# QUANTUM COHOMOLOGY OF THE MODULI SPACE OF STABLE BUNDLES OVER A RIEMANN SURFACE

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Abstract. We determine the quantum cohomology of the moduli space $M_\Sigma$ of odd degree rank two stable vector bundles over a Riemann surface $\Sigma$ of genus $g \geq 1$. This work together with [10] complete the proof of the existence of an isomorphism $QH^*(M_\Sigma) \cong HF^*(\Sigma \times \mathbb{S}^1)$.

## 1. Introduction

Let $\Sigma$ be a Riemann surface of genus $g \geq 2$ and let $M_\Sigma$ denote the moduli space of flat $SO(3)$-connections with nontrivial second Stiefel-Whitney class $w_2$.

This is a smooth symplectic manifold of dimension $6g-6$. Alternatively, we can consider $\Sigma$ as a smooth complex curve of genus $g$. Fix a line bundle $\Lambda$ on $\Sigma$ of degree 1, then $M_\Sigma$ is the moduli space of rank two stable vector bundles on $\Sigma$ with determinant $\Lambda$, which is a smooth complex variety of complex dimension $3g-3$. The symplectic deformation class of $M_\Sigma$ only depends on $g$ and not on the particular complex structure on $\Sigma$.

The manifold $X = M_\Sigma$ is a positive symplectic manifold with $\pi_2(X) = \mathbb{Z}$. For such a manifold $X$, its quantum cohomology, $QH^*(X)$, is well-defined (see [14] [15] [8] [12]). As vector spaces, $QH^*(X) = H^*(X)$ (rational coefficients are understood), but the multiplicative structure is different. Let $A$ denote the positive generator of $\pi_2(X)$, i.e. the generator such that the symplectic form evaluated on $A$ is positive. Let $N = c_1(X)[A] \in \mathbb{Z}_{>0}$. Then there is a natural $\mathbb{Z}/2N\mathbb{Z}$-grading for $QH^*(X)$, which comes from reducing the $\mathbb{Z}$-grading of $H^*(X)$. (For the case $X = M_\Sigma$, $N = 2$, so $QH^*(M_\Sigma)$ is $\mathbb{Z}/4\mathbb{Z}$-graded). The ring structure of $QH^*(X)$, called quantum multiplication, is a deformation of the usual cup product for $H^*(X)$. For $\alpha \in H^p(X)$, $\beta \in H^q(X)$, we

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define the quantum product of $\alpha$ and $\beta$ as

$$\alpha \cdot \beta = \sum_{d \geq 0} \Phi_{dA}(\alpha, \beta),$$

where $\Phi_{dA}(\alpha,\beta) \in H^{p+q-2Nd}(X)$ is given by $< \Phi_{dA}(\alpha,\beta), \gamma >= \Psi^X_{dA}(\alpha,\beta,\gamma)$, the Gromov-Witten invariant, for all $\gamma \in H^{\dim X - p - q + 2Nd}(X)$. One has $\Phi_0(\alpha,\beta) = \alpha \cup \beta$. The other terms are the quantum correction terms and they all live in lower degree parts of the cohomology groups. It is a fact [15] that the quantum product gives an associative and graded commutative ring structure.

To define the Gromov-Witten invariant, let $J$ be a generic almost complex structure compatible with the symplectic form. Then for every 2-homology class $dA$, $d \in \mathbb{Z}$, there is a moduli space $\mathcal{M}_{dA}$ of pseudoholomorphic rational curves (with respect to $J$) $f : \mathbb{P}^1 \to X$ with $f_*[\mathbb{P}^1] = dA$. Note that $\mathcal{M}_0 = X$ and that $\mathcal{M}_{dA}$ is empty for $d < 0$. For $d \geq 0$, the dimension of $\mathcal{M}_{dA}$ is $\dim X + 2Nd$. This moduli space $\mathcal{M}_{dA}$ admits a natural compactification, $\overline{\mathcal{M}}_{dA}$, called the Gromov-Uhlenbeck compactification [14] [15, section 3]. Consider now $r \geq 3$ different points $P_1, \ldots, P_r \in \mathbb{P}^1$. Then we have defined an evaluation map $ev : \mathcal{M}_{dA} \to X^r$ by $f \mapsto (f(P_1), \ldots, f(P_r))$. This map extends to $\overline{\mathcal{M}}_{dA}$ and its image, $ev(\overline{\mathcal{M}}_{dA})$, is a pseudo-cycle [15]. So for $\alpha_i \in H^{p_i}(M_\Sigma)$, $1 \leq i \leq r$, with $p_1 + \cdots + p_r = \dim X + 2Nd$, we choose generic cycles $A_i$, $1 \leq i \leq r$, representatives of their Poincaré duals, and set

$$(1)\, \Psi^X_{dA}(\alpha_1, \ldots, \alpha_r) =< A_1 \times \cdots \times A_r, [ev(\overline{\mathcal{M}}_{dA})] >= \# ev^*_{P_1}(A_1) \cap \cdots \cap ev^*_{P_r}(A_r),$$

where $\#$ denotes count of points (with signs) and $ev_{P_i} : \mathcal{M}_{dA} \to X$, $f \mapsto f(P_i)$. This is a well-defined number and independent of the particular cycles. Also, as the manifold $X$ is positive, $\mathrm{coker} L_f = H^1(\mathbb{P}^1, f^* c_1(X)) = 0$, for all $f \in \mathcal{M}_{dA}$ (see [14] for definition of $L_f$). By [14] the complex structure of $X$ is generic and we can use it to compute the Gromov-Witten invariants.

Also for $r \geq 2$, let $\alpha_i \in H^{p_i}(M_\Sigma)$, $1 \leq i \leq r$, then

$$\alpha_1 \cdots \alpha_r = \sum_{d \geq 0} \Phi_{dA}(\alpha_1, \ldots, \alpha_r),$$

where the correction terms $\Phi_{dA}(\alpha_1, \ldots, \alpha_r) \in H^{p_1 + \cdots + p_r - 2Nd}(X)$ are determined by $< \Phi_{dA}(\alpha_1, \ldots, \alpha_r), \gamma >= \Psi^X_{dA}(\alpha_1, \ldots, \alpha_r, \gamma)$, for any $\gamma \in H^{\dim X + 2Nd - (p_1 + \cdots + p_r)}(X)$.

Returning to our manifold $X = M_\Sigma$, there is a classical conjecture relating the quantum cohomology $QH^*(M_\Sigma)$ and the instanton Floer cohomology of the three manifold $\Sigma \times \mathbb{S}^1$, $HF^*(\Sigma \times \mathbb{S}^1)$ (see [10]). In [1] a presentation of $QH^*(M_\Sigma)$ was given using physical methods, and in [10] it was proved that such a presentation was a presentation of $HF^*(\Sigma \times \mathbb{S}^1)$ indeed. Here we determine a presentation of $QH^*(M_\Sigma)$ and prove the isomorphism $QH^*(M_\Sigma) \cong HF^*(\Sigma \times \mathbb{S}^1)$.

Siebert and Tian have an alternative program [16] to find the presentation of $QH^*(M_\Sigma)$, which goes through proving a recursion formula for the Gromov-Witten invariants of $M_\Sigma$ in terms of the genus $g$.

