Simple completable contractions of nilpotent Lie algebras

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Abstract

We study a certain class of non-maximal rank contractions of the nilpotent Lie algebra $\mathfrak{g}_m$ and show that these contractions are completable Lie algebras. As a consequence a family of solvable complete Lie algebras of non-maximal rank is given in arbitrary dimension. .
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1 Generalities

The notion of contraction of a Lie algebra (also called degeneration by some authors) was originally introduced by physicists (Segal) as a tool to relate classical and quatum mechanics. Inonu and Wigner used contractions attending to a particularization, namely, that a subalgebra remains fixed through the contraction. This concept, quite restrictive for some purposes, was later generalized by Saletan [13] and Levi-Nahas. The relation between contractions and deformation theory is an important, but not fully exploited question [7]. Orbit closures of Lie algebras, where contractions play a central role, is not an easy problem, and it constitutes an essential tool in the analysis of the componets of the varieties $\mathfrak{L}^n$ and $\mathfrak{N}^n$.

Let $\mathfrak{L}^n$ be the set of complex Lie algebra laws in dimension n. We identify each law with its structure constants C_{ij}^k on a fixed basis $\{X_i\}$ of $\mathbb{C}^n$. The Jacobi identities

$$\sum_{l=1}^{n} C_{ij}^k C_{kl}^s + C_{jk}^l C_{il}^s + C_{ki}^l C_{jl}^s = 0$$

for $1 \leq i \leq j < k \leq n,\ 1 \leq s \leq n$ show that $\mathfrak{L}^n$ is an algebraic variety. The nilpotent Lie algebra laws $\mathfrak{N}^n$ are a closed subset in $\mathfrak{L}^n$. The

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linear group $GL(n,\mathbb{C})$ acts on $\mathfrak{L}^n$ via changes of basis, i.e., $(g*\mu)(x,y) = g\left(\mu\left(g^{-1}(x), g^{-1}(y)\right)\right)$ for $g \in GL(n,\mathbb{C})$. Let $\mathcal{O}(\mu)$ denote the orbit of the law μ by this action, consisting of all structures in a single isomorphism class.

Definition. *A Lie algebra λ is a contraction of a law μ if $\lambda \in \overline{O(\mu)}$.*

Here the topology on the variety is either the metric topology or the Zariski topology. Both topologies lead to the same contractions. A contraction will be denoted by $\mu \rightarrowtail \lambda$. It follows that the entire orbit $O(\lambda)$ lies in the Zariski-closure of $O(\mu)$. In particular, the following condition implies $\mu \rightarrowtail \lambda$:

$$\exists g_t \in GL(n,\mathbb{C}(t)) \quad \text{such that} \quad \lim_{t\to 0} g_t * \mu = \lambda$$

Contractions are also transitive, i.e., if $\lambda \rightarrowtail \mu$ and $\mu \rightarrowtail \psi$, then $\lambda \rightarrowtail \psi$. Thus not any existing contraction must be shown directly.

For a Lie algebra λ to be a contraction of μ, the following conditions are necessary:

Lemma. *Let $\mu \rightarrowtail \lambda$. Then*

1. $\dim\ Der(\mu) < \dim\ Der(\lambda)$
2. $\dim\ [\lambda,\lambda] \leq \dim\ [\mu,\mu]$
3. $\dim\ Z(\lambda) \geq \dim\ Z(\mu)$
4. $rank(\lambda) \geq rank(\mu)$

where $Der(\mu)$ denotes the algebra of derivations, $Z(\mu)$ the center of the algebra μ and $rank(\mu)$ is the dimension of a maximal toral subalgebra of μ.

