

η/s and phase transitions

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We present a calculation of η/s for the meson gas (zero baryon number), with the viscosity computed within unitarized next-to-leading-order chiral perturbation theory, and confirm the observation that η/s decreases towards the possible phase transition to a quark-gluon plasma/liquid. The value is somewhat higher than previously estimated in leading-order χ PT. We also examine the case of atomic Argon gas to check the discontinuity of η/s across a first-order phase transition. Our results suggest employing this dimensionless number, sometimes called KSS number (in analogy with other ratios in fluid mechanics such as Reynolds number or Prandtl number) to pin down the phase transition and critical end point to a crossover in strongly interacting nuclear matter between the hadron gas and quark and gluon plasma/liquid.

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I. INTRODUCTION

It has been recently pointed out [1] that the ratio of the shear viscosity to entropy density, η/s , has an extremum at a phase transition, based on empirical information for several common fluids, and follow-up calculations by us [2] and other groups [3,4] have suggested that η/s in a hadron gas does indeed fall slightly with the temperature towards the predicted transition to a quark and gluon plasma or liquid phase. Renewed interest in this quantity arose after the KSS conjecture [5] about a possible lower bound $1/(4\pi)$ (the existence of a bound had already been put forward, on the basis of simple physical arguments, in [6]) and it is the subject of much current research in Heavy-Ion Collisions. The precise reach of the bound has been under recent discussion, [7,8], and there is much interest in finding theoretical or laboratory fluids that reach the minimum possible value of η/s [9,10].

There is good hope that η/s and even η by itself can be derived from particle and momentum distributions in heavy-ion collisions [11–13].

It has been shown through several examples that, empirically, η/s seems to have a discontinuity at a first order phase transition, but is continuous and has an extremum at a second-order phase transition or at a crossover.

Based on lattice data [14] it is believed that the phase transition between a gas of hadrons and a quark-gluon phase at zero baryon chemical potential is actually a crossover. The result of [1] η/s , however, presents a clear discontinuity. This is of course not a serious claim of that paper, but simply an artifact of the very crude approximations there employed. We here revisit the issue, improving as far as feasible on the hadron-side estimate, and further motivating the proposed behavior of η/s .

II. INVERSE AMPLITUDE METHOD IN χ PT AND HADRON PHASE TRANSITION

We here improve the very rough calculation of [1] for η/s on the hadron phase. We have calculated in [15] the

shear viscosity of a meson gas (that is, the hadron gas as a function of the temperature and approximate meson chemical potentials, at zero baryon chemical potential). That work employed the Inverse Amplitude Method (IAM) [16] that gives a good fit to the elastic phase shifts for meson-meson scattering at low momentum, respects unitarity, and is consistent with chiral perturbation theory at NLO [17]. The only explicit degrees of freedom are light pseudoscalar mesons (π, K, η), but elastic meson-meson resonances below 1 GeV appear through the phase shifts [18].

It is an elementary exercise to divide the calculated viscosity from that work by the entropy density of the free Bose gas, for N species

$$s = \frac{S}{V} = \frac{N}{6\pi^2 T^2} \int_0^\infty dp p^4 \frac{E - \mu}{E} \frac{e^{\beta(E-\mu)}}{[e^{\beta(E-\mu)} - 1]^2}, \quad (1)$$

and plot the result in Fig. 1. Incidentally, it can be seen in the figure that the holographic bound $\frac{\eta}{s} > 1/4\pi$ is not violated, which had been claimed in the literature (we reported this in [2] with independent confirmation by [19]). The reason is that in chiral perturbation theory alone the cross section grows unchecked, eventually violating the unitarity bound, which induces a very small viscosity. Of more interest for our discussion in this work is to examine the possible behavior across the phase transition. We take the simple estimate for η/s in the quark-gluon plasma from [1], but we use our much improved calculation for the low-temperature hadron side (those authors employ LO chiral perturbation theory without unitarization). The result is plotted in Fig. 2. In addition we plot also the phase-shift based phenomenological calculation of [4], that is consistent with ours but somewhat smaller.

