HEEGAARD DIAGRAMS FOR CLOSED 4-MANIFOLDS

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Deep results of Laudenbach and Poenaru are used to get an analogue of the 3-dimensional Heegaard diagrams for PL, closed, orientable 4-manifolds. The way to passing from one diagram to another representing the same 4-manifold is obtained, but in contrast with the 3-dimensional analogue, the problem of deciding whether a given diagram corresponds to a closed, 4-manifold is left open.

I. INTRODUCTION

Each closed, orientable, PL 4-manifold W^4 admits a handle presentation $W^4 = H^0 \cup \lambda H^1 \cup \mu H^2 \cup \gamma H^3 \cup H^4$ and we show that W^4 is uniquely determined by the cobordism between $\partial(H^0 \cup \lambda H^1)$ and $\partial(H^0 \cup \lambda H^1 \cup \mu H^2)$, defined by the 2-handles μH^2. This provides an analogue of the 3-dimensional Heegaard diagrams for closed, orientable, PL 4-manifolds.

We solve the equivalence relation problem for Heegaard diagrams (i.e. the way to pass from one diagram to another representing the same 4-manifold), but the problem of deciding whether a given diagram corresponds to a closed 4-manifold remains open. This is in contrast with the 3-dimensional analogue.

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We illustrate the equivalence relation for Heegaard diagrams, giving a proof that $2V^4 \cong S^4$ for Mazur manifolds V^4, without appealing to the structure of $V^4 \times I$.

We finish with some examples of Heegaard diagrams for closed 4-manifolds and we suggest some applications.

I am indebted to Robert Edwards, Charles Giffen, Cameron Gordon and Laurence Siebenmann for helpful conversations.

II. THE HOMEOTOPY GROUP OF $\lambda \# S^1 \times S^2$

This is calculated, in an implicit way, in [3; p. 342] and we include the proof for completeness. We define $\lambda \# S^1 \times S^2$ as a connected sum of λ copies of $S^1 \times S^2$ if $\lambda > 0$, and as S^3 if $\lambda = 0$.

THEOREM 1. The homeotopy group of $\lambda \# S^1 \times S^2$ (i.e. the group of orientation-preserving autohomeomorphisms of $\lambda \# S^1 \times S^2$, quotient the subgroup of those isotopic to identity) is generated by sliding 1-handles, twisting 1-handles, permuting 1-handles and rotations.

Remark. The definitions of the first three kinds of generators are in [3] . For the fourth one, see [2], (see also Section 6).

Proof. Let φ be an orientation-preserving autohomeomorphism of $\lambda \# S^1 \times S^2$. Because the first three kinds of generators induce a system of generators for the automorphism group of $\pi_1(\lambda \# S^1 \times S^2)$, we can suppose that φ induces the identity on the fundamental group of $\lambda \# S^1 \times S^2$. But then (Lemma 3 of [3]) φ induces also the identity on $\pi_2(\lambda \# S^1 \times S^2)$; in other words, φa_i is homotopic to a_i, for each i, where $A = \{a_1, \ldots, a_\lambda\}$ is a set of transverse spheres to the handles of $\lambda \# S^1 \times S^2$. According to [2; Theorem 1 and proof of Lemma 5.1], up to isotopy, we can suppose that φ is the identity in A. Hence, using [2; 5, 4], we can compose φ with a sequence of rotations to get a map isotopic to the identity.

We remark, as in [3], that all of these generators extend to $\lambda \# S^1 \times B^3$. Note also, that there is an autohomeomorphism of $\lambda \# S^1 \times B^3$ which is orientation-reversing in the boundary. So we have;

THEOREM 2. Given a 4-manifold M^4, with $\partial M^4 = \lambda \# S^1 \times S^2$, the manifold $M^4 \cup \lambda \# S^1 \times B^3$ is independent of the way of pasting the boundaries together.

Remark. The results in [3] are a consequence of Theorem 2.

COROLLARY 1. Each closed, orientable 4-manifold with handle presentation $W^4 = H^0 \cup \lambda H^1 \cup \mu H^2 \cup \gamma H^3 \cup H^4$ is completely determined by $H^0 \cup \lambda H^1 \cup \mu H^2$.

Conversely (except for S^4, where the analogue of Waldhausen's result for Heegaard splittings of S^3 is true[1]), we do not know how many embeddings of a bouquet of γ 1-spheres there are in a closed W^4, such that the complementary space of the thickened bouquet has a representation with *0-*, *1-*, and 2-handles. This seems to be an interesting question.