The paper is organised as follows. In section 2 we review the ordinary cohomology ring of $M_\Sigma$. In section 3 the moduli space of lines (rational curves representing $A$) in $M_\Sigma$ is described. This makes possible to compute the Gromov-Witten invariants $\Psi_A^{M_\Sigma}$, which determine the first quantum correction terms of the quantum products in $QH^*(M_\Sigma)$. Section 4 is devoted to this task. In [3] Donaldson uses this information alone to determine $QH^*(M_\Sigma)$ in the case of genus $g=2$. It is somehow natural to expect that this idea can be developed in the general case $g \geq 3$. In section 5 we give an explicit presentation of $QH^*(M_\Sigma)$ for $g \geq 3$ (theorem 20), concluding the proof of $QH^*(M_\Sigma) \cong HF^*(\Sigma \times \mathbb{S}^1)$ (corollary 21). The two main ingredients that we make use of are the $\mathrm{Sp}\,(2g,\mathbb{Z})$-decomposition of $H^*(M_\Sigma)$ under the action of the mapping class group (not ignoring the non-invariant part as it was customary) and a recursion similar to that in [16] (lemma 17). The difference with [16] lies in the fact that we fix the genus, so that we do not need to compare the Gromov-Witten invariants for moduli spaces of Riemann surfaces of different genus. Finally in section 6 we discuss the cases $g=1$ and $g=2$, which are slightly different.

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## 2. Classical cohomology ring of $M_\Sigma$

Let us recall the known description of the homology of $M_\Sigma$ [6] [17] [10]. Let $\mathcal{U} \to \Sigma \times M_\Sigma$ be the universal bundle and consider the Künneth decomposition of

$$c_2(\mathrm{End}_0\,\mathcal{U}) = 2[\Sigma] \otimes \alpha + 4\psi - \beta, \tag{2}$$

with $\psi = \sum \gamma_i \otimes \psi_i$, where $\{\gamma_1, \ldots, \gamma_{2g}\}$ is a symplectic basis of $H^1(\Sigma;\mathbb{Z})$ with $\gamma_i\gamma_{i+g} = [\Sigma]$ for $1 \leq i \leq g$ (also $\{\gamma_i^\#\}$ will denote the dual basis for $H_1(\Sigma;\mathbb{Z})$). Here we can suppose without loss of generality that $c_1(\mathcal{U}) = \Lambda + \alpha$ (see [17]). In terms of the map $\mu : H_*(\Sigma) \to H^{4-*}(M_\Sigma)$, given by $\mu(a) = -\frac{1}{4}\,p_1(\mathfrak{g}_\mathcal{U})/a$ (here $\mathfrak{g}_\mathcal{U} \to \Sigma \times M_\Sigma$ is the associated universal $SO(3)$-bundle, and $p_1(\mathfrak{g}_\mathcal{U}) \in H^4(\Sigma \times M_\Sigma)$ its first Pontrjagin class), we have

$$\begin{cases} \alpha = 2\,\mu(\Sigma) \in H^2 & \\ \psi_i = \mu(\gamma_i^\#) \in H^3, & 1 \leq i \leq 2g \\ \beta = -4\,\mu(x) \in H^4 & \end{cases}$$

where $x \in H_0(\Sigma)$ is the class of the point, and $H^i = H^i(M_\Sigma)$. These elements generate $H^*(M_\Sigma)$ as a ring [6] [19], and $\alpha$ is the positive generator of $H^2(M_\Sigma;\mathbb{Z})$. We can rephrase this as saying that there exists an epimorphism

$$\mathbb{A}(\Sigma) = \mathbb{Q}[\alpha, \beta] \otimes \Lambda(\psi_1, \dots, \psi_{2g}) \twoheadrightarrow H^*(M_\Sigma) \tag{3}$$

(the notation $\mathbb{A}(\Sigma)$ follows that of Kronheimer and Mrowka [7], although it is slightly different). Recall that $\deg(\alpha) = 2$, $\deg(\beta) = 4$ and $\deg(\psi_i) = 3$.

The mapping class group $\mathrm{Diff}(\Sigma)$ acts on $H^*(M_\Sigma)$, with the action factoring through the action of $\mathrm{Sp}\,(2g, \mathbb{Z})$ on $\{\psi_i\}$. The invariant part, $H_I^*(M_\Sigma)$, is generated by $\alpha$, $\beta$ and $\gamma = -2\sum_{i=0}^{g} \psi_i \psi_{i+g}$. Then there is an epimorphism

$$\mathbb{Q}[\alpha, \beta, \gamma] \twoheadrightarrow H_I^*(M_\Sigma) \tag{4}$$

which allows us to write

$$H_I^*(M_\Sigma) = \mathbb{Q}[\alpha, \beta, \gamma]/I_g,$$

where $I_g$ is the ideal of relations satisfied by $\alpha$, $\beta$ and $\gamma$. From [17], a basis for $H_I^*(M_\Sigma)$ is given by the monomials $\alpha^a \beta^b \gamma^c$, with $a + b + c < g$. For $0 \le k \le g$, the primitive component of $\Lambda^k H^3$ is

$$\Lambda_0^k H^3 = \ker(\gamma^{g-k+1} : \Lambda^k H^3 \to \Lambda^{2g-k+2} H^3).$$

The spaces $\Lambda_0^k H^3$ are irreducible $\mathrm{Sp}\,(2g, \mathbb{Z})$-modules, i.e. the transforms of any nonzero element of $\Lambda_0^k H^3$ under $\mathrm{Sp}\,(2g, \mathbb{Z})$ generate the whole of it. The description of the ideals $I_g$ and the cohomology ring $H^*(M_\Sigma)$ is given in the following

**Proposition 1** ([17] [6]). *Define $q_0^1 = 1$, $q_0^2 = 0$, $q_0^3 = 0$ and then recursively, for all $r \ge 1$,*

$$\begin{cases} q_{r+1}^1 = \alpha q_r^1 + r^2 q_r^2 \\ q_{r+1}^2 = (\beta + (-1)^{r+1} 8) q_r^1 + \frac{2r}{r+1} q_r^3 \\ q_{r+1}^3 = \gamma q_r^1 \end{cases}$$

*Then $I_g = (q_g^1, q_g^2, q_g^3) \subset \mathbb{Q}[\alpha, \beta, \gamma]$, for all $g \ge 1$. Note that $\deg(q_g^1) = 2g$, $\deg(q_r^2) = 2g + 2$ and $\deg(q_g^3) = 2g + 4$. Moreover the $Sp\,(2g, \mathbb{Z})$-decomposition of $H^*(M_\Sigma)$ is*

$$H^*(M_\Sigma) = \bigoplus_{k=0}^{g-1} \Lambda_0^k H^3 \otimes \mathbb{Q}[\alpha, \beta, \gamma]/I_{g-k}. \tag{5}$$

This proposition allows us to find a basis for $H^*(M_\Sigma)$ as follows. Let $\{x_i^{(k)}\}_{i \in B_k}$ be a basis of $\Lambda_0^k H^3$, $0 \le k \le g - 1$. Then

$$\{x_i^{(k)} \alpha^a \beta^b \gamma^c / k = 0, 1, \dots, g-1,\ a + b + c < g - k,\ i \in B_k\} \tag{6}$$

is a basis for $H^*(M_\Sigma)$. If we set

$$x_0^{(k)} = \psi_1 \psi_2 \cdots \psi_k \in \Lambda_0^k H^3, \tag{7}$$

then proposition 1 says that a complete set of relations satisfied in $H^*(M_\Sigma)$ are $x_0^{(k)} q_{g-k}^i$, $i = 1, 2, 3$, $0 \le k \le g$, and the $\mathrm{Sp}\,(2g, \mathbb{Z})$ transforms of these.