The calculation that [1] reported shows a discontinuous jump between the QGP and the hadron gas, whereas simple-minded nonrelativistic phenomenology would make us expect a continuous function with a minimum. Our improved hadron calculation still shows a discontinu-

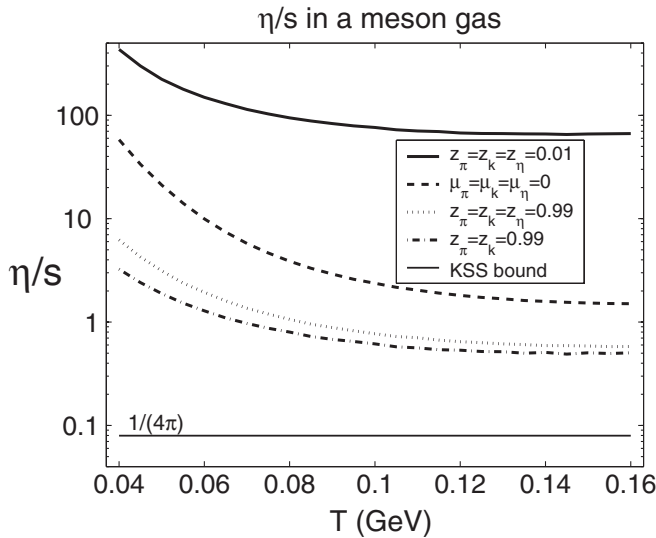


FIG. 1. The viscosity over entropy density of a meson gas in chiral perturbation theory unitarized by means of the IAM. (z represents the relativistic fugacity $e^{\beta(\mu-m)}$).

ity, although now the jump at the discontinuity has opposite sign (our viscosity is larger since the meson-meson cross section is smaller due to unitarity, instead of being an LO- χ PT polynomial). Since our estimate for η/s is now approximate only because of our use of the first order Chapman-Enskog expansion and the quantum Boltzmann equation, both of which are reasonable approximations, we feel further improvement on the hadron side will not restore continuity, and future work needs to concentrate on evaluating the viscosity from the QGP side.

To calculate the viscosity in a field theory, a possible and popular approach is to employ Kubo's formula in terms of field correlators. Another method, based on the Wigner function, is to write-down the hierarchy of BBGKY equations.

In either case one can perform a low-density expansion, leading to the use of the Boltzmann equation. Employing this on the hadron side, as opposed to the full hierarchy of BBGKY equations of kinetic theory, presumes the ‘‘molecular chaos’’ hypothesis of Boltzmann, which is tantamount to neglecting correlations between successive collisions. This requires the collisions to be well separated over the path of the particle, and induces a systematic error in the calculation of order $2/(m_\pi\lambda)$, where $1/m_\pi$ is the typical reach of the strong interaction, and λ the mean free path (controlled by the density). To keep this number below one requires small densities $n(T) < \frac{m_\pi}{2\sigma}$. If we take as a cross-section estimate 100 mbarn we see that the criterion is satisfied up to temperatures of order 140 MeV (where we stop our plot in Fig. 2).

We have also estimated the change in η/s caused by a small quark mass, by adapting the results of [20]. Those authors provide, within a 2PI formalism, the shear viscosity of the quark and gluon plasma of one fermion species as

