

HODGE THEORY FOR RIEMANNIAN SOLENOIDS

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ABSTRACT. A measured solenoid is a compact laminated space endowed with a transversal measure. The De Rham L^2 -cohomology of the solenoid is defined by using differential forms which are smooth in the leafwise directions and L^2 in the transversal direction. We develop the theory of harmonic forms for Riemannian measured solenoids, and prove that this computes the De Rham L^2 -cohomology of the solenoid. This implies in particular a Poincaré duality result.

Dedicated to the Memory of the 100th Anniversary of S. M. Ulam

1. INTRODUCTION

In this article we continue the geometric study of solenoids initiated in [6]. Solenoids are compact topological spaces which are locally modeled by the product of a k -dimensional ball by some transversal space and admit a transversal measure invariant by holonomy.

First we review how the De Rham cohomology theory, fundamental classes, and singular cohomology do extend to solenoids and preserve most of the functorial properties. The theory of bundles, connections and Chern classes also goes through.

For a solenoid S endowed with a transversal measure μ , we have a well defined De Rham L^2 -cohomology theory, which is suited to implement classical techniques from Harmonic and Functional Analysis. The De Rham L^2 -cohomology is defined by using forms which are smooth in the leaf-wise directions, and are L^2 -integrable with respect to μ in the transversal direction. We introduce the *reduced* De Rham L^2 -cohomology $\bar{H}_{DR}^*(S_\mu)$ as the quotient the De Rham L^2 -cohomology $H_{DR}^*(S_\mu)$ with the closure of $\{0\}$ (making it a Hausdorff topological space). See [5] for these notions.

Most of the formal theory of pseudo-differential operators can be carried out for spaces of sections of bundles which are L^2 -transversally. In order to

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extend classical Hodge theory as developed in [12], we define Sobolev spaces of sections using the L^2 transversal structure. Sobolev regularity lemma holds in this general setting but Rellich compact embedding does not in general. With these tools at hand we can develop harmonic theory for solenoids.

We obtain a Hodge theorem giving an isomorphism of the reduced De Rham cohomology with the space of harmonic forms. We prove the main results:

Theorem 1.1. *Let S_μ be a compact oriented solenoid endowed with a transversal measure μ . Let $\mathcal{K}^p(S_\mu)$ be the space of p -forms which are harmonic in the leaf-wise directions and L^2 -transversally (with respect to μ). Then there is an isomorphism:*

$$\bar{H}_{DR}^p(S_\mu) \cong \mathcal{K}^p(S_\mu).$$

Corollary 1.2. *The $*$ -Hodge operator gives an isomorphism (Poincaré duality) $*$: $\bar{H}_{DR}^p(S_\mu) \rightarrow \bar{H}_{DR}^{k-p}(S_\mu)$.*

Contrary to the classical situation for compact manifolds where Rellich theorem holds, here the spaces of harmonic forms are not necessarily finite dimensional. Actually, the situation for solenoids is more similar to that of Hodge theory for L^2 -forms for complete non-compact manifolds (see e.g. [1]). However, when the transversal measure μ is ergodic (see definition 2.3), we may expect finite-dimensionality (see question 7.9). We discuss at the end the simple example of Kronecker foliations on the torus.

2. SOLENOIDS

We review the basic notions on solenoids introduced in [6].

Definition 2.1. Let $k \geq 0$, $r \geq 1$, $s \geq 0$ with $r \geq s$. A k -solenoid of class $C^{r,s}$ is a compact Hausdorff space endowed with an atlas of flow-boxes $\mathcal{A} = \{(U_i, \varphi_i)\}$,

$$\varphi_i : U_i \rightarrow D^k \times K(U_i),$$

where D^k is the k -dimensional open ball, and $K(U_i) \subset \mathbb{R}^l$ is an open set, the transversal of the flow-box. The changes of charts $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1}$ are of the form $\varphi_{ij}(x, y) = (X(x, y), Y(y))$, where $X(x, y)$ is of class $C^{r,s}$ and $Y(y)$ is of class C^s .

We shall always use *good* flow-boxes. By this, we mean a flow-box $U = D^k \times K(U)$ whose closure is contained in another flow-box $V = D^k \times K(V)$. Therefore $\overline{K(U)} \subset K(V)$ is compact.

Let S be a k -solenoid, and $U \cong D^k \times K(U)$ be a flow-box for S . The sets $L_y = D^k \times \{y\}$ are called the (local) leaves of the flow-box. A leaf $l \subset S$ of

the solenoid is a connected k -dimensional manifold whose intersection with any flow-box is a collection of local leaves. The solenoid is oriented if the leaves are oriented (in a transversally continuous way).

A transversal for S is a subset T which is a finite union of transversals of flow-boxes. Given two local transversals T_1 and T_2 and a path contained in a leaf from a point of T_1 to a point of T_2 , there is a well-defined germ of holonomy map at this point, h , from T_1 to T_2 .

Definition 2.2. Let S be a k -solenoid. A transversal measure $\mu = (\mu_T)$ for S associates to any local transversal T a locally finite measure μ_T supported on T , which are invariant by the holonomy, i.e. if h is a germ of holonomy map, then $h_*\mu_{T_1} = \mu_{T_2}$.

We denote by S_μ a k -solenoid S endowed with a transversal measure $\mu = (\mu_T)$. We refer to S_μ as a measured solenoid. Observe that for any transversal measure $\mu = (\mu_T)$ the scalar multiple $c\mu = (c\mu_T)$, where $c > 0$, is also a transversal measure. Notice that there is no natural scalar normalization of transversal measures.

Definition 2.3. A measured solenoid S_μ is *ergodic* if for any transversal T , and any subset $A \subset T$ invariant by the holonomy, either $\mu_T(A) = 0$ or $\mu_T(T - A) = 0$.

A solenoid S is *uniquely ergodic* if it has a unique (up to scalars) transversal measure μ and its support is the whole of S .

A Riemannian solenoid is a solenoid endowed with a Riemannian metric in the tangent spaces of the leaves, and with smoothness of class $C^{r,s}$. Note that a Riemannian metric defines a volume form in each leaf. A *daval* measure ν on S ([6]) is a finite Borel measure on the solenoid which in any flow-box $U = D^k \times K(U)$, it decomposes as volume along leaves, $\nu = \text{vol}_{D^k} \times \mu_{K(U)}$. Such a measure defines a transversal measure and moreover there is a one-to-one correspondence between daval measures and transversal measures. In particular, if a Riemannian solenoid is uniquely ergodic, then there is a *unique* daval measure with total mass 1.

Now let M be a smooth manifold of dimension n (and class C^r). An immersion of a k -solenoid S into M , with $k < n$, is a smooth map $f : S \rightarrow M$ (of class $C^{r,s}$) such that the differential restricted to the tangent spaces of leaves has rank k at every point of S . If M is a Riemannian manifold, this endows S with the pull-back Riemannian structure.

Denote by $\mathcal{C}_k(M)$ the space of compactly supported currents of dimension k on M . We have the following definition.

Definition 2.4. Let S_μ be an oriented measured k -solenoid. An immersion $f : S \rightarrow M$ defines a generalized Ruelle-Sullivan current $(S_\mu, f) \in \mathcal{C}_k(M)$ as follows. Let $\{U_i\}$ be a finite covering of S by flow-boxes, and a partition of unity $\{\rho_i\}$ associated to it. For $\omega \in \Omega^k(M)$, we define

$$\langle (S_\mu, f), \omega \rangle = \sum_i \int_{K(U_i)} \left(\int_{L_y} \rho_i f^* \omega \right) d\mu_{K(U_i)}(y),$$

where L_y denotes the horizontal disk of the flow-box.

In [6] it is proved that (S_μ, f) is a closed current. Therefore, it defines a real homology class

$$(2.1) \quad [S_\mu, f] \in H_k(M, \mathbb{R}).$$

Ruelle and Sullivan defined in [11] this notion for the restricted class of solenoids embedded in M . In [8], it is proved that if $a \in H_k(M, \mathbb{R})$ is any real homology class, then there is an immersed oriented k -solenoid $f : S_\mu \rightarrow M$ such that $a = [S_\mu, f]$. Moreover, S can be chosen to be uniquely ergodic and with holonomy generated by a single map. In [9], we prove that the set of currents (S_μ, f) for immersed oriented uniquely ergodic k -solenoids with $a = [S_\mu, f]$ is actually dense in the space of closed currents $\alpha \in \mathcal{C}_k(M)$ representing a .

3. COHOMOLOGY OF SOLENOIDS

In general, and for the remainder of the article, we shall consider solenoids of class $C^{\infty,0}$.