III. HEEGAARD SPLITTINGS

Let $W^4 = H^0 \cup \lambda H^1 \cup \mu H^2 \cup \gamma H^3 \cup H^4$ and let V^4 be a collar of $\lambda \# S^1 \times S^2$ in $H^0 \cup \lambda H^1$. Then, $V^4 \cup \mu H^2$ is a cobordism essentially from $\lambda \# S^1 \times S^2$ to $\gamma \# S^1 \times S^2$ that we call $C^4(\lambda, \gamma)$, and we have:

[1] *Robert Edwards and Cameron Gordon have pointed out that that analogue is true also for simply-connected 4-manifolds.*

COROLLARY 2. The cobordism $C^4(\lambda,\gamma)$ determines W^4.

We call $(W^4, C^4(\lambda,\gamma))$ a *Heegaard-splitting* of W^4. This is an analogue of a Heegaard splitting in M^3. Two Heegaard-splittings $(W^4, C^4(\lambda,\gamma))$ and $(W'^4, C'^4(\lambda',\gamma'))$ are *equivalent* if there is a homeomorphism of pairs between them. We have just proved that each W^4 has at least one Heegaard-splitting.

IV. HEEGAARD-DIAGRAMS FOR 4-MANIFOLDS

Consider the pair $(\lambda \# S^1 \times S^2, w)$, where w is a framed link in $\lambda \# S^1 \times S^2$. We can associate to this pair a cobordism from $\lambda \# S^1 \times S^2$ to the 3-manifold M^3 obtained by surgery on the framed link w, as follows. Form $(\lambda \# S^1 \times S^2) \times I$ and attach a number of 2-handles along the framed curves w, which we suppose are living in the $\times\{1\}$ boundary component of $(\gamma \# S^1 \times S^2) \times I$. If the new end M^3 of this cobordism is $\gamma \# S^1 \times S^2$, for some γ, we call $(\lambda \# S^1 \times S^2, w)$ a *Heegaard-diagram.*

The splitting $(W^4, C^4(\lambda,\gamma))$, in Section 3, can be represented by Heegaard-diagram $(\lambda \# S^1 \times S^2, w)$, which turns out to be, at the same time, a representative for the closed, orientable 4-manifold W^4.[1]

Two pairs $(\lambda \# S^1 \times S^2, w)$ and $(\lambda' \# S^1 \times S^2, w')$ are *equivalent* if there is a homeomorphism of pairs preserving the framings. Because such a homeomorphism extends to $C(\lambda,\mu)$, we see that equivalent Heegaard-diagrams provide equivalent Heegaard-splittings.

[1] *The term diagram stands for a pair $(\lambda \# S^1 \times S^2; w)$ even when the other end M^3 of the cobordism is not $\gamma \# S^1 \times S^2$, for any γ. In this case, the diagram represents the bounded 4-manifold which is obtained by pasting $\chi \# S^1 \times B^3$ with the cobordism associated to the diagram, along their boundaries. We emphasize that a Heegaard diagram represents always a closed, orientable 4-manifold.*

The problem of enumerating of all possible Heegaard diagrams corresponds to the following

PROBLEM. When is a pair $(\lambda \# S^1 \times S^2, w)$ a Heegaard-diagram?[1]

V. MOVES ON A HEEGAARD-DIAGRAM

We want to know when two Heegaard-diagrams are representatives of the same 4-manifold. It turns out to be (Theorem 3, below) that this happens if and only if the diagrams become equivalent after applying some moves, which we describe in the following.

Move i) (stabilization). The diagram $(\lambda \# S^1 \times S^2, w)$ changes to $(\lambda \# S^1 \times S^2 \# S^1 \times S^2, w \cup \alpha)$, where α is $S^1 \times \{*\}$ with framing $S^1 \times \{**\}$, $\{*\}$ and $\{**\}$ being two points of S^2, in the last connected summand of course.

Move ii) (stabilization). The diagram changes to $(\lambda \# S^1 \times S^2, w \cup \alpha)$, where α is the boundary of a disc in $\lambda \# S^1 \times S^2$ which does not cut w. The framing of α is a concentric curve in that disc.

