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Uniform Approximation Theorems for Real–Valued Continuous Functions

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For a completely regular space X, $C(X)$ and $C^*(X)$ denote, respectively, the algebra of all real–valued continuous functions and bounded real–valued continuous functions over X. When X is not a pseudocompact space, i.e., if $C^*(X) \neq C(X)$, theorems about uniform density for subsets of $C^*(X)$ are not directly translatable to $C(X)$. In [1], Anderson gives a sufficient condition in order that certain rings of $C(X)$ be uniformly dense, but this condition is not necessary.

In this paper we study the uniform closure of a linear subspace of real–valued functions and we obtain, in particular, a necessary and sufficient condition of uniformly density in $C(X)$. These results generalize, for the unbounded case, those obtained by Blasco–Moltó for the bounded case [2]. The approximation technique used by them (essentially the same of Tietze [10], Mrowka [8] and Jameson [7]) is also the starting point for us.

In order to establish their results, Blasco–Moltó define a new concept, the S–separation, which is a suitable debilitation of that of complete separation. Here, we introduce a parallel concept, namely the "Lebesgue chain condition". From it we obtain our density theorem for linear subspaces of $C(X)$. Following the same structure of [2], we define the "property C". This property agrees in the bounded case with the property S of Blasco–Moltó. But, although the property S permits to characterize the linear subspaces (of bounded functions) whose uniform closure is a ring or a lattice containing all the real constant functions, the property C does not permit such characterization in the unbounded case. Nevertheless, by means of the property C we shall be able to characterize the linear subspaces of $F(X) = \mathbb{R}^X$ whose uniform closure is closed under composition with uniformly continuous functions over $\mathbb{R}$.

1. UNIFORM APPROXIMATION FOR LINEAR SUBSPACES.

For a set X, $F(X)$ (resp., $F^*(X)$) denotes the class of all (resp., all bounded) real-valued functions over X. $F(X)$ is a linear space by introducing pointwise addition and scalar multiplication. We let $\mathbb{R}$ the set of real numbers, $\mathbb{Z}$ the integer numbers, $\mathbb{N}$ the positive integer numbers and $\overline{\mathbb{R}}=\mathbb{R}\cup\{-\infty,\infty\}$.

DEFINITION 1.1. Let f be a function from $F(X)$. A Lebesgue chain of f we define as any countable cover $\{C_n\}_{n\in\mathbb{Z}}$ of X such that

$$C_n=\{x\in X:\alpha_{n-1}<f(x)<\alpha_{n+1}\}$$

and $\{\alpha_n\}_{n\in\mathbb{Z}}$ is a nondecreasing sequence in $\overline{\mathbb{R}}$ for which there exists $r>0$ satisfying $\alpha_n-\alpha_{n-1}\geqslant r$ provided α_n is a real number.

DEFINITION 1.2. A set $\mathfrak{F}$ of $F(X)$ satisfies the Lebesgue chain condition for a function f if there exists $k>0$ such that for every Lebesgue chain $\{C_n\}_{n\in\mathbb{Z}}$ of f there exists $g\in\mathfrak{F}$ such that $|g(x)-n|<k$ when $x\in C_n$.

PROPOSITION 1.3. *Let $\mathfrak{F}\subset F(X)$ a linear subspace over $\mathbb{R}$ and let $f\in F(X)$. If $\mathfrak{F}$ satisfies the Lebesgue chain condition for f then f belongs to the uniform closure of $\mathfrak{F}$.*

Recall that a cozero-set in a topological space X is a set of the form $\mathrm{coz}(f)=\{x\in X:f(x)\neq 0\}$ with $f\in C(X)$.

THEOREM 1.4. *Let X be a topological space and let $\mathfrak{F}\subset C(X)$ a linear subspace over $\mathbb{R}$. The following conditions are equivalent:*

a) $\mathfrak{F}$ is uniformly dense in $C(X)$.

b) For each countable 2-finite cover $\{C_n\}_{n\in\mathbb{Z}}$ (i.e., $C_n\cap C_m=\emptyset$ when $|n-m|>1$) of X by cozero-sets, there is $h\in\mathfrak{F}$ with $|h(x)-n|<2$ when $x\in C_n$.

Remark. The only results about uniform density in the unbounded case, that we know, are the Anderson's theorem [1] and some technics of uniform approximation by using partitions of unity. All of these results can be obtained now as a consequence of the precedent theorem.