3.1. De Rham cohomology. Let S be a solenoid (here, we allow S to be a *non-compact* solenoid). The space of p -forms $\Omega^p(S)$ consist of p -forms on leaves with function coefficients that are smooth on leaves and partial derivatives of all orders continuous transversally. Using the differential d in the leaf-wise directions, we obtain the De Rham differential complex $(\Omega^*(S), d)$. The De Rham cohomology groups of the solenoid are defined as the quotients

$$(3.1) \quad H_{DR}^p(S) := \frac{\ker(d : \Omega^p(S) \rightarrow \Omega^{p+1}(S))}{\text{im}(d : \Omega^{p-1}(S) \rightarrow \Omega^p(S))}.$$

We can also consider the spaces $\Omega_m^p(S)$ of differential forms with function coefficients that are smooth on leaves and measurable transversally (then the partial derivatives are automatically measurable transversally). Then define in the same way the De Rham measurable cohomology groups $H_{DRm}^p(S)$ using the complex $(\Omega_m^*(S), d)$. Note the natural map $H_{DR}^p(S) \rightarrow H_{DRm}^p(S)$.

Proposition 3.1. *Let $\underline{\mathbb{R}}_c$ and $\underline{\mathbb{R}}_m$ be respectively the sheaf of functions which are locally constant on leaves and transversally continuous, resp. measurable. Then we have isomorphisms*

$$H_{DR}^p(S) \cong H^p(S, \underline{\mathbb{R}}_c),$$

and

$$H_{DRm}^p(S) \cong H^p(S, \underline{\mathbb{R}}_m).$$

Proof. The proofs are similar. We prove the first isomorphism. It follows from the existence of the sheaf resolution

$$\underline{\mathbb{R}}_c \rightarrow \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \dots$$

The exactness of this complex of sheaves is a Poincaré lemma: for a small open set $U = D^k \times K(U)$, the complex $0 \rightarrow \underline{\mathbb{R}}_c(U) \rightarrow \Omega^0(U) \rightarrow \Omega^1(U) \rightarrow \dots$ is exact. If $d_x \alpha(x, y) = 0$ then $\alpha(x, y) = d_x \beta(x, y)$, for a collection of forms $\beta(x, y)$, $y \in K(U)$. We can choose $\beta(x, y)$ to depend continuously (or measurably for the proof of the second isomorphism) on y , as can be seen by the usual construction. Note that clearly if $f(x, y)$ is a function with $df = 0$ then $f(x, y)$ is locally constant on x . \square

Remark 3.2. The spaces $\Omega^p(S)$ are topological vector spaces. Therefore the De Rham cohomology (3.1) inherits a natural topology. In general, these spaces are infinite dimensional (even for compact solenoids). In some references, it is customary to take the closure of the spaces $\text{im } d$ in definition (3.1), obtaining the *reduced De Rham cohomology groups*

$$\bar{H}_{DR}^p(S) = \frac{\ker d|_{\Omega^p}}{\overline{\text{im } d|_{\Omega^{p-1}}}}.$$

This is equivalent to quotienting $H_{DR}^p(S)$ by $\overline{\{0\}}$, obtaining thus Hausdorff vector spaces.

We shall list some basic properties of the De Rham cohomology:

- (1) **Functoriality.** Let S_1, S_2 be two solenoids. A smooth map $f : S_1 \rightarrow S_2$ is a map sending leaves to leaves and transversally continuous. f defines a map on De Rham cohomologies, $f^* : H_{DR}^p(S_2) \rightarrow H_{DR}^p(S_1)$, by $f^*[\omega] = [f^*\omega]$. This applies in particular to an immersion of a solenoid into a smooth manifold $f : S \rightarrow M$, or to the inclusion of a leaf $i : l \rightarrow S$.
- (2) **Mayer-Vietoris sequence.** Let U, V be two open subsets of a solenoid S . There is a short exact sequence of complexes: $\Omega^\bullet(U \cup V) \rightarrow \Omega^\bullet(U) \oplus \Omega^\bullet(V) \rightarrow \Omega^\bullet(U \cap V)$. The only non-trivial point is the surjectivity of the last map, but it follows from the existence of a partition of unity

$\{\rho_U, \rho_V\}$ subordinated to $\{U, V\}$: any $\omega \in \Omega^\bullet(U \cap V)$ is the image of $(\rho_V\omega, -\rho_U\omega)$. Taking the associated long exact sequence, we get the Mayer-Vietoris exact sequence

$$\dots \rightarrow H_{DR}^p(U \cup V) \rightarrow H_{DR}^p(U) \oplus H_{DR}^p(V) \rightarrow H_{DR}^p(U \cap V) \rightarrow H_{DR}^{p+1}(U \cup V) \rightarrow \dots$$

- (3) Homotopy. A homotopy between two maps $f_0, f_1 : S_1 \rightarrow S_2$ is a map $F : S_1 \times [0, 1] \rightarrow S_2$ (where $S_1 \times [0, 1]$ is given the solenoid structure with leaves $l \times [0, 1]$, for $l \subset S_1$ a leaf of S_1) such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. We say that the maps f_0, f_1 are homotopic, written $f_0 \sim f_1$. In this case $f_0^* = f_1^* : H_{DR}^p(S_2) \rightarrow H_{DR}^p(S_1)$.

We prove this as follows: factor $f_0 = F \circ i_0$, and $f_1 = F \circ i_1$, where $i_t : S_1 \rightarrow S_1 \times [0, 1]$ is given as $i_t(x) = (x, t)$. Then $f_0^* = f_1^*$ follows from $i_0^* = i_1^*$. Let us check this. Consider $p : S_1 \times [0, 1] \rightarrow S_1$. Then $p \circ i_t = Id$. Let us see that $i_t^* : H_{DR}^p(S \times I) \rightarrow H_{DR}^p(S)$ is an isomorphism inverse to p^* . It is enough to see that $h^* = (i_t \circ p)^* : H_{DR}^p(S \times I) \rightarrow H_{DR}^p(S \times I)$ is the identity, $h(x, s) = (x, t)$, $t \in I$ fixed. For a closed p -form $\omega = \omega_1(x, s) + \omega_2(x, s) \wedge ds$, note that $0 = d\omega = d_x\omega_1 + (-1)^p \frac{\partial \omega_1}{\partial s} \wedge ds + d_x\omega_2 \wedge ds = 0$ implies that $d_x\omega_1 = 0$ and $\frac{\partial \omega_1}{\partial s} = (-1)^{p+1} d_x\omega_2$. Now $p^*i_t^*\omega = \omega_1(x, t)$. Hence

$$\begin{aligned} (Id - p^*i_t^*)\omega &= \omega_1(x, s) - \omega_1(x, t) + \omega_2(x, s) \wedge ds \\ &= \int_t^s \frac{\partial \omega_1}{\partial s}(x, u) du + \omega_2(x, s) \wedge ds = \\ &= (-1)^{p+1} \left(\int_t^s d_x\omega_2(x, u) du + (-1)^{p-1} \omega_2(x, s) \wedge ds \right) \\ &= (-1)^{p+1} d \left(\int_t^s \omega_2(x, u) du \right), \end{aligned}$$

as required.

- (4) We say that two solenoids S_1, S_2 are of the same homotopy type if there are maps $f : S_1 \rightarrow S_2, g : S_2 \rightarrow S_1$, such that $f \circ g \sim Id_{S_2}, g \circ f \sim Id_{S_1}$. Then the De Rham cohomology groups of S_1 and S_2 are isomorphic.

3.2. Fundamental classes. Let S be an *oriented* compact k -solenoid. The De Rham cohomology groups do not depend on any measure of S . If $\mu = (\mu_T)$ is a transversal measure, then the integral \int_{S_μ} descends to cohomology giving a map [6]

$$(3.2) \quad \int_{S_\mu} : H_{DR}^k(S) \rightarrow \mathbb{R}.$$

We define the solenoidal homology as

$$H_p(S, \mathbb{R}_c) := H^p(S, \mathbb{R}_c)^* = H_{DR}^p(S)^*.$$

Then the map (3.2) defines a homology class $[S_\mu] \in H^k(S, \mathbb{R}_c)^* = H_k(S, \mathbb{R}_c)$. We shall call this element the *fundamental class* of S_μ .

Any map $f : S_1 \rightarrow S_2$ defines a map $f^* : H_{DR}^p(S_2) \rightarrow H_{DR}^p(S_1)$ and hence, by dualizing, a map $f_* : H_p(S_1, \mathbb{R}_c) \rightarrow H_p(S_2, \mathbb{R}_c)$. Applying this to an immersion $f : S_\mu \rightarrow M$ of an oriented, measured, compact solenoid into a smooth manifold, then we have the equality

$$f_*[S_\mu] = [S_\mu, f] ,$$

with the generalized Ruelle-Sullivan class defined in (2.1).

Note that if S has a dense leaf (in particular when S it is minimal, i.e. all leaves are dense), then $H_0(S, \mathbb{R}_c) = \mathbb{R}$. On the other hand, the dimension of the top degree homology counts the number of mutually singular transverse measures on S .