Proofs of the assertions can be found in [5], [7] and [14].
For a nilpotent Lie algebra $\mathfrak{g}$, we denote the ideals of the central descending sequence by $C^i(\mathfrak{g})$, i.e., $C^i(\mathfrak{g}) = [\mathfrak{g}, C^{i-1}(\mathfrak{g})]$ for $i \geq 1$ and $C^0(\mathfrak{g}) = \mathfrak{g}$. If the algebra $\mathfrak{g}$ contracts to an algebra $\mathfrak{h}$, it follows from the lemma above that $dim\left(C^i(\mathfrak{g})\right) \geq dim\left(C^i(\mathfrak{g})\right)$. Therefore, if $\mathfrak{g}$ has nilindex p, then the nilindex of $\mathfrak{h}$ is $\leq p$.
Although we will not make explicit use of the characteristic sequence, it is convenient to recall this invariant: Consider a complex nilpotent Lie algebra $\mathfrak{g} = (\mathbb{C}^n, \mu)$. For each $X \in \mathbb{C}^n$ we denote $c(X)$ the ordered sequence of dimensions of Jordan blocks of the adjoint operator $ad_\mu(X)$.

Definition. *The characteristic sequence of $\mathfrak{g}$ is an isomorphism invariant $c(\mathfrak{g})$ defined as*

$$c(\mathfrak{g}) = \sup_{X \in \mathfrak{g} - C^1\mathfrak{g}} \{c(X)\}$$

where $C^1\mathfrak{g}$ denotes the derived subalgebra.
A characteristic sequence is called linear if there exists an integer n such that $c(\mathfrak{g}) = (n, 1, .., 1)$.

2 The algebras $\mathfrak{g}_m(q_1,..,q_k)$

In this section we analyze some families of nilpotent Lie algebras for which certain classes of contractions will be determined.

For $m \geq 4$ let $\mathfrak{g}_m$ be the Lie algebra whose structural equations are

$$\begin{aligned} d\omega_1 &= d\omega_2 = 0 \\ d\omega_j &= \omega_1 \wedge \omega_{j-1},\ 3 \leq j \leq 2m \\ d\omega_{2m+1} &= \sum_{j=2}^{m} (-1)^j \omega_j \wedge \omega_{2m+1-j} \end{aligned}$$

where $\{\omega_1,..,\omega_{2m+1}\}$ is a basis of $\left(\mathbb{C}^{2m+1}\right)^*$.

Proposition. *For any $m \geq 4$ the Lie algebra $\mathfrak{g}_m$ is naturally graded of characteristic sequence $(2m-1,1,1)$ satisfying the following property*

$$\begin{aligned} C_{\mathfrak{g}_m}\left(C^m(\mathfrak{g}_m)\right) &\supset C^m(\mathfrak{g}_m) \\ C_{\mathfrak{g}_m}\left(C^{m-1}(\mathfrak{g}_m)\right) &\not\supset C^{m-1}(\mathfrak{g}_m) \end{aligned}$$

This algebra, which has been analyzed in [3], can be characterized as follows:

Theorem. *For $m \geq 4$ any naturally graded, central extension $\mathfrak{g}$ of the filiform model Lie algebra L_{2m} whose nilindex is $(2m-1)$ and satisfies*

$$\begin{aligned} C_{\mathfrak{g}}\left(C^m(\mathfrak{g})\right) &\supset C^m(\mathfrak{g}) \\ C_{\mathfrak{g}}\left(C^{m-1}(\mathfrak{g})\right) &\not\supset C^{m-1}(\mathfrak{g}) \end{aligned}$$

is isomorphic to $\mathfrak{g}_m$.

Algebras satisfying the preceding "centralizer condition" arise from the analysis of gradations of nilradicals of Borel subalgebras of complex simple Lie algebras, and are defined by a modification of the graded structure of these algebras [3]. In particular, the deformations and extensions of $\mathfrak{g}_m$ have been studied in [4].