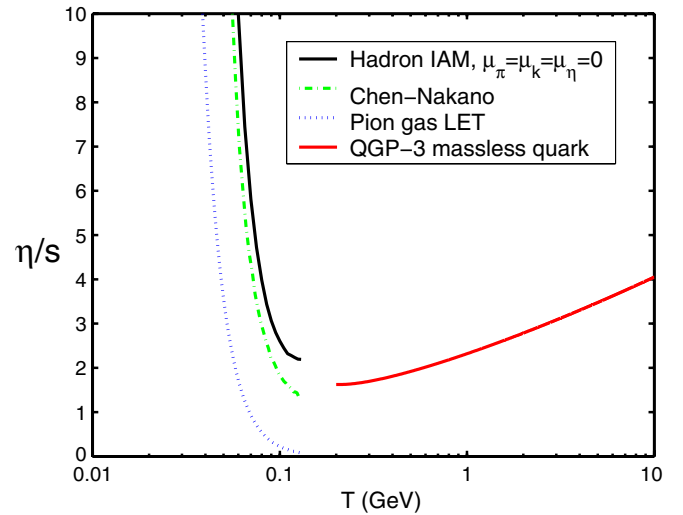


FIG. 2 (color online). We improve the hadron-side (low T) estimate of [1] that showed the jump in the η/s ratio in the transition from the hadron gas to the quark-gluon plasma, substituting the Low-Energy-Theorem of those authors (first-order chiral perturbation theory) by the Inverse Amplitude Method, that agrees with Chiral Perturbation Theory at NLO, and satisfies elastic unitarity. We confirm the result of those authors, although the actual numerical value of η/s is quite different (as should be expected from their calculation reaching temperatures $T \simeq 150$ MeV but with only the first-order interaction). One should note that, the calculation being performed at zero baryon chemical potential, based on lattice data that suggest a crossover between the hadron gas and the quark-gluon plasma, and from simple phenomenology, η/s should be continuous.

a function of the fermion mass divided by the temperature, for fixed coupling constant. Although we are employing, as Csernai *et al.* do, a coupling that runs with the scale (the temperature), the mass correction is small, so we can take $g = 2$ as fixed for a quick eyeball estimate. We normalize the viscosity of [20] to the value plotted in Fig. 2 at zero fermion mass, and then allow the fermion mass to vary. The results are now plotted in Fig. 3. We plot the extreme case of all three light quarks equally massive and with mass equaling $m_s = 120$ MeV. As can be seen, the difference to the massless case is irrelevant at current precision and does not change the fact that we cannot conclude as of yet whether the transition between a hadron gas and a quark-gluon plasma/liquid has a discontinuity in η/s or not. The reason that the fermion mass is not so relevant in the calculation is twofold. First, even at m_s , we have $T/m_s > 1$ for any value of $T > T_c$. Since kinetic momentum transport in a gas is dominated by the fraction of molecules (here, partons) with the largest energy allowed by the Boltzmann tail of the distribution $E > T$, and the momentum varies as

$$p = E \left(1 - \frac{m^2}{2E^2} + \dots \right),$$

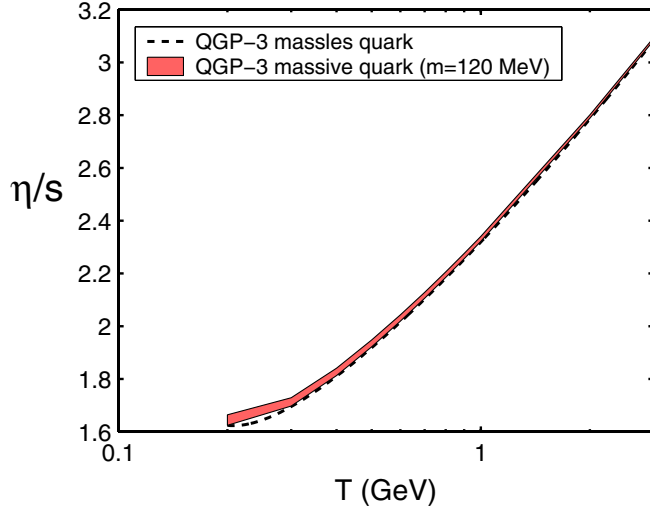


FIG. 3 (color online). We plot the dependence of η/s on the quark mass from quark-gluon plasma side by adapting the results from [20]. Note that given the nontrivial calculation there, we have slightly simplified by taking a constant g in the mass correction. The band in the figure corresponds to the interval $g \in [1, 2]$. We have also taken all quarks of equal mass $m_s = 120$ MeV as the maximum possible variation. As can be seen, the dependence is small and positive, bringing about even better agreement with the hadron-side Inverse Amplitude Method evaluation.

we see that the small parton mass makes just a correction to the momentum of each (efficient) carrier. The second reason is that the cross section, in a regime where perturbation theory is of any use, is weakly dependent on the fermion mass, with a slight dependence brought about by the logarithmic running of the quark-gluon vertex.