Theorem 3.3. *Let S be a compact, oriented k -solenoid. Then $H_k(S, \mathbb{R}_c)$ is isomorphic to the real vector space generated by all transversal measures.*

Proof. There is a well-defined linear map

$$\Psi : \mu \mapsto [S_\mu] \in H_k(S, \mathbb{R}_c)$$

which sends each signed transversal measure (with finite total mass in each transversal) to the associated fundamental class: Decompose $\mu = \mu_+ - \mu_-$ and integrate with respect to each measure. We prove first that Ψ is a bijection.

If $\Psi(\mu) = 0$ then

$$\int_{S_\mu} \omega = 0$$

for all k -forms $\omega \in \Omega^k(S)$. Let $U = D^k \times K(U)$ be a flow-box and let ω be a compactly supported k -form with integral 1 on D^k . For any function ϕ with $\text{supp } \phi \subset K(U)$, we have

$$0 = \int_{S_\mu} \phi \omega = \int_{K(U)} \phi \left(\int_{L_y} \omega \right) d\mu_{K(U)}(y) = \int_{K(U)} \phi d\mu_{K(U)}(y) .$$

Therefore $\mu_{K(U)} = 0$. So $\mu = 0$.

We prove that Ψ is onto. Let $C \in H_k(S, \mathbb{R}_c)$ be given. By definition $C : H_{DR}^k(S) \rightarrow \mathbb{R}$, so for any k -form ω , we have $C(\omega) \in \mathbb{R}$ and $C(d\eta) = 0$, for $\eta \in \Omega^{k-1}(S)$. Let $U = D^k \times K(U)$ be a flow-box, and fix a compactly supported k -form ω with integral 1 on D^k . Let ϕ be a continuous function with $\text{supp } \phi \subset K(U)$ and let $\tilde{\phi}$ be the corresponding function on the flow box $U = D^k \times K(U)$ constant on leaves. Now the map

$$\phi \mapsto C(\tilde{\phi} \omega) ,$$

is a continuous linear functional: if $\phi_n \rightarrow \phi$ in C^0 then $\tilde{\phi}_n \omega \rightarrow \tilde{\phi} \omega$ in $\Omega^k(S)$, so $C(\tilde{\phi}_n \omega) \rightarrow C(\tilde{\phi} \omega)$. Therefore by Riesz representation theorem it is represented by a signed measure $\mu_{K(U)}$ with finite total mass,

$$C(\tilde{\phi} \omega) = \int_{K(U)} \phi d\mu_{K(U)} .$$

The measure $\mu_{K(U)}$ does not depend on the choice of ω . Taking another ω' we have $\omega' - \omega = d\eta$, with η compactly supported in D^k . Hence $\tilde{\phi} \omega' - \tilde{\phi} \omega = d(\tilde{\phi} \eta)$ and $C(\tilde{\phi} \omega') = C(\tilde{\phi} \omega)$.

Also the constructed measure does not depend on the coordinates of the flow-box. Taking a change of chart $\Phi : D^k \times K(U) \rightarrow D^k \times K(U)$, we have that $C(\Phi^*(\tilde{\phi} \omega)) = C(\tilde{\phi} \omega)$ since $\Phi^*(\tilde{\phi} \omega) - \tilde{\phi} \omega = d\eta$ (both are compactly supported forms with leaf-wise integral 1). Finally, the (μ_T) are invariant by the holonomy. We only need to check the invariance by local holonomy, and this follows from the previous remark. \square

Remark 3.4. There is no Poincaré duality for $H_{DR}^*(S)$ in general. Moreover these spaces may be infinite dimensional (even for uniquely ergodic solenoids): if S is an n -torus foliated by irrational lines, then $H_{DR}^1(S)$ can be infinite-dimensional as the example in section 8 shows.

3.3. Singular cohomology. We consider the space $\text{Map}(I^n, S)$ of continuous maps $T : I^n \rightarrow S$ mapping into a leaf, and endow it with the uniform convergence topology. The degenerate maps (see [4]) form a closed subspace, therefore the quotient, $\text{Map}'(I^n, S)$, has a natural quotient topology. The space of singular chains $C_n(S)$ is the free abelian group generated by $\text{Map}'(I^n, S)$. There is a natural boundary map $\partial : C_n(S) \rightarrow C_{n-1}(S)$.

Let G be any topological abelian group. Define the cochains $C^n(S, G) = \text{Hom}_{\text{cont}}(C_n(S), G)$ as the continuous homomorphisms. That is, $\varphi : C_n(S) \rightarrow G$ such that if $T_k : I^n \rightarrow S$ are maps which converge to $T_o : I^n \rightarrow S$ in the uniform topology, then $\varphi(T_k) \rightarrow \varphi(T_o)$. Define the differential $\delta : C^n(S) \rightarrow C^{n+1}(S)$ by $\delta\varphi(T) = \varphi(\partial T)$. The solenoid singular cohomology of S with coefficients in G is defined as:

$$H^n(S, G) := \frac{\ker(\delta : C^n(S, G) \rightarrow C^{n+1}(S, G))}{\text{im}(\delta : C^{n-1}(S, G) \rightarrow C^n(S, G))} .$$

We have some basic properties:

- (1) **Functoriality.** Let $f : S_1 \rightarrow S_2$ be a solenoid map. Then there is a map $f_* : C_n(S_1) \rightarrow C_n(S_2)$, $f_*(T) = f \circ T$, and a map $f^* : C^n(S_2, G) \rightarrow C^n(S_1, G)$, $f^*(\varphi) = \varphi \circ f$. Clearly $f_*\delta = \delta f_*$, so the map descends to cohomology: $f^* : H^n(S_2, G) \rightarrow H^n(S_1, G)$.

- (2) Homotopy. Suppose that $f, g : S_1 \rightarrow S_2$ are two homotopic solenoid maps. The usual construction yields a chain homotopy H between f^* and g^* (one only have to check that this map sends continuous cochains into continuous cochains). Therefore $f^* = g^* : H^n(S_2, G) \rightarrow H^n(S_1, G)$.
- (3) If S_1, S_2 are of the same homotopy type, then $H^n(S_1, G) \cong H^n(S_2, G)$.
- (4) If $U = D^k \times K(U)$ is a flow-box, then U is of the same homotopy type than $\{*\} \times K(U)$. Therefore $H^n(U) = 0$ for $n > 0$, and $H^0(U) = \text{Map}_{cont}(K(U), G)$. In particular, this implies that

$$\underline{\mathbb{R}}_c \rightarrow C^0(-, \mathbb{R}) \xrightarrow{\delta} C^1(-, \mathbb{R}) \xrightarrow{\delta} \dots$$

is a resolution. Therefore there is an isomorphism $H^n(S, \mathbb{R}) \cong H^n(S, \underline{\mathbb{R}}_c)$.

- (5) Mayer-Vietoris. For two open sets U, V with $S = U \cup V$, define $C_n(S; U, V)$ as the subcomplex generated by those singular chains completely contained in either U or V . Define accordingly $C^n(S; U, V)$. It is not difficult to see that the restriction $C^n(S, G) \rightarrow C^n(S, G; U, V)$ is chain homotopy equivalence (by a process of subdivision of simplices, as in [4]). Therefore the exact sequence $0 \rightarrow C^n(S, G; U, V) \rightarrow C^n(U, G) \oplus C^n(V, G) \rightarrow C^n(U \cap V, G) \rightarrow 0$ gives rise to a long exact sequence:

$$\dots \rightarrow H^p(U \cup V, G) \rightarrow H^p(U, G) \oplus H^p(V, G) \rightarrow H^p(U \cap V, G) \rightarrow H^{p+1}(U \cup V, G) \rightarrow \dots$$

4. DE RHAM L^2 -COHOMOLOGY

Now consider a k -solenoid S with a transversal measure μ . There is a notion of cohomology which takes into account the transversal measure structure. For this, we work with forms which are L^2 -transversal relative to μ .