Let $m \geq 4$. For any sequence $3 \leq q_1 < q_2 < ... < q_k \leq m+1$ let $\mathfrak{g}_m(q_1,.,q_k)$ be the $(2m+1)$-dimensional Lie algebra whose Maurer-Cartan equations are:

$$\begin{aligned} d\omega_1 &= d\omega_2 = 0 \\ d\omega_{q_i} &= d\omega_{2m+2-q_i} = 0,\ 1 \leq i \leq k \\ d\omega_j &= \omega_1 \wedge \omega_{j-1},\ 3 \leq j \leq 2m,\ j \notin \{q_i, 2m+2-q_i\}_{1 \leq i \leq k} \\ d\omega_{2m+1} &= \sum_{j=2}^{m} (-1)^j \omega_j \wedge \omega_{2m+1-j} \end{aligned}$$

where $\{\omega_1,..,\omega_{2m+1}\}$ is a basis of $\left(\mathbb{C}^{2m+1}\right)^*$.

Lemma. *For $m \geq 4$ and $k \geq 1$ the Lie algebras $\mathfrak{g}_m(q_1,.,q_k)$ are nonsplit nilpotent of nonlinear characteristic sequence.*

Remark. *In general, the algebras $\mathfrak{g}_m(q_1,.,q_k)$ will not be naturally graded. In particular this happens whenever we have q, q' such that $q' = 1 + q$. In fact, the differential form $d\omega_{2m+1}$ determines the gradation in some sense [3]. These algebras are also interesting for the study of solvable rigid Lie algebras whose nilradical has linear characteristic sequence [2].*

Theorem. *For any $m \geq 4$ and $k \geq 1$ the Lie algebra $\mathfrak{g}_m(q_1,.,q_k)$ is a contraction of $\mathfrak{g}$.*

Proof. For any k-tuple $(q_1,..,q_k)$ let $S(m,q_1,..,q_k)$ be the following linear system

$$a_1 + a_{j-1} = a_j,\ 3 \leq j \leq 2m,\ j \notin \{q_i, 2m+2-q_i\}_{1\leq i\leq k} \tag{1}$$

$$a_1 + a_{j-1} - a_{j-1} = -1,\ j \in \{q_i, 2m+2-q_i\}_{1\leq i\leq k} \tag{2}$$

$$a_j + a_{2m+1-j} = a_{j+1} + a_{2m-j},\ 2 \leq j \leq m-1 \tag{3}$$

and let $S'(m,q_1,..q_k)$ be the system given by (1) and (2). We claim that any solution of $S'(m,q_1,..,q_k)$ is also a solution of $S(m,q_1,..,q_k)$. In fact, if $j \notin \{q_i, 2m+2-q_i\}_{1\leq i\leq k}$ then $2m+2-q_i \neq 2m+1-j$ for all i, and therefore we have

$$a_j + a_{2m+1-j} = a_{j+1} + a_{2m-j} = a_1 + a_j + a_{2m-j}$$

thus

$$a_{2m+1-j} = a_1 + a_{2m-j}$$

If $j \in \{q_i, 2m+2-q_i\}_{1\leq i\leq k}$ then $2m+2-q_i = 2m+1-j$ for some $i \in \{1,..,k\}$. Then

$$a_j + a_{2m+1-j} = a_{j+1} + a_{2m-j} = 1 + a_1 + a_j + a_{2m-j}$$

and simplifying

$$a_{2m+1-j} = 1 + a_1 + a_{2m-j}$$

This shows that the equations (3) are superfluous. Now, the system $S'(m,q_1,..q_k)$ clearly has integer solutions, depending on the parameters $a_2 = N_1$ and $a_3 = N_2$. Let $(a_1,..,a_{2m})$ be the solution corresponding to the values $N_1 = N_2 = 1$ and $T_{2m+1}(\mathbb{C}(t))$ denote the Borel subgroup of $GL(2m+1,\mathbb{C})$ consisting of lower triangular matrices. Define $f_{(q_1,..,q_k),t} \in T_{2m+1}(\mathbb{C}(t))$ by

$$\begin{cases} f_{(q_1,..,q_k),t}(X_i) = t^{a_i}X_i,\ i \neq 2m+1 \\ f_{(q_1,..,q_k),t}(X_{2m+1}) = t^{1+a_{2m-1}} \end{cases}$$