## 3. Holomorphic lines in $M_\Sigma$

In order to compute the Gromov-Witten invariants $\Psi_A^{M_\Sigma}$, we need to describe the space of lines, i.e. rational curves in $M_\Sigma$ representing the generator $A \in H_2(M_\Sigma;\mathbb{Z})$,

$$\mathcal{M}_A = \{f : \mathbb{P}^1 \to M_\Sigma / f \text{ holomorphic, } f_*[\mathbb{P}^1] = A\}.$$

Let us fix some notation. Let $J$ denote the Jacobian variety of $\Sigma$ parametrising line bundles of degree 0 and let $\mathcal{L} \to \Sigma \times J$ be the universal line bundle. If $\{\gamma_i\}$ is the basis of $H^1(\Sigma)$ introduced in section 2 then $c_1(\mathcal{L}) = \sum \gamma_i \otimes \phi_i \in H^1(\Sigma) \otimes H^1(J)$, where $\{\phi_i\}$ is a symplectic basis for $H^1(J)$. Thus $c_1(\mathcal{L})^2 = -2[\Sigma] \otimes \omega \in H^2(\Sigma) \otimes H^2(J)$, where $\omega = \sum_{i=1}^g \phi_i \wedge \phi_{i+g}$ is the natural symplectic form for $J$.

Consider now the algebraic surface $S = \Sigma \times \mathbb{P}^1$. It has irregularity $q = g \geq 2$, geometric genus $p_g = 0$ and canonical bundle $K \equiv -2\Sigma + (2g-2)\mathbb{P}^1$. Recall that $\Lambda$ is a fixed line bundle of degree 1 on $\Sigma$. Fix the line bundle $L = \Lambda \otimes \mathcal{O}_{\mathbb{P}^1}(1)$ on $S$ (we omit all pull-backs) with $c_1 = c_1(L) \equiv \mathbb{P}^1 + \Sigma$, and put $c_2 = 1$. The ample cone of $S$ is $\{a\mathbb{P}^1 + b\Sigma \,/\, a, b > 0\}$. Let $H_0$ be a polarisation close to $\mathbb{P}^1$ in the ample cone and $H$ be a polarisation close to $\Sigma$, i.e. $H = \Sigma + t\mathbb{P}^1$ with $t$ small. We wish to study the moduli space $\mathfrak{M} = \mathfrak{M}_H(c_1, c_2)$ of $H$-stable bundles over $S$ with Chern classes $c_1$ and $c_2$.

**Proposition 2.** *$\mathfrak{M}$ can be described as a bundle $\mathbb{P}^{2g-1} \to \mathfrak{M} = \mathbb{P}(\mathcal{E}_\zeta^\vee) \to J$, where $\mathcal{E}_\zeta$ is a bundle on $J$ with $\mathrm{ch}\,\mathcal{E}_\zeta = 2g + 8\omega$. So $\mathfrak{M}$ is compact, smooth and of the expected dimension $6g-2$. The universal bundle $\mathcal{V} \to S \times \mathfrak{M}$ is given by*

$$0 \to \mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{L} \otimes \lambda \to \mathcal{V} \to \Lambda \otimes \mathcal{L}^{-1} \to 0,$$

*where $\lambda$ is the tautological line bundle for $\mathfrak{M}$.*

*Proof.* For the polarisation $H_0$, the moduli space of $H_0$-stable bundles with Chern classes $c_1, c_2$ is empty by [13]. Now for $p_1 = -4c_2 + c_1^2 = -2$ there is only one wall, determined by $\zeta \equiv -\mathbb{P}^1 + \Sigma$ (here we fix $\zeta = 2\Sigma - L = \Sigma - c_1(\Lambda)$ as a divisor), so the moduli space of $H$-stable bundles with Chern classes $c_1, c_2$ is obtained by crossing the wall as described in [9]. First, note that the results in [9] use the hypothesis of $-K$ being effective, but the arguments work equally well with the weaker assumption of $\zeta$ being a good wall [9, remark 1] (see also [5] for the case of $q = 0$). In our case, $\zeta \equiv -\mathbb{P}^1 + \Sigma$ is a good wall (i.e. $\pm\zeta + K$ are both not effective) with $l_\zeta = 0$. Now with the notations of [9], $F$ is a divisor such that $2F - L \equiv \zeta$, e.g. $F = \Sigma$. Also $\mathcal{F} \to S \times J$ is the universal bundle parametrising divisors homologically equivalent to $F$, i.e. $\mathcal{F} = \mathcal{L} \otimes \mathcal{O}_{\mathbb{P}^1}(1)$. Let $\pi : S \times J \to J$ be the projection. Then $\mathfrak{M} = E_\zeta = \mathbb{P}(\mathcal{E}_\zeta^\vee)$, where

$$\mathcal{E}_\zeta = \mathcal{E}\mathrm{xt}^1_\pi(\mathcal{O}(L - \mathcal{F}), \mathcal{O}(\mathcal{F})) = R^1\pi_*(\mathcal{O}(\zeta) \otimes \mathcal{L}^2).$$

Actually $\mathfrak{M}$ is exactly the set of bundles $E$ that can be written as extensions

$$0 \to \mathcal{O}_{\mathbb{P}^1}(1) \otimes L \to E \to \Lambda \otimes L^{-1} \to 0$$

for a line bundle $L$ of degree 0. The Chern character is computed in [9, section 3] to be $\mathrm{ch}\, \mathcal{E}_\zeta = 2g + e_{K-2\zeta}$, where $e_\alpha = -2(\mathbb{P}^1 \cdot \alpha)\omega$ (the class $\Sigma$ defined in [9, lemma 11] is $\mathbb{P}^1$ in our case). Finally, the description of the universal bundle follows from [9, theorem 10]. □

**Proposition 3.** *There is a well defined map $\mathcal{M}_A \to \mathfrak{M}$.*

*Proof.* Every line $f : \mathbb{P}^1 \to M_\Sigma$ gives a bundle $E = (\mathrm{id}_\Sigma \times f)^* \mathcal{U}$ over $\Sigma \times \mathbb{P}^1$ by pulling-back the universal bundle $\mathcal{U} \to \Sigma \times M_\Sigma$. Then for any $t \in \mathbb{P}^1$, the bundle $E|_{\Sigma \times t}$ is defined by $f(t)$. Now, by equation (2), $p_1(E) = p_1(\mathcal{U})[\Sigma \times A] = -2\alpha[A] = -2$. Since $c_1(E) = (\mathrm{id}_\Sigma \times f)^* c_1(\mathcal{U}) = \Lambda + \Sigma$, it must be $c_2 = 1$. To see that $E$ is $H$-stable, consider any sub-line bundle $L \hookrightarrow E$ with $c_1(L) \equiv a\mathbb{P}^1 + b\Sigma$. Restricting to any $\Sigma \times t \subset \Sigma \times \mathbb{P}^1$ and using the stability of $E|_{\Sigma \times t}$, one gets $a \leq 0$. Then $c_1(L) \cdot \Sigma < \frac{c_1(E) \cdot \Sigma}{2}$, which yields the $H$-stability of $E$ (recall that $H$ is close to $\Sigma$). So $E \in \mathfrak{M}$. □