Definition 4.1. A function f is $L^2(\mu)$ -transversally smooth if in any (good) flow-box $U = D^k \times K(U)$ all partial derivatives on the first variable exist and are in $L^2(\mu_{K(U)})$, i.e. if we write f as $f(x, y)$ then for all $r \geq 0$,

$$\int_{K(U)} \|f(\cdot, y)\|_{C^r}^2 d\mu_{K(U)}(y) < \infty .$$

We consider the space of forms

$$\Omega_{L^2(\mu)}^p(S)$$

which are $L^2(\mu)$ -transversally smooth, i.e. locally these are forms $\alpha = \sum f_I(x, y) dx_I$, where f_I are $L^2(\mu)$ -transversally smooth functions. There is a well-defined differential along leaves $d : \Omega_{L^2(\mu)}^p(S) \rightarrow \Omega_{L^2(\mu)}^{p+1}(S)$ which defines the complex

$(\Omega_{L^2(\mu)}^*(S), d)$. We define the *De Rham L^2 -cohomology* vector space as the quotients

$$(4.1) \quad H_{DR}^p(S_\mu) := \frac{\ker(d : \Omega_{L^2(\mu)}^p(S) \rightarrow \Omega_{L^2(\mu)}^{p+1}(S))}{\operatorname{im}(d : \Omega_{L^2(\mu)}^{p-1}(S) \rightarrow \Omega_{L^2(\mu)}^p(S))}.$$

We also introduce the reduced De Rham L^2 -cohomology:

$$(4.2) \quad \bar{H}_{DR}^p(S_\mu) := \frac{\ker d}{\operatorname{im} d}.$$

Note that there are natural maps

$$H_{DR}^p(S) \rightarrow H_{DR}^p(S_\mu) \rightarrow H_{DRm}^p(S),$$

since $C^{\infty,0}$ -functions are $L^2(\mu)$ -transversally smooth. The integration map \int_{S_μ} is well-defined for forms in $\Omega_{L^2(\mu)}^k$, since a $L^2(\mu)$ -transversally smooth k -form is automatically $L^1(\mu)$ -transversal (all measures are finite measures on compact transversals). So we have $\int_{S_\mu} : H_{DR}^k(S_\mu) \rightarrow \mathbb{R}$.

Let $\underline{\mathbb{R}}_\mu$ be the sheaf of measurable functions which are locally constant on leaves and $L^2(\mu)$ -transversally. A standard Poincaré lemma shows that there is a resolution of sheaves

$$\underline{\mathbb{R}}_\mu \rightarrow \Omega_{L^2(\mu)}^0 \rightarrow \Omega_{L^2(\mu)}^1 \rightarrow \dots \rightarrow \Omega_{L^2(\mu)}^k.$$

So we get a natural isomorphism

$$H_{DR}^p(S_\mu) \cong H^p(S, \underline{\mathbb{R}}_\mu).$$

Definition 4.2. The ergodic dimension of μ is $1 \leq d(\mu) \leq +\infty$ defined to be the maximal number of mutually singular non-zero transversal measures μ_i such that

$$\mu \geq \sum_{i=1}^d c_i \mu_i,$$

with $c_i > 0$ in any transversal.

By classical ergodic theory, if the ergodic dimension $d(\mu)$ is finite then μ is a linear combination of exactly $d(\mu)$ ergodic transversal measures.

Lemma 4.3. *The ergodic dimension of μ is equal to the dimension of $H_{DR}^0(S_\mu)$,*

$$d(\mu) = \dim_{\mathbb{R}} H_{DR}^0(S_\mu).$$

In particular μ is ergodic if and only if $H_{DR}^0(S_\mu) \cong \mathbb{R}$.

Proof. Let $d \geq 1$ be finite and $d \leq d(\mu)$. By definition, $\mu \geq \sum_{i=1}^d c_i \mu_i$, where μ_i are ergodic measures and $c_i > 0$. Consider disjoint measurable subsets $S_i \subset S$ which are leaf-saturated, such that S_i is of total measure for μ_i and of zero measure for any μ_j with $j \neq i$ in any transversal. Then the characteristic functions χ_{S_i} are independent in $\Omega_{L^2(\mu)}^0(S)$, since $\mu(S_i) > 0$ for all i : if $f = \sum \lambda_i \chi_{S_i} = 0$ then $0 = \|f\|^2 = \int_{S_\mu} |f|^2 = \sum \lambda_i^2 \mu(S_i) \implies \lambda_i = 0$ for all i . (Here we fix an auxiliary Riemannian metric and consider the daval measures corresponding to the transversal measures.)

Thus $\dim_{\mathbb{R}} H_{DR}^0(S_\mu) = \dim_{\mathbb{R}} H^0(S, \underline{\mathbb{R}}_\mu) \geq d$. This proves the result when $d(\mu) = +\infty$.

Now assume that $d(\mu)$ is finite. Write $\mu = \sum_{i=1}^d c_i \mu_i$, where μ_i are ergodic measures, $d = d(\mu)$. Let S_i be as before. Let us prove that the (χ_{S_i}) do generate $H^0(S, \underline{\mathbb{R}}_\mu)$.

Let f be any measurable function locally constant on leaves and $L^2(\mu)$ -transversally. Note that f is automatically $L^2(\mu_j)$ -transversally. For a transversal T , the real function $x \mapsto \mu_{j,T}(T \cap f^{-1}((-\infty, x]))$ is a non-decreasing Heaviside function taking the values 0 and 1 (by ergodicity of μ_j). So this function has a jump at some $x_j \in \mathbb{R}$ that is independent of the transversal T (as holonomy shows). Then $F_{x_j} = f^{-1}(x_j)$ has total μ_j -measure. So $F_{x_j} = S_j$ up to a set of μ_j -measure zero, thus of μ -measure zero. That means that f is constant along each S_j up to a set of μ -measure zero. So $f = \sum x_j \chi_{S_j}$ in $L^2(\mu)$ (note that $S - (\cup S_j)$ is of zero μ -measure). \square

We review basic properties of the De Rham L^2 -cohomology:

- (1) There is not cup product, and therefore the $H_{DR}^*(S_\mu)$ are just vector spaces (not rings).
- (2) Functoriality. If $f : S_1 \rightarrow S_2$ is a solenoidal map, then we require that $\mu_2 = f_* \mu_1$. This means that for any local transversal T_1 of S_1 , $f(T_1)$ is a local transversal of S_2 and the transported measure $f_* \mu_1$ is a constant multiple of μ_2 on the transversal. Note that this is automatic when the solenoids are uniquely ergodic. Then for any form ω which is $L^2(\mu_2)$ -transversally smooth we have that $f^* \omega$ is $L^2(\mu_1)$ -transversally smooth.
- (3) Mayer-Vietoris. It holds exactly as in subsection 3.1.
- (4) Poincaré duality. We shall see that it holds for the reduced L^2 -cohomology for oriented ergodic solenoids (see corollary 7.8).

5. BUNDLES OVER SOLENOIDS

Let S be a k -solenoid. A vector bundle of rank n over S consists of a $(k+n)$ -solenoid E and a projection map $\pi : E \rightarrow S$ satisfying the following condition: there is an open covering U_α for S , and solenoid isomorphisms $\psi_\alpha : E_\alpha = \pi^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times \mathbb{R}^n = D^k \times K(U_\alpha) \times \mathbb{R}^n$, such that $\pi = pr_1 \circ \psi_\alpha$, where $pr_1 : U_\alpha \times \mathbb{R}^n \rightarrow U_\alpha$ denotes the projection, and the transition functions

$$\psi_\alpha \circ \psi_\beta^{-1} : (U_\beta \cap U_\alpha) \times \mathbb{R}^n \rightarrow (U_\beta \cap U_\alpha) \times \mathbb{R}^n$$

are of the form $(x, y, v) \mapsto (x, y, g_{\alpha\beta}(x, y)(v))$, where $g_{\alpha\beta}$ is a $C^{\infty,0}$ -smooth function from $U_\alpha \cap U_\beta$ to $GL(n)$.

Some points are easy to check:

- (1) The usual constructions of vector bundles remain valid here: direct sums, tensor products, symmetric and anti-symmetric products. Also there are notions of sub-bundle and of quotient bundle.
- (2) A section of a bundle $\pi : E \rightarrow S$ is a map $s : S \rightarrow E$ such that $\pi \circ s = Id$. We denote the space of sections as $\Gamma(E)$. By definition these are maps of class $C^{\infty,0}$.
- (3) If S_μ is a measured solenoid, and $E \rightarrow S$ is a vector bundle, then we have the notion of sections which are $L^2(\mu)$ -transversally smooth. Locally, in a chart $E_\alpha = D^k \times K(U) \times \mathbb{R}^n \rightarrow U_\alpha = D^k \times K(U)$, the section is written $s(x, y) = (x, y, v(x, y))$. We require that v is C^∞ on x and $L^2(\mu)$ on y . This does not depend on the chosen trivialization.
- (4) If $f : S_1 \rightarrow S_2$ is a solenoid map, and $\pi : E \rightarrow S_2$ is a vector bundle, then the pull-back $f^*E = \{(p, v) \in S_1 \times E \mid f(p) = \pi(v)\}$ is naturally a vector bundle over S_1 .
- (5) The tangent bundle TS of S is an example of vector bundle. We have bundles of (p, q) -tensors $TS^{\otimes p} \otimes (TS^*)^{\otimes q}$ on any solenoid S . In particular, we have bundles of p -forms (anti-symmetric contravariant tensors) $\bigwedge^p T^*S$. Its sections are the p -forms $\Omega^p(S)$.
- (6) A metric on a bundle E is a section of $\text{Sym}^2(E^*)$ which is positive definite at every point. A metric on S is a metric on the tangent bundle. An orientation of a bundle E is a continuous choice of orientation for each of the fibers of E . An orientation of S is an orientation of its tangent bundle.