and consider the Lie algebra $f_{(q_1,..,q_k),t}^{-1} * \mu$, where μ is the Lie algebra law associated to $\mathfrak{g}_m$. Then the structural equations of $f_{(q_1,..,q_k),t}^{-1} * \mu$ are given by

$$\begin{aligned}
d\omega_1 &= d\omega_2 = 0 \\
d\omega_j &= t^{a_1+a_{j-1}-a_j}\, \omega_1 \wedge \omega_{j-1},\ j \in \{q_i, 2m+2-q_i\}_{1\leq i\leq k} \\
d\omega_j &= \omega_1 \wedge \omega_{j-1},\ 3 \leq j \leq 2m,\ j \notin \{q_i, 2m+2-q_i\}_{1\leq i\leq k} \\
d\omega_{2m+1} &= \sum_{j=2}^{m} (-1)^j\, \omega_j \wedge \omega_{2m+1-j}
\end{aligned}$$

Now, as $a_1 + a_{j-1} - a_j = -1$ for $j \in \{q_i, 2m+2-q_i\}_{1\leq i\leq k}$ by the system $S'(m, q_1, .., q_k)$ (therefore by $S(m, q_1, .., q_k)$), it follows easily that

$$\lim_{t\to\infty} f_{(q_1,..,q_k),t}^{-1} * \mu = \mu(q_1, .., q_k)$$

where $\mu(q_1, .., q_k)$ is the law associated to $\mathfrak{g}_m(q_1, .., q_k)$. □

Corollary. *For* $m \geq 4$

$$\mu(q_1, .., q_k) \in O(\mu(q_1', .., q_k'))$$

if and only if $\{q_1, .., q_k\} = \{q_1', .., q_k'\}$.

Corollary. *For* $m \geq 4$ *the algebras* $\mathfrak{g}_m(q_1, .., q_k)$ *(included* $\mathfrak{g}$*) are nontrivial deformations of the algebra* $\mathfrak{h}_{m-1} \oplus \mathbb{C}^2$*, where* $\mathfrak{h}_{m-1}$ *is the* $(2m-1)$*-dimensional Heisenberg Lie algebra.*

Proof. Let $f_t \in T_{2m+1}(\mathbb{C}(t))$ be defined by

$$f_t(X_i) = t^{a_i} X_i,\ 1 \leq i \leq 2m+1$$

where the a_i satisfy the system $S'(m, q_1, .., q_m)$ and the additional equation

$$a_1 + a_{2m-1} = a_{2m} - 1$$

Then $\lim_{t\to\infty} f_t^{-1} * \mu \in O\left(\mathfrak{h}_{m-1} \oplus \mathbb{C}^2\right)$.
A similar reasoning shows that $\mathfrak{g}_m(q_1, .., q_k) \rightarrowtail \mathfrak{h}_{m-1} \oplus \mathbb{C}^2$. The result follows from the fact that a contraction defines a nontrivial deformation [7]. □

3 Applications to complete Lie algebras

Recall that a Lie algebra $\mathfrak{g}$ is called complete if it is centerless and any derivation is inner [8]. I recent years, a general theory of complete solvable Lie algebras whose nilradical is of maximal rank has been developed ([9], [10], [12]), and the existence of non-maximal rank algebras has been pointed out. In this section we will see how to obtain families of non-maximal rank completable Lie algebras considering the contractions above.

Let $\mathfrak{g}$ be a nilpotent Lie algebra and $Der(\mathfrak{g})$ its Lie algebra of derivations. A torus $\mathfrak{t}$ over $\mathfrak{g}$ is an abelian subalgebra of $Der(\mathfrak{g})$ consisting of semisimple derivations. The torus $\mathfrak{t}$ induces a natural represntation on $\mathfrak{g}$ such that