Now define $N$ as the set of extensions on $\Sigma$ of the form

$$0 \to L \to E \to \Lambda \otimes L^{-1} \to 0, \tag{8}$$

for $L$ a line bundle of degree 0. Then the groups $\mathrm{Ext}^1(\Lambda \otimes L^{-1}, L) = H^1(L^2 \otimes \Lambda^{-1}) = H^0(L^{-2} \otimes \Lambda \otimes K)$ are of constant dimension $g$. Moreover $H^0(L^2 \otimes \Lambda^{-1}) = 0$, so the moduli space $N$ which parametrises extensions like (8) is given as $N = \mathbb{P}(\mathcal{E}^\vee)$, where $\mathcal{E} = \mathcal{E}\mathrm{xt}^1_p(\Lambda \otimes \mathcal{L}^{-1}, \mathcal{L}) = R^1 p_*(\mathcal{L}^2 \otimes \Lambda^{-1})$, $p : \Sigma \times J \to J$ the projection. Then we have a fibration $\mathbb{P}^{g-1} \to N = \mathbb{P}(\mathcal{E}^\vee) \to J$. The Chern character of $\mathcal{E}$ is

$$\begin{aligned} \mathrm{ch}\,(\mathcal{E}) &= \mathrm{ch}\,(R^1 p_*(\mathcal{L}^2 \otimes \Lambda^{-1})) = -\mathrm{ch}\,(p_!(\mathcal{L}^2 \otimes \Lambda^{-1})) = \\ &= -p_*((\mathrm{ch}\ \mathcal{L})^2\,(\mathrm{ch}\ \Lambda)^{-1}\ \mathrm{Todd}\, T_\Sigma) = \\ &= -p_*((1 + c_1(\mathcal{L}) + \tfrac{1}{2} c_1(\mathcal{L})^2)^2 (1 - \Lambda)(1 - \tfrac{1}{2}K)) = \\ &= -p_*(1 - \tfrac{1}{2}K + 2c_1(\mathcal{L}) - 4\omega \otimes [\Sigma] - \Lambda) = g + 4\omega. \end{aligned} \tag{9}$$

It is easy to check that all the bundles in $N$ are stable, so there is a well-defined map

$$i : N \to M_\Sigma.$$

Now we wish to construct the space of lines in $N$. Note that $\pi_2(N) = \pi_2(\mathbb{P}^{g-1}) = \mathbb{Z}$, as there are no rational curves in $J$. Let $L \in \pi_2(N)$ be the positive generator. We want to describe

$$\mathcal{N}_L = \{ f : \mathbb{P}^1 \to N / f \text{ holomorphic, } f_*[\mathbb{P}^1] = L \}.$$

For the projective space $\mathbb{P}^n$, the space $H_1$ of lines in $\mathbb{P}^n$ is the set of algebraic maps $f : \mathbb{P}^1 \to \mathbb{P}^n$ of degree 1. Such an $f$ has the form $f[x_0, x_1] = [x_0 u_0 + x_1 u_1]$, $[x_0, x_1] \in \mathbb{P}^1$, where $u_0$, $u_1$ are linearly independent vectors in $\mathbb{C}^{n+1}$. So

$$H_1 = \mathbb{P}(\{(u_0, u_1)/u_0,\ u_1 \text{ are linearly independent}\}) \subset \mathbb{P}((\mathbb{C}^{n+1} \oplus \mathbb{C}^{n+1})^\vee) = \mathbb{P}^{2n+1}.$$

The complement of $H_1$ is the image of $\mathbb{P}^n \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^{2n+1}$, $([u], [x_0, x_1]) \mapsto [x_0 u, x_1 u]$, which is a smooth $n$-codimensional algebraic subvariety. So $\mathcal{N}_L$ can be described as the fibration

$$\begin{array}{ccccc} H_1 & \to & \mathcal{N}_L & \to & J \\ \cap & & \cap & & \| \\ \mathbb{P}^{2g-1} & \to & \mathbb{P}((\mathcal{E} \oplus \mathcal{E})^\vee) & \to & J \end{array} \tag{10}$$

*Remark* 4. Note that $\mathcal{E}_\zeta = R^1\pi_*(\mathcal{O}(\zeta) \otimes \mathcal{L}^2) = R^1\pi_*(\mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{L}^2 \otimes \Lambda^{-1}) = H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes R^1 p_*(\mathcal{L}^2 \otimes \Lambda^{-1}) = H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{E} \cong \mathcal{E} \oplus \mathcal{E}$. So $\mathfrak{M} = \mathbb{P}((\mathcal{E} \oplus \mathcal{E})^\vee)$, canonically.

**Proposition 5.** *The map $i : N \to M_\Sigma$ induces a map $i_* : \mathcal{N}_L \to \mathcal{M}_A$. The composition $\mathcal{N}_L \to \mathcal{M}_A \to \mathfrak{M}$ is the natural inclusion of* (10).

*Proof.* The first assertion is clear as $i$ is a holomorphic map. For the second, consider the universal sheaf on $\Sigma \times N$,

$$0 \to \mathcal{L} \otimes U \to \mathbb{E} \to \Lambda \otimes \mathcal{L}^{-1} \to 0, \tag{11}$$

where $U = \mathcal{O}_N(1)$ is the tautological bundle of the fibre bundle $\mathbb{P}^{g-1} \to N \to J$. Any element in $\mathcal{N}_L$ is a line $\mathbb{P}^1 \hookrightarrow N$, which must lie inside a single fibre $\mathbb{P}^{g-1}$. Restricting (11) to this line, we have an extension

$$0 \to L \otimes \mathcal{O}_{\mathbb{P}^1}(1) \to E \to \Lambda \otimes L^{-1} \to 0$$

on $S = \Sigma \times \mathbb{P}^1$, which is the image of the given element in $\mathfrak{M}$ (here $L$ is the line bundle corresponding to the fibre in which $\mathbb{P}^1$ sits). Now it is easy to check that the map $\mathcal{N}_L \to \mathfrak{M}$ is the inclusion of (10). □

**Corollary 6.** *$i_*$ is an isomorphism.*

*Proof.* By proposition 5, $i_*$ has to be an open immersion. The group $PGL(2, \mathbb{C})$ acts on both spaces $\mathcal{N}_L$ and $\mathcal{M}_A$, and $i_*$ is equivariant. The quotient $\mathcal{N}_L / PGL(2, \mathbb{C})$ is compact, being a fibration over the Jacobian with all the fibres the Grassmannian $\mathrm{Gr}\,(\mathbb{C}^2, \mathbb{C}^{g-1})$, hence irreducible. As a consequence $i_*$ is an isomorphism. □

*Remark* 7. Notice that the lines in $M_\Sigma$ are all contained in the image of $N$, which is of dimension $4g-2$ against $6g-6 = \dim M_\Sigma$. They do not fill all of $M_\Sigma$ as one would naively expect.