We define $\Omega^p(E) = \Gamma(\bigwedge^p T^*S \otimes E)$. A connection on a vector bundle $E \rightarrow S$ is a map

$$\nabla : \Gamma(E) \rightarrow \Omega^1(E),$$

such that $\nabla(f \cdot s) = f\nabla s + df \wedge s$. Consider a local trivialization in a flow-box U_α with coordinates (x, y) . Then $\nabla|_{U_\alpha} = d + a_\alpha$, where $a_\alpha \in \Omega^1(U_\alpha, \text{End } E)$. Under a change of trivialization $g_{\alpha\beta}$, for two trivializing open subsets U_α, U_β , we have the usual formula $a_\beta = g_{\alpha\beta}^{-1}a_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1}dg_{\alpha\beta}$.

A partition of unity argument proves that there are always connections on a vector bundle $E \rightarrow S$. The space of connections is an affine space over $\Omega^1(\text{End } E)$.

Given a connection ∇ on E , there is a unique map $d_\nabla : \Omega^p(E) \rightarrow \Omega^{p+1}(E)$, $p \geq 0$, such that $d_\nabla s = \nabla s$ for $s \in \Gamma(E)$, and $d_\nabla(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d_\nabla \beta$, for $\alpha \in \Omega^p(S)$, $\beta \in \Omega^q(E)$. It is easy to see that $\hat{F}_\nabla : \Gamma(E) \rightarrow \Omega^2(E)$, given by $\hat{F}_\nabla(s) = d_\nabla d_\nabla s$, has a tensorial character (i.e., it is linear on functions). Therefore there is a $F_\nabla \in \Omega^2(\text{End } E)$, called *curvature* of ∇ , such that $\hat{F}_\nabla(s) = F_\nabla \cdot s$. Locally on a trivialization U_α , we have the formula $F_\nabla = da_\alpha + a_\alpha \wedge a_\alpha$.

Given connections on vector bundles, there are induced connections on associated bundles (dual bundle, tensor product, direct sum, symmetric product, pull-back under a solenoid map, etc.). This follows in a straightforward way from the standard theory. In particular, if $l \hookrightarrow S$ is a leaf of a solenoid S , then we can perform the pull-back of the bundle and connection to the leaf, which consists on restricting them to l . This gives a bundle and connection of a complete k -dimensional manifold. Also, if $f : S \rightarrow M$ is an immersion of a solenoid in a smooth n -manifold, and $E \rightarrow M$ is a bundle with connection, then the pull-back construction produces a bundle with connection on S .

Consider a vector bundle $E \rightarrow S$ endowed with a metric. We say that a connection ∇ is compatible with the metric if it satisfies

$$d\langle s, t \rangle = \langle \nabla s, t \rangle + \langle s, \nabla t \rangle.$$

In the particular case of the tangent bundle TS of a Riemannian solenoid S , we have the Levi-Civita connection ∇^{LC} , which is the unique connection compatible with the metric and with torsion $T_\nabla(X, Y) = \nabla_X Y - \nabla_Y X = 0$. This is the Levi-Civita connection on each leaf, and the transversal continuity follows easily.

6. CHERN CLASSES

We can also define a complex vector bundle over a solenoid, by using \mathbb{C}^n as fiber, and taking the transition functions with values in $\text{GL}(n, \mathbb{C})$. An hermitian metric on a complex vector bundle is a positive definite hermitian form in each fiber with smoothness of type $C^{\infty, 0}$ on any local trivialization.

Let $E \rightarrow S$ be a complex vector bundle over a solenoid of rank n . Put a hermitian structure on E , and consider any hermitian connection ∇ on E . Then the curvature F_∇ is a 2-form with values in $\text{End } E$, i.e. $F_\nabla \in \Omega^2(\text{End } E)$. The Bianchi identity says

$$d_\nabla F_\nabla = 0.$$

This holds leaf-wise, so it holds on the solenoid.

Consider the elementary functions: $\text{Tr}_i : M_{r \times r} \rightarrow \mathbb{C}$, given by $\text{Tr}_i(A) = \text{Tr}(\bigwedge^i A)$. Then the Chern classes are

$$c_i(E) = \left[\text{Tr}_i \left(\frac{\sqrt{-1}}{2\pi} F_\nabla \right) \right] \in H_{DR}^{2i}(S).$$

These classes are well defined (since the forms inside are closed, which again follows by working on leaves) and do not depend on the connection (different connections give forms differing by exact forms), see [12, Chapter III].

We have some facts:

- (1) If M is a manifold, we recover the usual Chern classes.
- (2) If $f : S_1 \rightarrow S_2$ is a solenoid map, then $f^*c_i(E) = c_i(f^*E)$. In particular,
 - If $f : S \rightarrow M$ is an immersion of a solenoid in a manifold and $E|_S = f^*E$, then $c_i(E|_S) = f^*c_i(E)$.
 - If $j : l \rightarrow S$ is the inclusion of a leaf, then $c_i(E|_l) = j^*c_i(E)$.

Question 6.1. Are the Chern classes defined as elements in $H^{2i}(S, \underline{\mathbb{Z}})$?

This question has an affirmative answer for the case of line bundles (complex vector bundles of rank 1). A line bundle $L \rightarrow S$ is given by its transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow S^1$ which are of class $C^{\infty,0}$. Therefore the line bundles on S are parametrized by $H^1(S, C^{\infty,0}(S^1))$, where $C^{\infty,0}(S^1)$ is the sheaf which assigns $U \mapsto C^{\infty,0}(U, S^1)$.

Note that there is an exact sequence of sheaves:

$$0 \rightarrow \underline{\mathbb{Z}} \rightarrow C^{\infty,0}(\mathbb{R}) \rightarrow C^{\infty,0}(S^1) \rightarrow 0.$$

Here the sheaf $\underline{\mathbb{Z}}$ is the locally constant sheaf. As $C^{\infty,0}(\mathbb{R})$ is a fine sheaf (it has partitions of unity), it is acyclic. So the map

$$\delta : H^1(S, C^{\infty,0}(S^1)) \rightarrow H^2(S, \underline{\mathbb{Z}})$$

is an isomorphism. There is a natural map $\alpha : H^2(S, \underline{\mathbb{Z}}) \rightarrow H^2(S, \underline{\mathbb{R}}_c)$. It is easy to see that $\alpha(\delta([L])) = c_1(L)$, by using the transition functions $g_{\alpha\beta}$ to construct a suitable connection on L with which we compute the curvature (as in the case of manifolds, see [12]).

7. HODGE THEORY

7.1. Sobolev norms. Let S_μ be a compact Riemannian k -solenoid which is oriented and endowed with a transversal measure. We denote the associated (finite) daval measure also by μ . Now consider a vector bundle $E \rightarrow S$ and endow it with a metric. The space of sections of class $C^{\infty,0}$ is denoted $\Gamma(S, E)$. The space of $L^2(\mu)$ -transversally smooth sections (sections of class C^∞ along leaves and L^2 in the transversal directions) is denoted by $\Gamma_{L^2(\mu)}(S, E)$.

Now let us introduce suitable completions of these spaces of sections. Fix a connection ∇ for E and the Levi-Civita connection for TS . There is an L^2 -norm on sections of E , given by

$$(s, t)_E = \int_S \langle s, t \rangle d\mu.$$

We can complete the spaces of sections to obtain spaces of L^2 -sections $L^2_\mu(S, E)$. We consider also Sobolev norms $W^{l,2}$ as follows. Take s a section of E . Then we set

$$\|s\|_{W^{l,2}_\mu}^2 = \int_S \sum_{i=0}^l |\nabla^i s|^2 d\mu.$$

Completing with respect to this norm gives a Hilbert space consisting of sections with regularity $W^{l,2}$ on leaves and $L^2(\mu)$ -transversally, denoted $W^{l,2}_\mu(S, E)$. These spaces do not depend on the choice of metrics and connections.

For future use, we also introduce the norms C^r_μ , which give spaces of sections with C^r -regularity on leaves and $L^2(\mu)$ -transversally. Take s a section of E . Assume it has support in a flow box $U = D^k \times K(U)$, and assume that E has been trivialized by an orthonormal frame. Then

$$\|s\|_{C^r_\mu}^2 = \int_{K(U)} \|s(\cdot, y)\|_{C^r}^2 d\mu_{K(U)}(y).$$

These norms are patched (via partitions of unity, in a non-canonical way) to get a norm on the spaces of sections on the whole solenoid. The topology defined by this norm is independent of the partition of unity. The spaces of sections are denoted $C^r_\mu(S, E)$. Note that $\bigcap_{r \geq 0} C^r_\mu(S, E) = L^2_\mu(S, E)$.