$$\mathfrak{g} = \sum_{\alpha \in \mathfrak{t}^*} \mathfrak{g}_\alpha$$

where $\mathfrak{t}^* = Hom(\mathfrak{t}, \mathbb{C})$ and $\mathfrak{g}_a = \{X \in \mathfrak{g} \mid [t, X] = \alpha(t) X,\ \forall t \in \mathfrak{t}\}$. If $\mathfrak{t}$ is maximal for the inclusion relation, by the conjugation theorems of Morozov, its common dimension is a numerical invariant called the rank of $\mathfrak{g}$, denoted $rank(\mathfrak{g})$. An algebra is called of maximal rank if $rank(\mathfrak{g}) = b_1 = \dim H^1(\mathfrak{g}, \mathbb{C})$. Following Favre [6], a weight system of $\mathfrak{g}$ is given by

$$P\mathfrak{g}(\mathfrak{t}) = \{(\alpha, d\alpha) \mid \alpha \in t^* \text{ such that } \mathfrak{g}_\alpha \neq 0,\ da = \dim \mathfrak{g}_\alpha\}$$

We also recall the folowing results:

Proposition. *Let $\mathfrak{g}$ be of maximal rank and $\mathfrak{t}$ a maximal torus. Then the semidirect product $\mathfrak{t} \oplus \mathfrak{g}$ is complete solvable.*

Definition. *The algebra $\mathfrak{g}$ is called completable if $\mathfrak{t} \oplus \mathfrak{g}$ is complete for a maximal torus $\mathfrak{t}$, and simple completable if it is not the direct sum of nontrivial completable Lie algebras.*

The main result we will use is a slight modification of the next

Theorem. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ a Cartan subalgebra. Assume that following conditions are satisfied:*

1. *$\mathfrak{h}$ is abelian*
2. *$\mathfrak{g} = \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$ with $\Delta \subset \mathfrak{h}^* - \{0\}$*
3. *there is a basis $\{\alpha_1, .., \alpha_l\}$ of $\mathfrak{h}^*$ n Δ such that $\dim \mathfrak{g}_{\pm\alpha_j} \leq 1$ for $1 \leq j \leq l$ and $[\mathfrak{g}_{\alpha_j}, \mathfrak{g}_{-\alpha_j}] \neq 0$ if $-\alpha_j \in \Delta$*
4. *$\mathfrak{h}$ and $\left\{\mathfrak{g}_{\pm\alpha_j},\ 1 \leq j \leq l\right\}$ generate $\mathfrak{g}$*

Then $\mathfrak{g}$ is a complete Lie algebra.

The last conditions can be replaced by by more general statements [11]

3' there is a enerating system $\{\alpha_1, .., \alpha_l\}$ of $\mathfrak{h}^*$ in Δ such that $\dim \mathfrak{g}_{\alpha_j} = 1$ for all j and $\mathfrak{h}, \mathfrak{g}_{\alpha_1}, .., \mathfrak{g}_{\alpha_l}$ genrate $\mathfrak{g}$.

4' Let $0 \neq x_j \in \mathfrak{g}_{\alpha_j}$ and a basis $\{\alpha_1, .., \alpha_r\}$ of $\mathfrak{h}^*$. For $r+1 \leq s \leq l$,

$$\alpha_s = \sum_{i=1}^{t} k_{is} \alpha_{j_i} - \sum_{i=1+t}^{r} k_{is} \alpha_{j_i}$$

where $k_{is} \in \mathbb{N} \cup \{0\}, (j_1, .., j_r)$ is a permutation of $(1, .., r)$, and there is a formula

$$\begin{aligned} & [\underbrace{x_{j_1}, .., x_{j_1}}_{k_{1s}}, ..\underbrace{x_{j_t}, .., x_{j_t}}_{k_{ts}}, ..., x_{k_m}] \\ = \quad & [\underbrace{x_{j_t+1}, .., x_{j_t+1}}_{k_{t+1s}}, ..\underbrace{x_{j_r}, .., x_{j_r}}_{k_{rs}}, x_s, x_{k_1} ..., x_{k_m}] \end{aligned}$$

without regard to the order or the way of bracketing, where $m \neq 0$ if $t = r$.