Still, given the large uncertainties in our knowledge of the quark-gluon medium created in heavy-ion collisions, that make it difficult to match with the hadron side, we also study a simple nonrelativistic system where the jump in η/s at the phase transition is very clear.

III. LIQUID-GAS PHASE TRANSITION IN ATOMIC ARGON

In prior works it has been pointed out that experimental data suggest that first-order phase transitions present a discontinuity in η/s and second-order phase transitions (and maybe crossovers) present a minimum. We will examine one case a little closer, for a liquid-gas phase transition in the atomic Argon gas, where we will calculate the η/s ratio theoretically and compare to data. The empirical data that has been brought forward was based on atomic Helium and molecular Nitrogen and Water. Quantum effects are very strong in the first at low temperatures where the phase transition occurs, and the later have relatively strong interactions.

Instead we choose Argon due to its sphericity and closed-shell atomic structure, that make it a case very close

to a hard-sphere system. Thus, Argon is the perfect theoretical laboratory, and sufficient data has been tabulated due to its use as a cryogenic fluid.

The gas phase is therefore well described in terms of hard-sphere interactions. In elementary kinetic theory one neglects any correlation between successive scatterings. The viscosity follows then the formula

$$\eta_{\text{gas}} = \frac{5}{16d^2} \sqrt{\frac{mT}{\pi}}, \quad (2)$$

where $d = 3.42 \times 10^5$ fm is the viscosity diameter of the Argon atom and $m = 37.3$ GeV its mass [21].

Experimental data is quoted as function of the temperature for fixed pressure. The particle density is then fixed by the equation of state; therefore a chemical potential needs to be introduced. In order to calculate the entropy density we again use Eq. (1) with $N = 1$.

As said, we keep the pressure P constant, and the chemical potential μ varies then within the temperature range. In order to obtain μ we simply invert (numerically) the function $P(T = 1/\beta, m, \mu)$ at fixed temperature. The expression for the pressure consistent with the entropy above is

$$P(T, m, \mu) = -\frac{T}{2\pi^2} \int_0^\infty dp p^2 \log[1 - e^{\beta(\mu - E)}] \quad (3)$$

(we have neglected in both cases the effect of the Bose-Einstein condensate since the gas liquefies before this is relevant). The problem has then been reduced to computing the viscosity at the given temperature and chemical potential, which we do employing our computer program for the meson gas in the Chapman-Enskog approximation, with minimum modifications.

We change variables to absorb the scale and make the integrand of order 1 to

$$\bar{\mu} \equiv \frac{\mu - m}{T}, \quad x \equiv \frac{p^2}{mT}. \quad (4)$$

Thus, the final expressions for the entropy density and pressure from Eqs. (1) and (3) (once integrated by parts) are

$$P = \frac{1}{12\pi^2} m^{3/2} T^{5/2} \int_0^\infty dx x^{3/2} \frac{1}{e^{x/2 - \bar{\mu}} - 1}, \quad (5)$$

$$s_{\text{gas}} = \frac{1}{12\pi^2} (mT)^{3/2} \int_0^\infty dx x^{3/2} \left(\frac{x}{2} - \bar{\mu}\right) \frac{e^{x/2 - \bar{\mu}}}{(e^{x/2 - \bar{\mu}} - 1)^2}. \quad (6)$$

To treat the liquid Ar phase there is not a very rigorous theory. This is because in liquids the momentum transfer mechanism is quite complex and does involve the interaction between molecules. Here, our choice of a noble gas is of help since long-range interactions are absent. It is common to resort to semiempirical formulas with unclear

theory support, or work with formal expressions of difficult applicability. We compromise by combining the Van der Waals equation of state (that ultimately encodes the Lennard-Jones theory for the interatomic potential), and use the Eyring liquid theory [22].