We can define the norm $W^{s,2}_\mu$ by using Fourier transforms. For this we have to restrict to a flow-box $U = D^k \times K(U)$. We Fourier-transform the section $s(x, y)$ in the leaf-wise directions, to get $\hat{s}(\xi, y)$, and then take the integral

$$\int_{K(U)} \left(\int (1 + |\xi|^2)^l |\hat{s}(\xi, y)|^2 d\xi \right) d\mu_{K(U)}(y).$$

Proposition 7.1 (Sobolev). $W^{s,2}_\mu(S, E) \subset C^p_\mu(S, E)$, for $s > [k/2] + p + 1$.

This is similar to Proposition 1.1 in Chapter IV of [12]. The proof carries over to the solenoid situation verbatim. As a consequence,

$$\bigcap_{r \geq 0} W_\mu^{r,2}(S, E) = \Gamma_{L^2(\mu)}(S, E).$$

7.2. Pseudodifferential operators. Let E, F be two vector bundles over S of ranks n, m respectively. A differential operator L of order l is an operator

$$L : \Gamma(S, E) \rightarrow \Gamma(S, F)$$

which locally on a flow-box $U = D^k \times K(U)$ is of the form

$$L(s) = \sum_{|\alpha| \leq l} A_\alpha(x, y) D^\alpha s,$$

where A_α are $(n \times m)$ -matrices of functions (with regularity $C^{\infty,0}$) and $\alpha = (\alpha_1, \dots, \alpha_k)$ is a multi-index, with $|\alpha| = \sum \alpha_i$, $D^\alpha = \frac{\partial^{|\alpha|}}{\partial^{x_1 \alpha_1} \dots \partial^{x_k \alpha_k}}$. Note that a differential operator gives rise to differential operators on each leaf. Moreover, L extends to

$$L : W_\mu^{p,2}(S, E) \rightarrow W_\mu^{p-l,2}(S, F).$$

The usual properties, like the existence of adjoints, extend to this setting.

The symbol of a differential operator on a solenoid is defined in the same fashion as for the case of manifolds, and coincides with the symbol of the differential operator on the leaves. We recall that the symbol $\sigma_l(L) \in \text{Hom}(\pi^* E, \pi^* F)$, $\pi : TS \rightarrow S$, has the form

$$\sigma_l(L)(x, y, v) = \sum_{|\alpha|=l} A_\alpha(x, y) v_1^{\alpha_1} \dots v_k^{\alpha_k}.$$

The properties of the symbol map, such as the rule of the symbol of the composition of differential operators, or the symbol of the adjoint, hold here. This is just the fact that they can be done leaf-wise, and the continuous transversality is easy to check.

Differential operators can be generalized to pseudodifferential operators as in the case of manifolds. A pseudodifferential operator of order l on a flow-box $U = D^k \times K(U)$ is an operator

$$L(p) : \Gamma_c(U, E) \rightarrow \Gamma(U, F)$$

which sends a (compactly supported) section $s(x, y)$ to

$$L(p)s(x, y) = \int p(x, \xi, y) \hat{s}(\xi, y) e^{i(x, \xi)} d\xi,$$

where $\hat{s}(\xi, y)$ is the (leaf-wise) Fourier transform, and $p(x, \xi, y)$ is a function defined in $D^k \times \mathbb{R}^k \times K(U)$, smooth on x and ξ , continuous on y , and satisfying:

- $|D_x^\beta D_\xi^\alpha p(x, \xi, y)| \leq C_{\alpha\beta l}(1 + |\xi|)^{l-|\alpha|}$, for constants $C_{\alpha\beta l}$,
- the limit $\sigma_l(p)(x, \xi, y) = \lim_{\lambda \rightarrow \infty} \frac{p(x, \lambda\xi, y)}{\lambda^l}$ exists,
- $p(x, \xi, y) - \sigma_l(p)(x, \xi, y)$ should be of order $\leq l - 1$ for $|\xi| \geq 1$.

A pseudodifferential operator of order l on S is an operator $L : \Gamma(S, E) \rightarrow \Gamma(S, F)$ which is locally of the form $L(p_U)$ for some p_U as above. The symbol of L is $\sigma_l(L) = \sigma_l(p_U)$ for a local representative $L|_U = L(p_U)$. This symbol is well-defined and independent of choices, which is a delicate point but it is analogous to the case of manifolds (see [12]). The usual properties of the symbol map (composition, adjoint) hold here.

A pseudodifferential operator of order l is an operator of order l , i.e., it extends as a continuous map to

$$L : W_\mu^{p,2}(S, E) \rightarrow W_\mu^{p-l,2}(S, E).$$

This is done as in Theorem 3.4 of [12, Ch. IV], by noting that $\|L(p)s(\cdot, y)\|_{W_\mu^{p-l,2}} \leq C\|s(\cdot, y)\|_{W_\mu^{p,2}}$, where C is a constant depending on $C_{\alpha\beta l}$.

The key of the theory is the fact that we can construct a pseudodifferential operator given a symbol $\sigma_l(L)$.

Proposition 7.2. *Let S be a compact solenoid. Then there is an exact sequence $0 \rightarrow \text{OP}_{l-1}(E, F) \rightarrow \text{PDiff}_l(E, F) \rightarrow \text{Symb}_l(E, F) \rightarrow 0$, where $\text{OP}_{l-1}(E, F)$ is the space of operators of order $l-1$, $\text{PDiff}_l(E, F)$ the space of pseudodifferential operators of order l , and $\text{Symb}_l(E, F)$ the space of symbols of order l .*

7.3. Elliptic operator theory for solenoids. We say that a pseudodifferential operator $L : E \rightarrow F$ of order l is elliptic if the symbol $\sigma_l(L)$ satisfies that $\sigma_l(L)(x, v) : E_x \rightarrow F_x$ is an isomorphism for each $x \in S, v \in T_x S, v \neq 0$.

Theorem 7.3. *Let L be an elliptic pseudodifferential operator of order l . Then there exists a pseudo-inverse, a pseudodifferential operator \tilde{L} of order $-l$ such that $L \circ \tilde{L} = \text{Id} + K_1$ and $\tilde{L} \circ L = \text{Id} + K_2$, where K_1, K_2 are operators of order -1 .*

This is done as in Theorem 4.4 of [12, Ch. IV]. The basic idea is to construct a pseudo-inverse by using Proposition 7.2. Note that K_1, K_2 are not usually compact operators (this is due to the failure of the Rellich lemma in our situation), so we will not have finite-dimensionality of the kernel and cokernel of elliptic operators.

Corollary 7.4. *Let L be an elliptic pseudodifferential operator of order l , and let $\mathcal{K}_{L_s} = \ker(L : W_\mu^{s,2}(S, E) \rightarrow W_\mu^{s-l,2}(S, F))$. Then $\mathcal{K}_{L_s} \subset \Gamma_{L^2(\mu)}(S, E)$, and it is independent of s .*

Proof. Let $\sigma \in W_\mu^{2,s}(S, E)$ such that $L\sigma = 0$. Then $\sigma = (\tilde{L} \circ L - K_2)(\sigma) = -K_2\sigma \in W_\mu^{2,s+1}(S, E)$. Working inductively, $\sigma \in \bigcap_{r \geq 0} W_\mu^{2,r}(S, E) = \Gamma_{L^2(\mu)}(S, E)$. The second assertion is clear. \square

An operator $L : \Gamma(E) \rightarrow \Gamma(E)$ is called self-adjoint if $L^* = L$. If L is an elliptic self-adjoint operator, then there is a pseudo-inverse G which is self-adjoint (just take the pseudo-inverse \tilde{L} provided by Theorem 7.3 and let $G = (\tilde{L} + \tilde{L}^*)/2$). Then we have that $L \circ G = G \circ L$, because

$$\langle (L \circ G - G \circ L)s, s \rangle = \langle Gs, Ls \rangle - \langle Ls, Gs \rangle = 0.$$

In particular, $K_1 = K_2$ in Theorem 7.3.

For self-adjoint operators, we have the following result

Theorem 7.5. *Let L be an elliptic self-adjoint operator of order l . Then*

$$W_\mu^{s,2}(S, E) = \ker L \oplus \overline{\operatorname{im} L}.$$

and an analogous result for $\Gamma_{L^2(\mu)}(S, E)$.

Proof. Clearly, $\ker L$ is a closed subspace. Let $s_1 \in \ker L$ and $s_2 \in \operatorname{im} L$, say $s_2 = L(t)$. From this we have $\langle s_1, s_2 \rangle = \langle s_1, L(t) \rangle = \langle L(s_1), t \rangle = 0$. The decomposition $\ker L \oplus \overline{\operatorname{im} L}$ is therefore orthogonal.

