{w,d}(x^r \alpha^{d-2r}) &= \epsilon_S(w) \sum_b (-1)^{h(\zeta)+1} \binom{d-2r}{b} a^{d-2r-b} \left(-\frac{1}{4}\right)^r e_\alpha^{b-2} \\ &\cdot \left[-4r[S]e_\alpha^2 \cdot s_{q-b} + \binom{b}{2} \alpha^2 \cdot s_{q-b+2} + \binom{b}{1} \alpha(-e_\alpha) \cdot s_{q-b+2} + e_\alpha^2 \cdot s_{q-b+2} \right]. \end{aligned}$$

Substituting the values of s_n from (6) and using remark 9, we get

$$\begin{aligned} \delta_{S,\zeta}^{w,d}(x^r \alpha^{d-2r}) &= \varepsilon(\zeta, w) \sum_b (-1)^{r+d+1} 2^{3q-b-d} \binom{d-2r}{b} (\zeta \cdot \alpha)^{d-2r-b} \left[(6\zeta^2 + 2K^2 \right. \\ &\quad \left. - 24q - 8r)(\Sigma \cdot \alpha)^b \frac{(\Sigma \cdot \zeta)^{q-b}}{(q-b)!} + 16(\zeta \cdot \alpha) \binom{b}{1} (\Sigma \cdot \alpha)^{b-1} \frac{(\Sigma \cdot \zeta)^{q-b+1}}{(q-b+1)!} + \right. \\ &\quad \left. + 32\alpha^2 \binom{b}{2} (\Sigma \cdot \alpha)^{b-2} \frac{(\Sigma \cdot \zeta)^{q-b+2}}{(q-b+2)!} \right] \int_J \omega^q. \end{aligned}$$

Reagrouping the terms we get the desired result. \square

This result agrees with theorems 6.4 and 6.5 in [5] particularising for $q = 0$ and $r = 0, 1$. We see from theorem 16 that the difference terms $\delta_{S,\zeta}^{w,d}$ do not satisfy in general the simple type condition [10].

Remark 17. L. Göttsche and the author have obtained the same formula of theorem 16 in some examples, like $\mathbb{C}\mathbb{P}^1 \times C_1$ (C_1 being an elliptic curve) using the simple type condition in limiting chambers. These arguments will appear elsewhere.

5. GENERAL CASE

We do not want to enter into more detailed computations of the wall-crossing formulae, but just to remark that the pattern laid in [5] together with theorem 7 can be used here to obtain partial information of $\delta_{S,\zeta}^{w,d}$. For instance, we write

$$S_{j,b} = \sum_k ([\mathcal{Z}_{l_\zeta-k}]/\alpha + [\mathcal{Z}_k]/\alpha)^j \cdot e_\alpha^b \cdot s_{2l_\zeta-j+q-b} (\mathcal{E}_\zeta^{l_\zeta-k,k} \oplus (\mathcal{E}_{-\zeta}^{k,l_\zeta-k})^\vee),$$

so that corollary 8 says $\delta_{S,\zeta}^{w,d}(\alpha^d) = \varepsilon(\zeta, w) \sum (-1)^{d+q+l_\zeta+j} \frac{d!}{j!b!(d-j-b)!} a^{d-j-b} S_{j,b}$. Then we can obtain (compare [5, proposition 5.12])

$$\begin{aligned} S_{2l_\zeta, q} &= \frac{(2l_\zeta)!}{l_\zeta!} (\alpha^2)^{l_\zeta} e_\alpha^q \\ S_{2l_\zeta-1, q} &= (-4) \frac{(2l_\zeta)!}{l_\zeta!} (\alpha^2)^{l_\zeta-1} a e_\alpha^q \\ S_{2l_\zeta, q-1} &= 4 \frac{(2l_\zeta)!}{l_\zeta!} (\alpha^2)^{l_\zeta} e_\alpha^{q-1} e_\zeta \end{aligned}$$

As an easy consequence of this we get (compare [5, theorems 5.13 and 5.14])

Corollary 18. *Let ζ be a wall of type (w, p_1) . Let $\alpha \in H_2(S; \mathbb{Z})$ and $a = (\zeta \cdot \alpha)/2$. Then $\delta_{S,\zeta}^{w,d}(\alpha^d)$ is congruent (modulo $a^{d-2l_\zeta-q+2}$) with*

$$\begin{aligned} &\varepsilon(\zeta, w) (-1)^{d+l_\zeta} 2^q \left[a^{d-2l_\zeta-q} \frac{d!}{l_\zeta!(d-2l_\zeta-q)!} (\alpha^2)^{l_\zeta} (\Sigma \cdot \alpha)^q + \right. \\ &\left. + 4a^{d-2l_\zeta-q+1} \frac{d! q}{l_\zeta!(d-2l_\zeta-q+1)!} (\alpha^2)^{l_\zeta} (\Sigma \cdot \alpha)^{q-1} (\Sigma \cdot \zeta) \right] \text{vol}. \end{aligned}$$

Corollary 19. *In the conditions of the previous corollary, suppose furthermore $d-2r \geq 2l_\zeta + q$. Then $\delta_{S,\zeta}^{w,d}(x^r \alpha^{d-2r})$ is congruent (modulo $a^{d-2r-2l_\zeta-q+2}$) with*

$$\begin{aligned} &\varepsilon(\zeta, w) (-1)^{d+l_\zeta+r} 2^{q-2r} \left(-\frac{1}{4}\right)^r \left[a^{d-2r-2l_\zeta-q} \frac{(d-2r)!}{l_\zeta!(d-2r-2l_\zeta-q)!} (\alpha^2)^{l_\zeta} (\Sigma \cdot \alpha)^q + \right. \\ &\left. + 4a^{d-2r-2l_\zeta-q+1} \frac{(d-2r)! q}{l_\zeta!(d-2r-2l_\zeta-q+1)!} (\alpha^2)^{l_\zeta} (\Sigma \cdot \alpha)^{q-1} (\Sigma \cdot \zeta) \right] \text{vol}. \end{aligned}$$