The Eyring theory is a vacancy theory of liquids. Each molecule composing the liquid has gaslike degrees of freedom when it jumps into a vacant hole, and solidlike degrees of freedom when fully surrounded by other molecules.

This model approach yields a partition function Z for a one-species liquid (in natural units)

$$Z = \left\{ \frac{e^{E_s/N_A T}}{(1 - e^{-\theta/T})^3} \times \left(1 + n \frac{V - V_s}{V_s} e^{-(aE_s V_s / (V - V_s) N_A T)} \right) \right\}^{(N_A V_s / V)} \times \left\{ \frac{e(2\pi m T)^{3/2} V}{(2\pi)^3 N_A} \right\}^{(N_A (V - V_s) / V)}, \quad (7)$$

from which one can derive complete statistical information about the system [23]. One can recognize in the second brace the partition function of a nonrelativistic gas for the fraction of atoms with gaslike behavior. The first brace corresponds to the solidlike behavior. The first factor is the partition function of a three-dimensional harmonic oscillator. The second term is a correction due to the translation degree of freedom, by which an atom can displace to a neighboring vacancy.

The shear viscosity, (like Z itself), turns out to be a weighted average between the viscosity of solidlike (first line) and gaslike (second line) degrees of freedom of the liquid's particles:

$$\eta_{\text{liq}} = \frac{N_A 2\pi}{V} \frac{1}{(1 - e^{-\theta/T})} \frac{6}{n\kappa} \frac{V}{V - V_s} e^{(a'E_s V_s / (V - V_s) N_A T)} + \frac{V - V_s}{V} \frac{5}{16d^2} \sqrt{\frac{mT}{\pi}}, \quad (8)$$

and to complete the model, θ , n , a , a' , κ , E_s and V_s are given in Table I for gaseous Argon. N_A is Avogadro's number. The V_s/V solidlike volume fraction controls the weighted average. Note that if this ratio approaches 1, the viscosity diverges as appropriate for a rigid solid [24].

The entropy is calculated as usual taking a derivative of the Helmholtz free energy ($A \equiv -T \log Z$),

$$S = \frac{\partial(T \log Z)}{\partial T}. \quad (9)$$

For our purposes we also need the liquid density which is easily estimated by means of the Van der Waals equation of state, that is of some applicability in the liquid phase. This equation takes into account the volume excluded by the particles and the attraction between them. In the simplest form the Van der Waals equation is

TABLE I. Liquid Argon parameters which appears in Eqs. (8) and (9). All these constants are given in [22]. However, κ has been modified because we use Eq. (2) instead of the formula that appears in [22] for the hard-sphere gas case.

Parameter	Value
θ	5.17 meV
n	10.80
$a = a'$	0.00534
κ	0.667
E_s	0.082 eV/particle
V_s	4.16×10^{16} fm ³ /particle

$$\left(n_{\text{gas}} + n_{\text{liq}}^2 \frac{a}{T} \right) (1 - n_{\text{liq}} b) = n_{\text{liq}}, \quad (10)$$

where n_{gas} and n_{liq} are the particle density of gas input and liquid output Argon, respectively; T is the temperature, $2b = 4\pi d^3/3$ is the covolume, that is, the excluded volume by the particle (here we take d as a mean value of the viscosity radius and the gas radius) and $a = 27T_c/64P_c$ is a measure of the particle attraction ($T_c = 150.87K$, $P_c = 4.898\text{MPa}$). Equation (10) is a cubic equation in n_{liq} which gives reasonably good results despite its simplicity. For this reason, we think that it is not necessary to derive a new state equation from the Helmholtz free energy.