## 4. Computation of $\Psi_A^{M_\Sigma}$

The manifold $N$ is positive with $\pi_2(N) = \mathbb{Z}$ and $L \in \pi_2(N)$ is the positive generator. Under the map $i : N \to M_\Sigma$, we have $i_* L = A$. Now $\dim N = 4g-2$ and $c_1(N)[L] = c_1(\mathbb{P}^{g-1})[L] = g$. So quantum cohomology of $N$, $QH^*(N)$, is well-defined and $\mathbb{Z}/2g\mathbb{Z}$-graded. From corollary 6, it is straightforward to prove

**Lemma 8.** *For any $\alpha_i \in H^{p_i}(M_\Sigma)$, $1 \le i \le r$, such that $p_1 + \cdots + p_r = 6g-2$, it is $\Psi_A^{M_\Sigma}(\alpha_1, \dots, \alpha_r) = \Psi_L^N(i^*\alpha_1, \dots, i^*\alpha_r)$.* $\square$

It is therefore important to know the Gromov-Witten invariants of $N$, i.e. its quantum cohomology. From the universal bundle (11), we can read the first Pontrjagin class $p_1(\mathfrak{g}_{\mathbb{E}}) = -8[\Sigma] \otimes \omega + h^2 - 2[\Sigma] \otimes h + 4h \cdot c_1(\mathcal{L}) \in H^4(\Sigma \times N)$, where $h = c_1(U)$ is the hyperplane class. So on $N$ we have

$$\left\{ \begin{array}{l} \alpha = 2\mu(\Sigma) = 4\omega + h \\ \psi_i = \mu(\gamma_i^\#) = -h \cdot \phi_i \\ \beta = -4\mu(x) = h^2 \end{array} \right. \tag{12}$$

Let us remark here that $h^2$ denotes ordinary cup product in $H^*(N)$, a fact which will prove useful later. Now let us compute the quantum cohomology ring of $N$. The cohomology of $J$ is $H^*(J) = \Lambda H_1$, where $H_1 = H_1(\Sigma)$. Now the fibre bundle description $\mathbb{P}^{2g-1} \to N = \mathbb{P}(\mathcal{E}^\vee) \to J$ implies that the usual cohomology of $N$ is $H^*(N) = \Lambda H_1[h]/ < h^g + c_1 h^{g-1} + \cdots + c_g = 0 >$, where $c_i = c_i(\mathcal{E}) = \frac{4^i}{i!}\omega^i$, from (9). As the quantum cohomology has the same generators as the usual cohomology and the relations are a deformation of the usual relations [18], it must be $h^g + c_1 h^{g-1} + \cdots + c_g = r$ in $QH^*(N)$, with $r \in \mathbb{Q}$. As in [15, example 8.5], $r$ can be computed to be 1. So

$$QH^*(N) = \Lambda H_1[h]/ < h^g + c_1 h^{g-1} + \cdots + c_g = 1 > . \tag{13}$$

**Lemma 9.** *For any $s \in H^{2g-2i}(J)$, $0 \le i \le g$, denote by $s \in H^{2g-2i}(N)$ its pullback to $N$ under the natural projection. Then the quantum product $h^{2g-1+i}s$ in $QH^*(N)$ has component in $H^{4g-2}(N)$ equal to $\frac{(-8)^i}{i!}\omega^i \wedge s$ (the natural isomorphism $H^{4g-2}(N) \cong H^{2g}(J)$ is understood).*

*Proof.* First note that for $s_1, s_2 \in H^*(J)$ such that their cup product in $J$ is $s_1 s_2 = 0$, then the quantum product $s_1 s_2 \in QH^*(N)$ vanishes. This is so since every rational line in $N$ is contained in a fibre of $\mathbb{P}^{2g-1} \to J \to N$.

Next recall that $h^{g-1+i}s$ has component in $H^{4g-2}(N)$ equal to $s_i(\mathcal{E}) \wedge s = \frac{(-4)^i}{i!}\omega^i \wedge s$. Then multiply the standard relation (13) by $h^{g-1+i}s$ and work by induction on $i$. For $i = 0$ we get $h^{2g-1}s = h^{g-1}s$ and the assertion is obvious. For $i > 0$,

$$h^{2g-1+i}s + h^{2g-2+i}c_1 s + \cdots + h^{2g-1}c_i s = h^{g-1+i}s.$$

So the component of $h^{2g-1+i}s$ in $H^{4g-2}(N)$ is

$$-\sum_{j=1}^{i}\frac{(-8)^{i-j}}{(i-j)!}\omega^{i-j}c_j s+\frac{(-4)^i}{i!}\omega^i s=\frac{(-8)^i}{i!}\omega^i s-\sum_{j=0}^{i}\frac{(-8)^{i-j}}{(i-j)!}\frac{4^j}{j!}\omega^i s+\frac{(-4)^i}{i!}\omega^i s=\frac{(-8)^i}{i!}\omega^i s.$$

□

**Lemma 10.** *Suppose $g>2$. Let $\alpha^a\beta^b\psi_{i_1}\cdots\psi_{i_r}\in\mathbb{A}(\Sigma)$ have degree $6g-2$. Then*

$$\Psi_L^N(\alpha,\overset{(a)}{\ldots},\alpha,\beta,\overset{(b)}{\ldots},\beta,\psi_{i_1},\ldots,\psi_{i_r}) =< (4\omega+X)^a(X^2)^b\phi_{i_1}\cdots\phi_{i_r}X^r,[J]>,$$

*evaluated on $J$, where $X^{2g-1+i}=\frac{(-8)^i}{i!}\omega^i\in H^*(J)$.*

*Proof.* By definition the left hand side is the component in $H^{4g-2}(N)$ of the quantum product $\alpha^a\beta^b\psi_{i_1}\cdots\psi_{i_r}\in QH^*(N)$. From (12), this quantum product is $(4\omega+h)^a(h^2)^b(-h\phi_{i_1})\cdots(-h\phi_{i_r})$, upon noting that when $g>2$, $\beta=h^2$ as a quantum product as there are no quantum corrections because of the degree. Note that $r$ is even, so the statement of the lemma follows from lemma 9. □

Now we are in the position of relating the Gromov-Witten invariants $\Psi_A^{M_\Sigma}$ with the Donaldson invariants for $S=\Sigma\times\mathbb{P}^1$ (for definition of Donaldson invariants see [4] [7]).

**Theorem 11.** *Suppose $g>2$. Let $\alpha^a\beta^b\psi_{i_1}\cdots\psi_{i_r}\in\mathbb{A}(\Sigma)$ have degree $6g-2$. Then*

$$\Psi_A^{M_\Sigma}(\alpha,\overset{(a)}{\ldots},\alpha,\beta,\overset{(b)}{\ldots},\beta,\psi_{i_1},\ldots,\psi_{i_r})=(-1)^{g-1}D_{S,H}^{c_1}((2\Sigma)^a(-4pt)^b\gamma_{i_1}^{\#}\cdots\gamma_{i_r}^{\#}),$$

*where $D_{S,H}^{c_1}$ stands for the Donaldson invariant of $S=\Sigma\times\mathbb{P}^1$ with $w=c_1$ and polarisation $H$.*

*Proof.* By definition, the right hand side is $\epsilon_S(c_1)<\alpha^a\beta^b\psi_{i_1}\cdots\psi_{i_r},[\mathfrak{M}]>$, where $\alpha=2\mu(\Sigma)\in H^2(\mathfrak{M})$, $\beta=-4\mu(x)\in H^4(\mathfrak{M})$, $\psi_i=\mu(\gamma_i^{\#})\in H^3(\mathfrak{M})$. Here the factor $\epsilon_S(c_1)=(-1)^{\frac{K_S c_1+c_1^2}{2}}=(-1)^{g-1}$ compares the complex orientation of $\mathfrak{M}$ and its natural orientation as a moduli space of anti-self-dual connections [4]. By [9, theorem 10], this is worked out to be $(-1)^{g-1}<(4\omega+X)^a(X^2)^b\phi_{i_1}\cdots\phi_{i_r}X^r,[J]>$, where $X^{2g-1+i}=s_i(\mathcal{E}_\zeta)=\frac{(-8)^i}{i!}\omega^i$. Thus the theorem follows from lemmas 8 and 10. □

*Remark* 12. The formula in theorem 11 is not right for $g=2$, as in such case, the quantum product $h^2\in QH^*(N)$ differs from $\beta$ by a quantum correction.