6. CONJECTURE

It is natural to propose the following

Conjecture. Let X be an oriented compact four-manifold with $b^+ = 1$ and $b_1 = 2q$ even. Let $w \in H^2(X; \mathbb{Z})$. Choose $\Sigma \in H^2(X)$ generating the image of $\wedge : H^1(X) \otimes H^1(X) \rightarrow H^2(X)$. Define $\omega \in H^2(J)$ such that $e_\alpha = -2(\Sigma \cdot \alpha)\omega$ and put $\text{vol} = \int_J \frac{\omega^n}{n!}$. If ζ defines a wall, then the wall-crossing difference term $\delta_{X,\zeta}^{w,d}(x^r \alpha^{d-2r})$ only depends on $w, d, r, b_1 = 2q, b_2, \zeta^2, \alpha^2, (\zeta \cdot \alpha)$ and $(\Sigma \cdot \alpha)^i (\Sigma \cdot \zeta)^{q-i} \text{vol}$, $0 \leq i \leq q$. The coefficients are universal on X .

This is quite a strong conjecture and one can obviously write down weaker versions. It would allow one to carry out similar arguments to those in [6] and therefore to find out the general shape of the wall-crossing formulae for arbitrary X , involving modular forms. One should be able to determine then all wall-crossing formulae from particular

cases. This and applications to computing the invariants of $\mathbb{C}\mathbb{P}^1 \times C_g$ (C_g the genus g Riemann surface) will be carried out in following joint work with L. Göttsche.

APPENDIX. ALGEBRAIC SURFACES WITH $p_g = 0$ AND $-K$ EFFECTIVE

From [1], the algebraic surfaces with $p_g = 0$ and $-K$ effective are $\mathbb{C}\mathbb{P}^2$, ruled surfaces and blow-ups of these. For the case $q = 0$, we have thus $\mathbb{C}\mathbb{P}^2$, the Hirzebruch surfaces and their blow-ups. Not all blow-ups have $-K$ effective, but they are always deformation equivalent to one with $-K$ effective. For the case $q > 0$, the minimal models are ruled surfaces over a surface C_g of genus $g \geq 1$. They have $c_1^2 = 8(1 - g)$. Let $S \rightarrow C_g$ be a ruled surface. It has $b_2 = 2$ and $b_1 = 2g$, so $g = q$. Let f be the class of the fibre and $\sigma = \sigma_{-N}$ the class of the section with negative self-intersection $\sigma_{-N}^2 = -N \leq 0$. Then there is a section σ_N homologically equivalent to $\sigma_{-N} + Nf$ with square N . Write $X = \mathbb{P}(V^\vee)$, for $V \rightarrow C_g$ a rank two bundle. Then $K = \mathbf{a}f - 2\sigma$, with $\mathbf{a} = \sigma^2 + K_{C_g}$ a divisor on C_g (see [7, section 5.2]). Therefore $-K$ is effective if and only if $-\mathbf{a}$ is effective. The section σ corresponds to a sub-line bundle $L \hookrightarrow V$ with $\mathcal{O}_{C_g}(\sigma^2) = L^{-2} \otimes \det(V)$. Then $-\mathbf{a}$ is effective when $L^2 \otimes \det(V)^{-1} \otimes K_{C_g}^{-1}$ has sections. We can find examples for any N as long as $N \geq 2(g - 1)$. Again, the non-minimal examples are blow-ups of these, and can be found to have $-K$ effective.

For fixed $q = g > 0$, there are only two deformation classes of minimal ruled surfaces, corresponding to two diffeomorphism types, the two different \mathbb{S}^2 -bundles over C_g , one with even w_2 , the other with odd w_2 .

- **N even:** S is diffeomorphic to $S_0 = \mathbb{C}\mathbb{P}^1 \times C_g$ (and the canonical classes correspond). Let C be the homology class of $\text{pt} \times C_g$ coming from the diffeomorphism. Then σ is homologous to $C - \frac{N}{2}f$. The ample cone C_S of S is generated by f and $\sigma_N = C + \frac{N}{2}f$ (i.e. it is given by $\mathbb{R}^+f + \mathbb{R}^+\sigma_N$). Note that the bigger N , the smaller the ample cone. The wall-crossing terms $\delta_{S,\zeta}^{w,d}$ do not depend on the complex structure of S , so our results for the case $-K$ effective give the wall-crossing terms for S_0 for any wall inside C_S . Letting $N = 2(g - 1)$, we actually compute $\delta_{S,\zeta}^{w,d}$ for any $\zeta = a\mathbb{C}\mathbb{P}^1 - bC$ with $a, b > 0$, $a > b(g - 1)$ (note that all these walls are good).
- **N odd:** S is diffeomorphic to the non-trivial \mathbb{S}^2 -bundle over C_g . Arguing as above, we compute the wall-crossing terms $\delta_{S,\zeta}^{w,d}$ for any $\zeta = a\mathbb{C}\mathbb{P}^1 - b\sigma_{-(2g-1)}$ with $a, b > 0$, $a > b\frac{2g-1}{2}$.

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