It remains to prove that $(\ker L)^\perp \subset \overline{\operatorname{im} L}$. Equivalently, $(\operatorname{im} L)^\perp \subset \ker L$. Let $s \in (\operatorname{im} L)^\perp$. Then $\langle L(s), s_2 \rangle = \langle s, L(s_2) \rangle = 0$, for all s_2 . Hence $L(s) = 0$. \square

A complex of differential operators is a sequence

$$\Gamma(E_0) \xrightarrow{L_0} \Gamma(E_1) \xrightarrow{L_1} \dots \xrightarrow{L_{m-1}} \Gamma(E_m),$$

where E_i are vector bundles, and L_i are differential operators such that $L_i \circ L_{i-1} = 0$. The complex is called elliptic if the sequence of symbols

$$\pi^* E_0 \xrightarrow{\sigma(L_0)} \pi^* E_1 \xrightarrow{\sigma(L_1)} \dots \xrightarrow{\sigma(L_{m-1})} \pi^* E_m,$$

is exact for each $v \neq 0$. We define the cohomology of the complex as

$$H^q(S, E) = \frac{\ker(L_q : \Gamma(E_q) \rightarrow \Gamma(E_{q+1}))}{\operatorname{im}(L_{q-1} : \Gamma(E_{q-1}) \rightarrow \Gamma(E_q))},$$

and the L^2 -cohomology by

$$H^q(S_\mu, E) = \frac{\ker(L_q : \Gamma_{L^2(\mu)}(E_q) \rightarrow \Gamma_{L^2(\mu)}(E_{q+1}))}{\operatorname{im}(L_{q-1} : \Gamma_{L^2(\mu)}(E_{q-1}) \rightarrow \Gamma_{L^2(\mu)}(E_q))}.$$

The *reduced* L^2 -cohomology is

$$\bar{H}^q(S_\mu, E) = \frac{\ker L_q}{\overline{\operatorname{im} L_{q-1}}}.$$

This is the group $H^q(S_\mu, E)$ quotiented by the closure of $\{0\}$, making it a Hausdorff space.

We construct the Laplacian operators of the elliptic complex as follows:

$$\Delta_j = L_j^* L_j + L_{j-1} L_{j-1}^* : \Gamma_{L^2(\mu)}(E_j) \rightarrow \Gamma_{L^2(\mu)}(E_j).$$

These are self-adjoint elliptic operators. There is an associated operator G given by Theorem 7.5. Denote

$$\mathcal{H}^j(E) = \ker \Delta_j.$$

And note that $\Delta_j s = 0$ if and only if $L_j s = 0$ and $L_j^* s = 0$. We remove the subindex j from now on.

Theorem 7.6. *We have the following:*

- (1) $\overline{\text{im } \Delta} = \overline{\text{im } L} \oplus \overline{\text{im } L^*}$, and it is an orthogonal decomposition.
- (2) $\Gamma_{L^2(\mu)}(S, E_j) = \mathcal{H}^j(E) \oplus \overline{\text{im } L} \oplus \overline{\text{im } L^*}$.
- (3) *There is a canonical isomorphism $\mathcal{H}^j(E) \cong \bar{H}^j(S_\mu, E)$.*

Proof. (1) The inclusion \subset is clear. Note that the decomposition is orthogonal: $\langle L(s_1), L^*(s_2) \rangle = \langle LL(s_1), s_2 \rangle = 0$, for all s_1, s_2 .

For the reverse inclusion, let us check that $\text{im } L \subset \overline{\text{im } \Delta}$ (the other inclusion is similar). This is equivalent to proving that $\ker \Delta \subset (\text{im } L)^\perp$. But $\ker \Delta = \ker L \cap \ker L^* \subset \ker L^* \subset (\text{im } L)^\perp$, so the result follows.

(2) follows from (1) and Theorem 7.5.

(2) Consider the map $\mathcal{H}^j(E) \rightarrow \bar{H}^j(S_\mu, E)$. This is well defined because if $\Delta s = 0$, $s \in \Gamma_{L^2(\mu)}(S, E_j)$ then $0 = \langle s, \Delta s \rangle = \langle s, (LL^* + L^*L)s \rangle = \langle L^*s, L^*s \rangle + \langle Ls, Ls \rangle \implies Ls = L^*s = 0$.

It is injective. If $s = Lt$ and $\Delta s = 0$ then $0 = L^*s = L^*Lt$, so $\|Lt\|^2 = \langle L^*Lt, t \rangle = 0$, so $s = Lt = 0$.

It is surjective. Take a class $[s] \in \bar{H}^j(S_\mu, E)$. Decompose $s = s_1 + s_2 + s_3$, where $s_1 \in \mathcal{H}^j(E)$, $s_2 \in \overline{\text{im } L}$ and $s_3 \in \overline{\text{im } L^*}$. Now $0 = Ls = Ls_3$, so $s_3 \perp \text{im } L^*$. Therefore $s_3 = 0$. So $s = s_1 + s_2$ and $[s] = [s_1]$ in $\bar{H}^j(S_\mu, E)$, where $s_1 \in \mathcal{H}^j(E)$. \square

7.4. Harmonic theory. The Riemannian metric and the orientation give rise to a natural volume form along leaves $\text{vol} \in \Omega^k(S)$. The usual Hodge-* operator (see [12]) can be defined for forms on S , actually, it is the * operator on leaves. This operator $* : \Omega^p(S) \rightarrow \Omega^{k-p}(S)$ is defined by

$$\alpha \wedge * \beta = (\alpha, \beta) \text{vol},$$

for $\alpha, \beta \in \Omega^p(S)$, where (\cdot, \cdot) is the point-wise metric induced on forms. Note that $*$ extends to $*$: $\Omega_{L_\mu^2}^p(S) \rightarrow \Omega_{L_\mu^2}^{k-p}(S)$, since it is leaf-wise isometric. Note that $\text{vol} = *1$.

It is easy to check that $d^* = \pm *d*$. The Laplacian is defined as $\Delta = dd^* + d^*d$. Note that if $\Delta s = 0$ then $(s, \Delta s) = (s, dd^*s) + (s, d^*ds) = (d^*s, d^*s) + (ds, ds) = \|d^*s\|^2 + \|ds\|^2$. So $d^*s = 0$ and $ds = 0$. We define the space of harmonic forms:

$$\mathcal{K}^j(S_\mu) = \mathcal{H}_\Delta(\wedge^j T^*S).$$

Then the theory of elliptic operators says the following

Theorem 7.7. *We have:*

- *The space of harmonic sections $\mathcal{K}^j(S_\mu) \subset \Omega_{L^2(\mu)}^j(S)$.*
- *There is a natural isomorphism $\bar{H}_{DR}^j(S_\mu) \cong \mathcal{K}^j(S_\mu)$.*

Corollary 7.8. *Poincaré duality:*

$$* : \mathcal{K}^p(S_\mu) \rightarrow \mathcal{K}^{k-p}(S_\mu)$$

is an isomorphism.

If S is ergodic, then $H_{DR}^0(S_\mu) \cong H_{DR}^k(S_\mu) \cong \mathbb{R}$ (with the isomorphism given by integration \int_{S_μ}). Therefore

$$\int_{S_\mu} : \bar{H}_{DR}^p(S_\mu) \otimes \bar{H}_{DR}^{k-p}(S_\mu) \rightarrow \mathbb{R}$$

is a perfect pairing.

In general, the spaces $\mathcal{K}^p(S_\mu)$ are not in general finite dimensional. For instance, take a solenoid which is a fibration, i.e., S is a compact $(n+k)$ -manifold such that there is a submersion $\pi : S \rightarrow B$ onto an n -dimensional manifold, and the transversal measure is induced by a measure μ on B . Then we have a fiber bundle $\mathbb{H}^p \rightarrow B$ such that $\mathbb{H}_y^p = H^p(F_y)$, $F_y = \pi^{-1}(y)$, $y \in B$. Then $\mathcal{K}^p(S_\mu) \cong L_\mu^2(\mathbb{H}^p)$.

Nonetheless, we propose the following:

Question 7.9. *If S_μ is an ergodic solenoid with controlled growth, are the spaces $\mathcal{K}^p(S_\mu)$ of finite dimension?*

A controlled solenoid has transversal measures and is defined in [7]. For such a solenoid, any leaf has an exhaustion K_n by compact sets such that for any flow-box in a finite covering, the number of local leaves which intersect partially K_n is negligible. In this case, the normalized measure supported on K_n converges to a daival measure (giving rise to a transversal measure, as usual). If the solenoid S_μ is ergodic, then for almost all μ -leaves, this limit coincides

with μ . So in this situation, we can understand the behaviour of harmonic forms by restricting to a leaf, and then study the spaces of harmonic forms on this leaf (which is a complete manifold with a quasi-periodic behaviour).