*Remark* 13. Suppose $g\geq 2$ and let $\alpha^a\beta^b\psi_{i_1}\cdots\psi_{i_r}\in\mathbb{A}(\Sigma)$ have degree $6g-6$. Then

$$\begin{aligned}\Psi_0^{M_\Sigma}(\alpha,\overset{(a)}{\ldots},\alpha,\beta,\overset{(b)}{\ldots},\beta,\psi_{i_1},\ldots,\psi_{i_r}) &= \epsilon_S(\mathbb{P}^1)<\alpha^a\beta^b\psi_{i_1}\cdots\psi_{i_r},[M_\Sigma]>=\\ &= -D_{S,H}^{\mathbb{P}^1}((2\Sigma)^a(-4\mathrm{pt})^b\gamma_{i_1}^{\#}\cdots\gamma_{i_r}^{\#}),\end{aligned}$$

as the moduli space of anti-self-dual connections on $S$ of dimension $6g-6$ is $M_\Sigma$.

## 5. Quantum cohomology of $M_\Sigma$

It is natural to ask to what extent the first quantum correction determines the full structure of the quantum cohomology of $M_\Sigma$. In [3], Donaldson finds the first quantum correction for $M_\Sigma$ when the genus of $\Sigma$ is $g=2$ and proves that this is enough to find the quantum product. Now it is our intention to show how the Gromov-Witten invariants $\Psi_A^{M_\Sigma}$ determine completely $QH^*(M_\Sigma)$. First we check an interesting fact.

**Lemma 14.** *Let $g \geq 3$. Then $\gamma = -2\sum \psi_i\psi_{i+g}$ as elements in $QH^*(M_\Sigma)$ (i.e. using the quantum product in the right hand side).*

*Proof.* Let $\hat{\gamma} = -2\sum \psi_i\psi_{i+g} \in QH^*(M_\Sigma)$. In principle, it is $\hat{\gamma} = \gamma + s\alpha$, for some $s \in \mathbb{Q}$. Let us show that $s=0$. Multiplying by $\alpha^{3g-4}$, we have $\hat{\gamma}\alpha^{3g-4} = \gamma\alpha^{3g-4} + s\alpha^{3g-3}$. Considering the component in $H^{6g-6}(M_\Sigma)$ and using lemma 8, we have

$$-2\sum \Psi_L^N(\alpha, \overset{(3g-4)}{\ldots}, \alpha, \psi_i, \psi_{i+g}) = \Psi_L^N(\alpha, \overset{(3g-4)}{\ldots}, \alpha, \gamma) + s < \alpha^{3g-3}, [M_\Sigma] > .$$

Now, in $N$, $\gamma$ is the cup product $-2\sum \phi_i\phi_{i+g}h^2$. It is easy to check that this coincides with the quantum product $-2\sum \phi_i\phi_{i+g}h^2$. For $g>3$ it is evident because of the degree. For $g=3$ there might be a quantum correction in $H^0(N)$, but this is $-2\sum \Psi_L^N(\phi_i, \phi_{i+g}, h, h, \mathrm{pt}) = 0$ (since lines are contained in the fibres). Now lemma 10 and its proof imply that $-2\sum \Psi_L^N(\alpha, \ldots, \alpha, \psi_i, \psi_{i+g}) = \Psi_L^N(\alpha, \ldots, \alpha, \gamma)$, so $s=0$. □

We are pursuing to prove an isomorphism between $QH^*(M_\Sigma)$ and $HF^*(\Sigma \times \mathbb{S}^1)$, the

instanton Floer homology of the three manifold $\Sigma \times \mathbb{S}^1$. First recall the main result contained in [10].

**Theorem 15** ([10])**.** *Define $R_0^1 = 1$, $R_0^2 = 0$, $R_0^3 = 0$ and then recursively, for all $r \geq 1$,*

$$\begin{cases} R_{r+1}^1 = \alpha R_r^1 + r^2 R_r^2 \\ R_{r+1}^2 = (\beta + (-1)^{r+1}8)R_r^1 + \frac{2r}{r+1}R_r^3 \\ R_{r+1}^3 = \gamma R_r^1 \end{cases}$$

*Put $I_r' = (R_r^1, R_r^2, R_r^3) \subset \mathbb{Q}[\alpha,\beta,\gamma]$, $r \geq 0$. Then the $Sp\,(2g,\mathbb{Z})$-decomposition of $HF^*(\Sigma \times \mathbb{S}^1)$ is*

$$HF^*(\Sigma \times \mathbb{S}^1) = \bigoplus_{k=0}^{g-1} \Lambda_0^k H^3 \otimes \mathbb{Q}[\alpha,\beta,\gamma]/I_{g-k}'.$$

The elements $R_r^1$, $R_r^2$ and $R_r^3$ are deformations graded mod 4 of $q_r^1$, $q_r^2$ and $q_r^3$, respectively. This means that we can write

$$R_r^i = \sum_{j \geq 0} R_{r,j}^i, \tag{14}$$

where $\deg(R^i_{r,j}) = \deg(q^i_r) - 4j$, $j \geq 0$, and $R^i_{r,0} = q^i_r$. In the case of $QH^*(M_\Sigma)$ we shall have

**Proposition 16.** *The* $Sp\,(2g,\mathbb{Z})$*-decomposition of* $QH^*(M_\Sigma)$ *is*

$$QH^*(M_\Sigma) = \bigoplus_{k=0}^{g-1} \Lambda_0^k H^3 \otimes \mathbb{Q}[\alpha,\beta,\gamma]/J_{g-k},$$

*where* $J_r$ *is generated by three elements* $Q^1_r$*,* $Q^2_r$ *and* $Q^3_r$*, which are deformations graded mod 4 of* $q^1_r$*,* $q^2_r$ *and* $q^3_r$*, respectively.*

*Proof.* The action of $\mathrm{Sp}\,(2g,\mathbb{Z})$ on $M_\Sigma$ being symplectic (see [16, section 3.1]), we have an epimorphism of rings (with $\mathrm{Sp}\,(2g,\mathbb{Z})$-actions) like in (3)

$$\mathbb{A}(\Sigma) \twoheadrightarrow QH^*(M_\Sigma).$$

This induces an epimorphism on the invariant parts

$$\mathbb{Q}[\alpha,\beta,\gamma] \twoheadrightarrow QH^*_I(M_\Sigma),$$

where $\gamma = -2\sum_{i=0}^{g}\psi_i\psi_{i+g}$ (see lemma 14). Therefore we have maps

$$\Lambda_0^k(\psi_1,\dots,\psi_{2g}) \otimes \mathbb{Q}[\alpha,\beta,\gamma] \to QH^*(M_\Sigma). \tag{15}$$

Let $V_k$ be the image of the map (15). As $\Lambda_0^k H^3$, $0 \leq k \leq g-1$, are inequivalent irreducible $\mathrm{Sp}\,(2g,\mathbb{Z})$-modules, the subspaces $V_k$ are pairwise orthogonal. On the other hand, the existence of the basis (6) of $H^*(M_\Sigma)$ and the results in [18] imply that $\{x_i^{(k)}\alpha^a\beta^b\gamma^c / k = 0,1,\dots,g-1,\ a+b+c < g-k,\ i \in B_k\}$ (where quantum products are now understood) is a basis of $QH^*(M_\Sigma)$. So the subspaces $V_k$ generate $QH^*(M_\Sigma)$, i.e.

$$QH^*(M_\Sigma) = \bigoplus_{k=0}^{g-1} V_k. \tag{16}$$

Actually this decomposition coincides with the decomposition (5). This is proved by giving a definition of $V_k$ independent of the ring structure (cup product or quantum product). For instance, say that $V_k$ is the space generated by elements which are orthogonal to $V_0, \dots, V_{k-1}$ and such that the $\mathrm{Sp}\,(2g,\mathbb{Z})$-module generated by them have dimension equal to $\dim \Lambda_0^k H^3$.