Theorem [11]. *Let $\mathfrak{g}$ be a Lie algebra satisfying conditions $(1), (2), (3')$ and $(4')$. Then $\mathfrak{g}$ is a complete Lie algebra.*

We obtain one more consecuence of the theorem in the preceding section

Corollary. *For $m \geq 4$ and $k \geq 1$*

$$rank\left(\mathfrak{g}_m\left(q_1, .., q_k\right)\right) > 2$$

Proof. The fact follows from the linear system $\left(S\left(\mathfrak{g}_m\left(q_1, .., q_k\right)\right)\right)$ associated to the algebras [1] and the fact that $b_1\left(\mathfrak{g}_m\left(q_1, .., q_k\right)\right) > b_1\left(\mathfrak{g}_m\right) = 2$, where $b_1(\mathfrak{g}) = \dim\ H^1(\mathfrak{g}, \mathbb{C})$. □

Proposition. *Let $m \geq 4$ and $k \geq 1$ a weight system of $\mathfrak{g}_m\left(q_1, .., q_k\right)$ is given by*

$$P\mathfrak{g}_m\left(\mathfrak{t}_m\right) = \left\{\left(\alpha_i, d\alpha_i\right)_{1 \leq i \leq 2m+1}\right\}$$

where $\dim \mathfrak{g}_{\alpha_i} = 1$ for all i and the weights $\{\alpha_1, .., \alpha_{2m+1}\}$ satisfy the following linear system

$$\begin{aligned} \alpha_1 + \alpha_{j-1} = a_j, \qquad & 3 \leq j \leq 2m,\ j \notin \{q_i, 2m+2-q_i\}_{1 \leq i \leq k} \\ a_{2m-t} + -\alpha_{t+1} - \alpha_m = \alpha_{m+1}, \qquad & 1 \leq t \leq m-2 \end{aligned} \qquad (S_1)$$

In particular, $rank\left(\mathfrak{g}_m\left(q_1, .., q_k\right)\right) \leq m+1$.

Proof. The system (S_1) coincides with the linear system $S\left(\mathfrak{g}_m\left(q_1, .., q_k\right)\right)$ associated to the nilpotent Lie algebra $\mathfrak{g}_m\left(q_1, .., q_k\right)$, thus the α_i correspond to eigenvalues of semisimple derivations [1]. Therefore we can isolate $\alpha_{j_1}, .., \alpha_{j_s}$ $(1 \leq j_1 < j_2 .. < j_s < 2+2k)$ such that for any $j \in \{1, , , 2m+1\} - \{j_1, .., j_s\}$ we have

$$\alpha_j = \sum_{t=1}^{s} a_j^t \alpha_{j_t},\ a_j^t \in \mathbb{C}$$

Now, using the standard techniques [1] it is routine to verify that for $1 \leq t \leq s$ the derivations $f_{j_t} \in Der\left(\mathfrak{g}_m\left(q_1, .., q_k\right)\right)$ given by

$$f_{j_t}\left(X_i\right) = a_i^t X_i,\ 1 \leq i \leq 2m+1$$

define a maximal torus $\mathfrak{t}_m(q_1,..,q_k)$ of $\mathfrak{g}_m(q_1,..,q_k)$. Thus the weight system is given as above.
Clearly any weight space $\mathfrak{g}_{\alpha_j}$ is at most one dimensional. For the last assertion, observe that for $k=m+1$ the system (S_1) is

$$\begin{aligned}\alpha_1+\alpha_{2m-1} &= \alpha_{2m}\\ \alpha_t+\alpha_{2m+1-t} &= \alpha_m+\alpha_{m+1},\ 2\leq t\leq m-1\end{aligned}$$

and that the rank is the maximal possible, namely $m+1$. □

Theorem. *For $m\geq 4$ and $k\geq 1$ the semidirect products*

$$\mathfrak{r}_m(q_1,..,q_k)=\mathfrak{t}_m(q_1,..,q_k)\oplus\mathfrak{g}_m(q_1,..,q_k)$$

are solvable and complete.