Move iii) (band move, see [1] and Fig. 1.) The curve $\alpha \in w$, with framing a, changes to (α', a') by using another curve $\beta \in w$ with framing b, as follows. Let A (resp. A') be the framing annulus with $\partial A = \alpha \cup a$ (resp. $\partial A' = \beta \cup b$), and let $\hat{A}$ be a proper collar of b in A'. Let R be a ribbon connecting a with $\partial\hat{A} - b$, as in Figure 1, and let R' be a ribbon concentric to R. Then $(A \cup \hat{A} - R) \cup (R - R')$ is an annulus; the boundary containing a part of α (resp. a) is α' (resp. a').

[1] *Kirby calculus [1] is very useful for deciding this in concrete examples, (see example in section 7).*

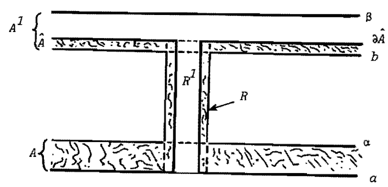


Fig. 1.

Now we state:

THEOREM 3. The diagrams $(\lambda \# S^1 \times S^2, w)$ *and* $(\lambda' \# S^1 \times S^2, w')$ *represent the same 4-manifold if and only if they become equivalent after applying a sequence of moves i), ii), iii), or their inverses.*

Proof. If $C^4(\lambda,\gamma)$ is the cobordism associated with the diagram $(\lambda \# S^1 \times S^2, w)$ we see that move iii), corresponding to a sliding of 2-handles [1], does not change the type of $C^4(\lambda,\gamma)$. Moves i) and ii) do change this type but do not change the 4-manifold W^4 associated with the diagram. To see this, note first that move ii) is the dual of a move i), and that (after filling the cobordism up with $\lambda \# S^1 \times B^3$) move i) correspond to a birth of a pair of cancelling handles in a handle presentation of W^4.

Now, we prove the other implication. We fill up $C^4(\lambda,\gamma)$ with $\lambda \# S^1 \times B^3$ and $\gamma \# S^1 \times B^3$ to get the closed 4-manifold W^4. We choose an arbitrary handle structure $H^0 \cup \lambda H^1$ (resp. $H^0 \cup \gamma H^1$) for $\lambda \# S^1 \times B^3$ (resp. $\gamma \# S^1 \times B^3$) to get a handle structure $H^0 \cup \lambda H^1 \cup \mu H^2 \cup \gamma H^3 \cup H^4$ for W^4, where μ is the number of components of the link w.

Doing the same with the other diagram $(\lambda' \# S^1 \times S^2, w')$ we get another handle presentation for W^4.

As in [1], we see that these two handle presentations are related by a sequence of the following moves:

1) Births and deaths of complementary (i.e. $(i, i+1)$-) handle pairs.

2) Handle slidings.

Also, as in [1], we can suppose that, in such a sequence of moves, there are no births or deaths of $(0,1)$-handle pairs or $(3,4)$-handle pairs.

Now a birth of a $(1,2)$-handle pair changes the diagram associated with the handle presentation by a move i). The 1-handle slidings do not change the equivalence class of the diagram. The 2-handle slidings are moves iii) [1]. The 3-handle slidings do not change the diagram. A birth of a $(2,3)$-handle pair modifies the cobordism $C^4(\lambda,\gamma)$ by the addition of a 2-handle along the boundary of a disc which is contained in $\gamma \# S^1 \times S^2$. We can push this disc off the 2-handles of the cobordism using moves iii): this time the diagram is changed by moves iii) and ii).

VI. A PARTICULAR MODEL

If we fix a particular model M_λ for $\lambda \# S^1 \times S^2$ we can define a Heegaard diagram (M_λ, w), where now the framed curves w live in M_λ.

For example, take $R^3 + \infty$ minus 2λ open balls $B = \{B_1, B_1', \ldots, B_\lambda, B_\lambda'\}$ of radius 1/4, B_i having its center at $(2i-1,0,0)$ and B_i' at $(2i,0,0)$. Then M_λ is the result of identifying the boundaries of B_i and B_i' by reflection in the plane $x = 2i - 1/2$.

We define some moves in this model which suffice to generate the group of autohomeomorphisms of M_λ, up to isotopy.

Move iv) (twisting 1-handles). This move takes place in a ball containing B_i and B_i' but meeting no others: outside of this ball, the move is the identity. The move is a permutation of the pair $(\partial B_i, \partial B'_i)$ accomplished by rotation of 180° around the axis $\{x = 2i - 1/2,\ y = 0\}$.