Our second purpose is to describe the kernel of (15), i.e. the relations satisfied by the elements of $\Lambda_0^k H^3$, $\alpha$, $\beta$ and $\gamma$. The results in [18] imply that we only need to write the relations of $V_k \subset H^*(M_\Sigma)$ in terms of the quantum product. Fix $k$, and recall $x_0^{(k)} = \psi_1\psi_2\cdots\psi_k \in \Lambda_0^k H^3$ from (7). By section 2 the relations in $V_k$ are given by $x_0^{(k)} q_{g-k}^i$, $i = 1, 2, 3$, and its $\mathrm{Sp}\,(2g, \mathbb{Z})$-transforms. We rewrite these relations in terms of the quantum product, using the basis (6), as

$$x_0^{(k)} q_{g-k}^i = \sum_{a+b+c<g-k} x_{abc} \alpha^a \beta^b \gamma^c \in QH^*(M_\Sigma) \tag{17}$$

where $x_{abc} \in \Lambda_0^k H^3$, and the monomials in the right hand side have degree strictly less that the degree of the left hand side. Now we want to prove that $x_{abc}$ are all multiples of $x_0^{(k)}$. Suppose not. Then it is easy to see that there exists $\phi \in \mathrm{Sp}\,(2g, \mathbb{Z})$ satisfying $\phi(x_0^{(k)}) = x_0^{(k)}$ and $\phi(x_{abc}) \neq x_{abc}$. Consider (17) minus its transform under $\phi$. This is a relation between the elements of the basis of $V_k$, which is impossible.

Therefore (17) can be rewritten as

$$x_0^{(k)} (q_{g-k}^i + Q_{g-k,1}^i + Q_{g-k,2}^i + \cdots) = 0,$$

where $\deg Q_{g-k,j}^i = \deg q_{g-k}^i - 4j$, $j \geq 1$. This finishes the proof. □

**Lemma 17.** *$\gamma J_k \subset J_{k+1} \subset J_k$, for $k = 0, 1, \ldots, g-1$.*

*Proof.* Let $f \in J_k \subset \mathbb{Q}[\alpha, \beta, \gamma]$. By definition (proposition 16) this means that the quantum product $\psi_1 \cdots \psi_{g-k} f = 0$. Using the action of $\mathrm{Sp}\,(2g, \mathbb{Z})$ we have $\psi_1 \cdots \psi_{g-k-1} \psi_i f = 0$, for $g - k \leq i \leq g$. Thus $\psi_1 \cdots \psi_{g-(k+1)} \gamma f = 0$, i.e. $\gamma f \in J_{k+1}$. For the second inclusion, let $f \in J_{k+1}$. Then $\psi_1 \cdots \psi_{g-(k+1)} f = 0$ and hence $\psi_1 \cdots \psi_{g-k} f = 0$, i.e. $f \in J_k$. □

**Proposition 18.** *There are numbers $c_r, d_r \in \mathbb{Q}$, $1 \leq r \leq g-1$, such that for $0 \leq r \leq g-1$ it is*

$$\begin{cases} Q_{r+1}^1 = \alpha Q_r^1 + r^2 Q_r^2 \\ Q_{r+1}^2 = (\beta + c_{r+1}) Q_r^1 + \frac{2r}{r+1} Q_r^3 \\ Q_{r+1}^3 = \gamma Q_r^1 + d_{r+1} Q_r^2 \end{cases}$$

*Proof.* Completely analogous to the proof of [10, theorem 10]. □

**Proposition 19.** *For all $1 \leq r \leq g$, $c_r = (-1)^{r+g+1} 8$ and $d_r = 0$.*

*Proof.* We write $R_{g-k}^i = \sum_{j \geq 0} R_{g-k,j}^i$ and $Q_{g-k}^i = \sum_{j \geq 0} Q_{g-k,j}^i$, as in (14), for $0 \leq k \leq g-1$. Then $R_{g-k,0}^i = Q_{g-k,0}^i = q_{g-k}^i$. The coefficients $c_r$ and $d_r$ are determined by the first correction term $Q_{r,1}^i$ of $Q_r^i$. By the definition of $R_r^i$ in theorem 15, we only need to check that $R_{g-k,1}^i = (-1)^g Q_{g-k,1}^i$, for $i = 1, 2, 3$, $0 \leq k \leq g-1$.

Fix $i$ and $k$. Recall $x_0^{(k)} = \psi_1 \cdots \psi_k \in \Lambda_0^k H^3$. By theorem 15, $x_0^{(k)} R_{g-k}^i = 0 \in HF^*(\Sigma \times \mathbb{S}^1)$. Pick an arbitrary $f = \alpha^a \beta^b \psi_{i_1} \cdots \psi_{i_r} \in \mathbb{A}(\Sigma)$ of degree $6g - 2 - \deg(x_0^{(k)} q_{g-k}^i)$. In $HF^*(\Sigma \times \mathbb{S}^1)$ the pairing $< x_0^{(k)} R_{g-k}^i, f >= 0$, i.e. $D_{S,H}^{(w,\Sigma)}(\bar{x}_0^{(k)} \bar{R}_{g-k}^i \bar{f}) =$

0, where $\bar{R}^i_{g-k} = R^i_{g-k}(2\Sigma, -4x, -2\sum \gamma_i^{\#}\gamma_{g+i}^{\#})$, and analogously for $\bar{f}$ and $\bar{x}_0^{(k)}$ (for the notation $D^{(w,\Sigma)}$ see [10]). This means that

$$D^{\mathbb{P}^1}_{S,H}(\bar{x}_0^{(k)}\bar{R}^i_{g-k,1}\bar{f}) + D^{c_1}_{S,H}(\bar{x}_0^{(k)}\bar{R}^i_{g-k,0}\bar{f}) = 0.$$

From theorem 11 and remark 13 we have that the component in $H^{6g-6}(M_\Sigma)$ of the quantum product $-x_0^{(k)}R^i_{g-k,1}f + (-1)^{g-1}x_0^{(k)}R^i_{g-k,0}f \in QH^*(M_\Sigma)$ vanishes, i.e.

$$- < x_0^{(k)}R^i_{g-k,1}, f > +(-1)^{g-1} < x_0^{(k)}R^i_{g-k,0}, f >= 0 \tag{18}$$

in $QH^*(M_\Sigma)$.