Proof. It is easy to see that for any $f\in Der(\mathfrak{r}_m(q_1,..,q_k))$ we have

$$f(\mathfrak{t}_m(q_1,..,q_k))\subset\mathfrak{g}_m(q_1,..,q_k)$$

This is a direct consequence of the particular action of $\mathfrak{t}_m(q_1,..,q_k)$ over $\mathfrak{g}_m(q_1,..,q_k)$. If $\mathfrak{t}_m(q_1,..,q_k)=\{h_1,..,h_s\}$, then for any $1\leq t\leq s$ there exists a permutation $\sigma\in\mathcal{S}_s$ such that $f_{j_t}=ad(h_t)$. The nilradical is clearly generated by $\{X_1,X_2,X_{q_i},X_{2m+2-q_i}\}_{1\leq i\leq k}$, where $\{X_1,..,X_{2m+1}\}$ is a dual basis to $\{\omega_1,..,\omega_{2m+1}\}$. It follows that the algebra $\mathfrak{r}_m(q_1,..,q_k)$ satisfies the conditions of the preceding theorem, so that it is complete. □

Lemma. *If $k=1$ and $q_1\neq m+1$ or $k\geq 2$, the $\mathfrak{g}_m(q_1,..,q_k)$ is not of maximal rank.*

Proof. The proof is an immediate consequence of the weight system. For $k=1$ and $q_1=m+1$ we have $rank(S_1)=b_1=3$. Observe that this is the only case where $q_i=2m+2-q_i$. □

Corollary. *For any odd dimension $n\geq 9$ there exist completable Lie algebras of non-maximal rank.*

Thus the contractions of the algebra $\mathfrak{g}_m$ (it is itself completable since it is of maximal rank) are completable of non-maximal rank, up to an exception. The families above expand the examples obtained in [10],[12] for non-maximal rank. In fact, since the algebras $\mathfrak{g}_m(q_1,..,q_k)$ are nonsplit, we obtain even more:

Corollary. *For $m\geq 4$ and $k\geq 1$ the algebras $\mathfrak{g}_m(q_1,..,q_k)$ are simple completable.*

References

[1] J. M. Ancochea, M. Goze. Le rang du système lineaire des racines d'une algèbre de Lie resoluble complexe. Comm. Algebra **20** (1992), 875-887.

[2] J. M. Ancochea, R. Campoamor. On Lie algebras whose nilradical is (n-p)-filiform. Comm. Algebra **29** (2001), 427-450.

[3] J. M. Ancochea, R. Campoamor. On certain families of naturally graded Lie algebras. J. Pure Appl. Algebra, *to appear.*

[4] J. M. Ancochea, R. Campoamor. Nonfiliform characteristically nilpotent and complete Lie algebras. Algebra. Colloq., *to appear.*

[5] D. Burde. Degenerations of nilpotent Lie algebras. J. Lie Theory **9** (1999), 193-202.

[6] G. Favre Système des poids sur une algèbre de Lie nilpotente. Manuscripta Math.**9** (1973), 53-90.

[7] A. Fialowski, J. O'Halloran. A comparison of deformations and orbit closure. Comm. Algebra **(**18) (1990), 4121-4140.

[8] G. F. Leger. Derivations of Lie algebras III. Duke Math. J. **30** (1963), 637-645.

[9] D. J. Meng, L. S. Zhu. Solvable complete Lie algebras I. Comm. Algebra **24** (1996), 4181-4197.

[10] D. J. Meng, L. S. Zhu. Solvable complete Lie algebras II. Algebra Colloq. **5** (1998), 289-296.

[11] D. J. Meng. Complete Lie algebras and Heisenberg Lie algebras. Comm. Algebra **22** (1994), 5509-5524.

[12] D. J. Meng. The complete Lie algebras with abelian nilpotent radical. Acta Math. Sinica **34** (1991), 191-202.

[13] E. J. Saletan. Contractions of Lie groups. J. Math. Phys. **2** (1961), 1-21.

[14] C. Seeley. Degenerations of central quotients. Arch. Math. **56** (1991), 236-241.