Putting all together we are able to calculate the η/s ratio in both liquid and gas states. The final result is plotted in Fig. 4 where a good agreement with the experimental data of [25] is shown. One can see how the KSS bound is

Argon viscosity/entropy around the liquid-gas phase transition

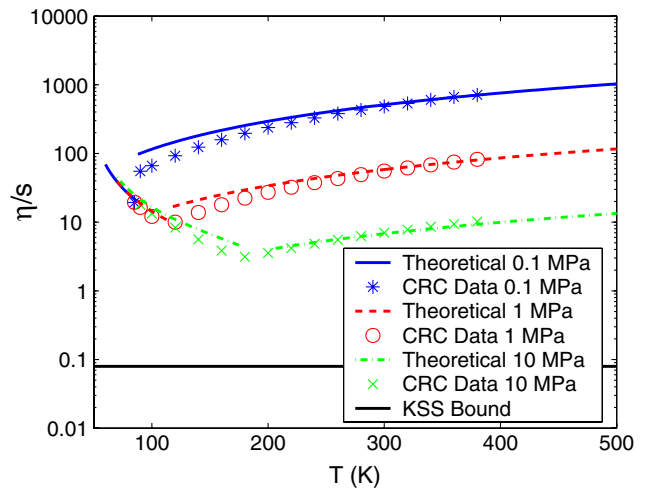


FIG. 4 (color online). η/s (a pure number in natural units) for atomic Argon in the liquid and gas phases near the phase transition. Lines correspond to theoretical calculation described in the text, symbols are the experimental values from [25]. Note that η/s is quite independent of the pressure in the liquid phase, and that the theoretical curves calculated from the liquid side and gas side do get closer together with increasing pressure, suggesting as the data that indeed, η/s will be continuous in the crossover regime.

maintained. Moreover, one can observe that for the liquid-gas phase transition η/s presents a minimum and discontinuity at the phase transition (below the critical pressure, P_c). Above this pressure, a minimum is still seen but the function is continuous.

IV. CONCLUSIONS AND OUTLOOK

In this article we have argued, in agreement with previous authors, how it is likely that η/s is a reasonable derived observable in relativistic heavy-ion collisions to pin down the phase transition and possible critical end point between a hadron gas and the quark and gluon plasma/liquid. We have contributed an evaluation of the hadron-side η/s that simultaneously encodes basic theoretical principles such as chiral symmetry and unitarity, and also produces a practical and good fit of the pion scattering phase shifts, by means of the Inverse Amplitude Method. In so doing we have updated our past meson gas work. Our conclusions are in qualitative agreement with those of [26].

Since our lack of understanding of the nonperturbative dynamics on the high- T side of the phase transition to the quark-gluon phase prevents us from matching asymptotic behavior of η/s at high T with the hadron gas, we have studied this KSS number in a related Sigma Model. We find numerically, and confirm with an analytical estimate, that keeping the s -channel amplitude one can isolate a minimum, and within reasonable calculational uncertainties, this coincides with the known phase transition of the model. A complete analysis is to be reported elsewhere.

Since we are not in possession of a good program that can proceed to finite baryon density, we leave this for further investigation. Meanwhile we have investigated the past observation that in going from a crossover to a first-

order phase transition, η/s changes behavior, from having a continuous minimum to presenting a discontinuity. We choose, as very apt for theoretical study, atomic Argon. We employ standard gas kinetic theory above the critical temperature and the Eyring theory of liquids in the liquid phase. Whereas the discontinuity in η/s is very clear for low pressures, theory and data are close to matching (showing continuity) at high pressures where a crossover between the two phases is seen in the phase diagram.

The conclusion is that indeed the minimum of the η/s and the temperature of the phase transition might well be proportional. Whether the proportionality constant is exactly one could only be established by an exact calculation of the viscosity which is not theoretically at hand.

As a consequence, we provide further theory hints to the currently proposed method to search for the critical end point in hot hadron matter. If, as lattice gauge theory suggests, a smooth crossover occurs between the hadron phase and the quark-gluon phase then, at least under the conditions in the Relativistic Heavy Ion Collider where the baryon number is small at small rapidity, one expects to see a minimum of viscosity over entropy density. In the FAIR experimental program however it might be possible to reach the critical end point given the higher baryon density (since the energy per nucleon will be smaller), and whether the phase transition is then first or second order can be inferred from the possibility of a discontinuity of η/s .

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