The example in section 8 shows that the question has a negative answer if we ask instead for $H_{DR}^p(S_\mu)$.

8. EXAMPLE: KRONECKER SOLENOIDS

Let us develop the example of Kronecker solenoids, i.e. the flat torus with a linear foliation. Let $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ with a flat euclidean metric on \mathbb{T}^n . Consider a foliation given by a k -dimensional linear subspace $W \subset \mathbb{R}^n$. That is, fix an orthonormal basis w_1, \dots, w_n , so that $W = \langle w_1, \dots, w_k \rangle$. Consider the solenoid S whose leaves are the images of k -planes of \mathbb{R}^n parallel to W under $\pi : \mathbb{R}^n \rightarrow \mathbb{T}^n$.

The solenoid is minimal with all leaves dense in \mathbb{T}^n if there is no hyperplane $H = \sum m_i x_i = 0$ with $m_i \in \mathbb{Z}$ so that $W \subset H$. Equivalently, for any integer vector $m \in \mathbb{Z}^n$, we have that $\langle m, w_j \rangle \neq 0$ for some $j = 1, \dots, k$. Equivalently,

$$(8.1) \quad \sum \langle m, w_j \rangle^2 \neq 0 .$$

Notice that all leaves have the same topological type and are simply connected and diffeomorphic to \mathbb{R}^k if and only if $W \cap \mathbb{Z}^n = \{0\}$. We shall suppose that the solenoid is minimal, but we allow $W \cap \mathbb{Z}^n \neq \{0\}$.

In this situation, we have affine transversals $T \subset W^\perp = \langle w_{k+1}, \dots, w_n \rangle$ and the holonomy are irrational translations, therefore by Haar theorem the unique transversal measure is the Lebesgue measure on the transversal. Thus S is uniquely ergodic, and the unique invariant measure is the Haar measure on \mathbb{T}^n , image of the Lebesgue measure on \mathbb{R}^n .

We can characterize the space of functions C^∞ on leaves and L^2 -transversally by their Fourier expansion. All such functions are in particular L^2 on the torus, so they have a Fourier expansion ($x = (x_1, \dots, x_n) \in \mathbb{T}^n$, $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$)

$$f(x) = \sum_{m \in \mathbb{Z}^n} a_m e^{2\pi i \langle m, x \rangle} ,$$

where $\sum |a_m|^2 < \infty$.

To be smooth on leaves is equivalent to be $W^{l,2}$ on leaves for all l . Let us take a derivative along some w_j , $j = 1, \dots, k$.

$$D_{w_j} f = \sum_{m \in \mathbb{Z}^n} \langle m, w_j \rangle a_m e^{2\pi i \langle m, x \rangle}$$

So the condition C^∞ on leaves and L^2 -transversally is

$$\sum_{m \in \mathbb{Z}^n} \left| \prod_{j=1}^k \langle m, w_j \rangle^{r_j} a_m \right|^2 < \infty$$

for all $r_1, \dots, r_k \geq 0$.

Note that the Laplacian on functions is the usual Laplacian on the leaves direction, so that

$$\Delta f = - \sum_{j=1}^k D_{w_j}^2 f = - \sum_{m \in \mathbb{Z}^n} \left(\sum_{j=1}^k \langle m, w_j \rangle^2 \right) a_m e^{2\pi i \langle m, x \rangle}.$$

So $\Delta f = 0 \iff f = \text{constant}$, using (8.1).

Now consider p -forms, $0 \leq p \leq k$. A p -form $\omega = \sum_{|J|=p} f_J d\varpi_J$, where $\varpi_1, \dots, \varpi_k$ are coordinates on the leaves, dual to the basis w_1, \dots, w_k , and $J = (j_1, \dots, j_p)$, $1 \leq j_1 < \dots < j_p \leq k$, $d\varpi_J = d\varpi_{j_1} \wedge \dots \wedge d\varpi_{j_p}$. For the flat metric, $\Delta \omega = \sum_{|J|=p} \Delta f_J d\varpi_J$. So $\Delta \omega = 0 \iff f_J = \text{constant}$.

In conclusion,

$$\mathcal{K}^p(S_\mu) = \bigwedge^p W^*,$$

which is of dimension $\binom{k}{p}$.

Note that the *-Hodge operator satisfies $*\varpi_J = \pm \varpi_{J^c}$, where J^c is the complement $\{1, \dots, k\} - J$. So $*$: $\mathcal{K}^p(S_\mu) \rightarrow \mathcal{K}^{k-p}(S_\mu)$ is the natural isomorphism $\bigwedge^p W^* \cong \bigwedge^{k-p} W^*$.

The case of the Kronecker 1-solenoid. Let us see the particular case of the Kronecker 1-solenoid with $w_1 = \alpha = (\alpha_1, \dots, \alpha_n)$ and

$$\dim_{\mathbb{Q}} \langle \alpha_1, \dots, \alpha_n \rangle = n.$$

As above, $\mathcal{K}^0(S_\mu) = \mathbb{R}$ and $\mathcal{K}^1(S_\mu) = \mathbb{R}$.

Let us compute $H_{DR}^0(S_\mu)$ and $H_{DR}^1(S_\mu)$. For a function $f = \sum a_m e^{2\pi i \langle m, x \rangle}$,

$$df = \sum \langle m, \alpha \rangle a_m e^{2\pi i \langle m, x \rangle}.$$

So $df = 0 \iff f = \text{constant}$. Then $H_{DR}^0(S_\mu) = \mathbb{R}$.

Now a 1-form $\omega = g d\varpi_1$, $g = \sum b_m e^{2\pi i \langle m, x \rangle}$, is exact if and only if

$$g = df = \sum \langle m, \alpha \rangle a_m e^{2\pi i \langle m, x \rangle},$$

i.e.

$$a_m = \frac{b_m}{\langle m, \alpha \rangle}, \quad m \in \mathbb{Z}^n,$$

should be in L^2 . So

$$(8.2) \quad H_{DR}^1(S_\mu) = \frac{\{(b_m) \mid \sum |\langle m, \alpha \rangle^r b_m|^2 < \infty, \forall r \geq 0\}}{\{(b_m) \mid \sum |\langle m, \alpha \rangle^r b_m|^2 < \infty, \forall r \geq -1\}}.$$

This space is always infinite dimensional. We can construct an uncountable basis by using Minkowski's diophantine approximation theorem: there are infinitely many integer vectors $m \in \mathbb{Z}^n$ such that

$$|\langle m, \alpha \rangle| < \frac{1}{\|m\|}.$$

Then we can choose an uncountable number of *disjoint* sequences $(m_k) \subset \mathbb{Z}^n$ such that $\langle m_k, \alpha \rangle \rightarrow 0$ when $k \rightarrow +\infty$. For each such sequence take a Fourier series with support on it (that is $b_m = 0$ if $m \neq m_k$) such that $\sum_k b_{m_k}^2 < +\infty$ but

$$\sum_k \left(\frac{b_{m_k}}{\langle m_k, \alpha \rangle} \right)^2 = +\infty.$$

We can take for example $b_{m_k} = \langle m_k, \alpha \rangle$. Then clearly (b_{m_k}) lies in (8.2). In this way we get uncountably many independent elements in (8.2).

As an aside, note that the De Rham cohomology $H_{DR}^1(S)$ is also of infinite dimension (as it is proved in [3] by sheaf theoretic methods). However, if we use forms with regularity $C^{\infty, \infty}$, using the criterion characterising C^∞ as those with polynomially decaying Fourier coefficients, we can compute that

$$H^1(S, \mathbb{R}_\infty) \cong \frac{\{(b_m) \mid \|b_m\| = O((1 + \|m\|)^{-r}), \forall r > 0\}}{\{(b_m) \mid \|\langle m, \alpha \rangle^{-1} b_m\| = O((1 + \|m\|)^{-r}), \forall r > 0\}}$$

(here \mathbb{R}_∞ is the sheaf of functions locally constant on leaves and C^∞ transversally). For a vector $\alpha \in DC(\gamma, \tau)$ satisfying a diophantine condition, we have that there exist $\tau > 0$ and $\gamma > 0$ such that for all $m \in \mathbb{Z}^n$ we have

$$|\langle m, \alpha \rangle| \geq \frac{\gamma}{\|m\|^{n+\tau}}.$$

(The subset of diophantine vectors $DC(\tau) = \bigcup_{\gamma > 0} DC(\gamma, \tau) \subset \mathbb{R}^n$ is of full Lebesgue measure.) Then it follows that when α is diophantine, $H^1(S, \mathbb{R}_\infty)$ is one dimensional, and otherwise, when α is Liouville, it is infinite dimensional ([10],[2]). To impose such type of transversal regularity in this problem does not seem to be natural.

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