2. CHARACTERIZATION OF THE LEBESGUE CHAIN CONDITION. THE PROPERTY C.

This section is devoted to characterize the Lebesgue chain condition in terms of composition with certain class of continuous functions over $\mathbb{R}$. This useful characterization will permite us to find some algebraic properties of the uniform closure of the linear subspaces.

PROPOSITION 2.1. *Let* $\mathfrak{F} \subset F(X)$ *a linear subspace over* $\mathbb{R}$ *and let* $f \in \mathfrak{F}$. *The following conditions are equivalent:*

a) $\mathfrak{F}$ *satisfies the Lebesgue chain condition for f.*

b) $\varphi \circ f \in \overline{\mathfrak{F}}$ *(uniform closure of* $\mathfrak{F}$*) for every* φ *uniformly continuous and monotonic function over* $\mathbb{R}$.

c) $\varphi \circ f \in \overline{\mathfrak{F}}$ *for every* φ *uniformly continuous function over* $\mathbb{R}$.

DEFINITION 2.2. A subset $\mathfrak{F}$ of $F(X)$ has the property C, whenever $\mathfrak{F}$ satisfies the Lebesgue chain condition for every $f \in \mathfrak{F}$.

From Proposition 2.1 or also Theorem 1.4, it follows that every linear subspace which is uniformly dense in $C(X)$ has the property C. The next theorem say us that the Lebesgue chain condition characterizes the uniform closure of linear subspaces with the property C.

THEOREM 2.3. *Let* $\mathfrak{F} \subset F(X)$ *a linear subspace over* $\mathbb{R}$ *with the property* C *and let* $f \in F(X)$. *Then* $f \in \overline{\mathfrak{F}}$ *if and only if* $\mathfrak{F}$ *satisfies the Lebesgue chain condition for f.*

THEOREM 2.4. *Let* $\mathfrak{F} \subset F(X)$ *a linear subspace over* $\mathbb{R}$. *The following conditions are equivalent:*

a) $\mathfrak{F}$ *has the property* C.

b) $\overline{\mathfrak{F}}$ *is closed under composition with uniformly continuous and monotonic functions over* $\mathbb{R}$.

c) $\overline{\mathfrak{F}}$ *is closed under composition with uniformly continuous functions over* $\mathbb{R}$.

d) $\overline{\mathfrak{F}}$ *has the property* C.

e) There exists $k > 0$ *such that* $\mathfrak{F}$ *satifies the Lebesgue chain condition for every* $f \in \mathfrak{F}$ *and with the same k.*

From this result it follows that the uniform closure of a linear subspace with the property C is, in particular, a linear sublattice containing all the real constant functions. This algebraic property is equivalent in the bounded case to the property of being a subring containing all the real constant functions (Blasco–Moltó [2]). But in the general case that equivalence does not remain valid. For instance if $\mathfrak{F}$ is the set of all uniformly continuous functions over $\mathbb{R}$

then it is easy to check that $\mathfrak{F}$ is a linear sublattice with the property C but not a subring. Related with these topics we have the following result.

PROPOSITION 2.5. *If the uniform closure of a linear subspace* $\mathfrak{F}$ *of* $F(X)$ *is a subring and a sublattice containing all the real constant functions, then* $\mathfrak{F}$ *has the property* C.

Finally note that if $\mathfrak{F}$ satisfies the above conditions then $\overline{\mathfrak{F}}$ is closed under composition with a large class of continuous functions over $\mathbb{R}$. In fact, this class contains the polynomials functions, the uniformly continuous functions and the continuous and bounded functions but it is not all $C(\mathbb{R})$ as the next example shows.

EXAMPLE 2.6. Let $\mathfrak{F}$ be the subset of $C(\mathbb{R})$ defined by

$$\mathfrak{F} = \{\textstyle\sum_{i=1}^{n} f_i p_i \,/\, f_i \in C_0(\mathbb{R}) \text{ and } p_i \text{ is a polynomial, } i=1,\dots,n\}$$

where $C_0(\mathbb{R})$ denotes the set of all continuous functions over $\mathbb{R}$ vanishing at infinity. Thus, $\mathfrak{F}$ is an uniformly closed subring and sublattice containing all the real constant functions which is not closed under composition with all $C(\mathbb{R})$.

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