On the other hand, proposition 16 says that $x_0^{(k)}Q^i_{g-k} = 0 \in QH^*(M_\Sigma)$. Multiplying by $f$, $x_0^{(k)}Q^i_{g-k}f = 0$, so the component in $H^{6g-6}(M_\Sigma)$ of the quantum product $x_0^{(k)}Q^i_{g-k,1}f + x_0^{(k)}Q^i_{g-k,0}f \in QH^*(M_\Sigma)$ is zero. Thus

$$< x_0^{(k)}Q^i_{g-k,1}, f > + < x_0^{(k)}Q^i_{g-k,0}, f >= 0 \tag{19}$$

in $QH^*(M_\Sigma)$.

Equations (18) and (19) imply together that

$$< x_0^{(k)}Q^i_{g-k,1}, f >= (-1)^g < x_0^{(k)}R^i_{g-k,1}, f >, \tag{20}$$

for any $f \in \mathbb{A}(\Sigma)$ of degree $6g - 2 - \deg(x_0^{(k)}q^i_{g-k}) = 6g - 6 - \deg(x_0^{(k)}Q^i_{g-k,1})$. As we are considering the pairing on classes of complementary degree, equation (20) holds in $H^*(M_\Sigma)$ as well. By section 2, $Q^i_{g-k,1} \equiv (-1)^g R^i_{g-k,1} \pmod{I_{g-k}}$. Considering the degrees, it must be $Q^1_{g-k,1} = (-1)^g R^1_{g-k,1}$ and $Q^2_{g-k,1} = (-1)^g R^2_{g-k,1}$. For $i = 3$, the difference $Q^3_{g-k,1} - (-1)^g R^3_{g-k,1}$ is a multiple of $q^1_{g-k}$. The vanishing of the coefficient of $\alpha^{g-k}$ for both $Q^3_{g-k}$ and $R^3_{g-k}$ (see theorem 15 and equation (17)) implies $Q^3_{g-k,1} = (-1)^g R^3_{g-k,1}$. □

Putting all together we have proved the following

**Theorem 20.** *The quantum cohomology of $M_\Sigma$, for $\Sigma$ a Riemann surface of genus $g \geq 3$, has a presentation*

$$QH^*(M_\Sigma) = \bigoplus_{k=0}^{g-1} \Lambda_0^k H^3 \otimes \mathbb{Q}[\alpha, \beta, \gamma]/J_{g-k}.$$

*where $J_r = (Q^1_r, Q^2_r, Q^3_r)$ and $Q^i_r$ are defined recursively by setting $Q^1_0 = 1$, $Q^2_0 = 0$, $Q^3_0 = 0$ and putting for all $r \geq 0$*

$$\begin{cases} Q^1_{r+1} = \alpha Q^1_r + r^2 Q^2_r \\ Q^2_{r+1} = (\beta + (-1)^{r+g+1}8)Q^1_r + \frac{2r}{r+1}Q^3_r \\ Q^3_{r+1} = \gamma Q^1_r \end{cases}$$

**Corollary 21.** *Let $\Sigma$ be a Riemann surface of genus $g \geq 3$. Then there is an isomorphism*

$$QH^*(M_\Sigma) \stackrel{\cong}{\to} HF^*(\Sigma \times \mathbb{S}^1).$$

*For $g$ even, the isomorphism sends $(\alpha, \beta, \gamma) \mapsto (\alpha, \beta, \gamma)$. For $g$ odd, the isomorphism sends $(\alpha, \beta, \gamma) \mapsto (\sqrt{-1}\,\alpha, -\beta, -\sqrt{-1}\,\gamma)$.*

*Proof.* This is a consequence of the descriptions of $QH^*(M_\Sigma)$ and $HF^*(\Sigma \times \mathbb{S}^1)$ in theorem 20 and theorem 15, respectively. □

*Remark* 22. Alternatively, we can say that for any $g \geq 1$ there is an isomorphism $QH^*(M_\Sigma) \stackrel{\cong}{\to} HF^*(\Sigma \times \mathbb{S}^1)$, taking $(\alpha, \beta, \gamma) \mapsto (\sqrt{-1}^g \alpha, \sqrt{-1}^{2g} \beta, \sqrt{-1}^{3g} \gamma)$.

## 6. The cases $g = 1$ and $g = 2$

Let us review the cases of genus $g = 1$ and $g = 2$ in the view of theorem 20. These cases are somehow atypical, as the generators precise the introduction of

quantum corrections, a fact already noted in [1].

*Example* 23. Let $\Sigma$ be a Riemann surface of genus $g = 1$. Then $M_\Sigma$ is a point and we can write

$$QH^*(M_\Sigma) = \mathbb{Q}[\alpha, \hat{\beta}, \gamma]/(\alpha, \hat{\beta} + 8, \gamma),$$

where we have defined $\hat{\beta} = \beta - 8$. This agrees with theorem 20 but with corrected generators. Again $QH^*(M_\Sigma) \stackrel{\cong}{\to} HF^*(\Sigma \times \mathbb{S}^1)$, where $(\alpha, \hat{\beta}, \gamma) \mapsto (\sqrt{-1}\,\alpha, -\beta, -\sqrt{-1}\,\gamma)$.

*Example* 24. Let $\Sigma$ be a Riemann surface of genus $g = 2$. The quantum cohomology ring $QH^*(M_\Sigma)$ has been computed by Donaldson [3], using an explicit description of $M_\Sigma$ as the intersection of two quadrics in $\mathbb{P}^5$. Let $h_2$, $h_4$ and $h_6$ be the integral generators of $QH^2(M_\Sigma)$, $QH^4(M_\Sigma)$ and $QH^6(M_\Sigma)$, respectively. Then, with our notations, $\alpha = h_2$, $\beta = -4h_4$ and $\gamma = 4h_6$ (see [1]). Define $\hat{\gamma} = -2\sum \psi_i \psi_{i+g} \in QH^*(M_\Sigma)$. The computations in [3] yield $\hat{\gamma} = \gamma - 4\alpha$ (compare with lemma 14). Put $\hat{\beta} = \beta + 4$. It is now easy to check that the relations found in [3] can be translated to

$$QH^*(M_\Sigma) = \left(H^3 \otimes \mathbb{Q}[\alpha, \hat{\beta}, \hat{\gamma}]/(\alpha, \hat{\beta} - 8, \hat{\gamma})\right) \oplus \mathbb{Q}[\alpha, \hat{\beta}, \hat{\gamma}]/(Q_2^1, Q_2^2, Q_2^3),$$

where $Q_2^1 = \alpha^2 + \hat{\beta} - 8$, $Q_2^2 = (\hat{\beta} + 8)\alpha + \hat{\gamma}$ and $Q_2^3 = \alpha\hat{\gamma}$ (defined exactly as in theorem 20, but with corrected generators). Now $QH^*(M_\Sigma) \stackrel{\cong}{\to} HF^*(\Sigma \times \mathbb{S}^1)$, where $(\alpha, \hat{\beta}, \hat{\gamma}) \mapsto (\sqrt{-1}\,\alpha, -\beta, -\sqrt{-1}\,\gamma)$.

The artificially introduced definition of $\hat{\beta}$ is due to the same phenomenon which causes the failure of lemma 10 for $g = 2$, i.e. the quantum product $h^2$ differs from $\beta$ in (12) (defined with the cup product) because of a quantum correction in $QH^*(N)$ which appears when $g = 2$.

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