Move v) (permuting 1-handles). The support of the move is a ball containing B_i, B'_i, B_{i+1} and B'_{i+1}, but meeting no others. The pair $(\partial B_i, \partial B'_i)$ goes to $(\partial B'_{i+1}, \partial B_{i+1})$ by rotation around the axis $\{x = 2i + 1/2,\ y = 0\}$.

Move vi) (sliding 1-handles). Take a ball C in $R^3 + \infty$ containing in its interior only a single ball B_i (or B'_i) and such that ∂C cuts B only in two discs indentified by reflection (in $B_j \cup B'_j$, for instance). The support of the move is $C - B_i$ (which is a punctured solid torus in M_λ). We perform the move by pushing B_i along C, going through $\partial B_j \equiv \partial B'_j$ and coming back to the original position.

Move vii) (rotation). The support is a regular neighborhood of $\partial B_i \equiv \partial B'_i$ in M_λ, and consists in a complete rotation of ∂B_i around an axis. (i.e. this is the "suspension" of a Dehn-Lickorish twist).

Move viii) (symmetry). This is a reflection in the (x,y)-plane

Because these moves generate the group of autohomeomorphisms of $\lambda \# S^1 \times S^2$, up to isotopy, we have:

THEOREM 3'. The diagrams (M_λ, w), $(M_{\lambda'}, w')$ represent the same 4-manifold if and only if they become isotopic after applying a sequence of moves i), ii), ..., viii).

VII. THE DUAL DIAGRAM

Let $H^0 \cup \lambda H^1 \cup \mu H^2 \cup \gamma H^3 \cup H^4$ be a handle presentation for a closed, orientable, PL, 4-manifold W^4. There is a dual presentation with γ 1-handles, μ 2-handles and λ 3-handles. These two presentations define two Heegaard diagrams which are mutually *dual*.

Let $(\lambda \# S^1 \times S^2; w)$ be a diagram. A *framed meridian* for the component w_i of w is a meridian of $\partial U(w_i)$, framed by a parallel meridian.

THEOREM 4. If a Heegaard diagram $(\lambda \# S^1 \times S^2; w)$ is given, the associated dual diagram is $(M^3; n)$, where M^3 is obtained from $\lambda \# S^1 \times S^2$ by surgery in the framed w, and n is a system of framed meridians for w.

Proof. M^3 is $\partial(H^0 \cup \lambda H^1 \cup \mu H^2)$, which is obtained by surgery in the framed w. Let H^2_i be one of the components of μH^2. Then H^2_i is attached to M^3 by the cocore which is framed by the product structure of H^2_i. Isotoping this framed cocore out of H^2_i it becomes a framed meridian for the attaching sphere of H^2_i, which is a component of w.

Example. The dual diagram of the Heegaard diagram of Figure 2a is obtained in Figures 2b to 2e. In the proof we change a 1-handle by $\bigcirc^0$ [1] because this also represents $S^1 \times S^2$. Note that we are also proving that the original diagram is in fact a Heegaard diagram (compare footnote 3).

[1] *The notation $\bigcirc^n$ means that the framing has linking number n with the curve, in a regular neighbourhood of a disc bounding the curve.*

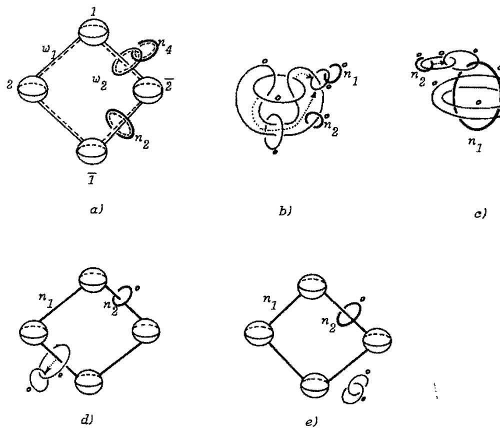


Fig. 2.

Remark. In the last example we illustrate the fundamental feature of Kirby's calculus [1], i.e. given a Heegaard diagram $(S^3;w)$ of a manifold W^4 (which necessarily is simply connected), by adition of k copies of $\bigcirc^{\pm 1}$ (see footnote 4) and moves we get $(S^3;\bigcirc^{\pm 1},\overset{m}{\ldots},\bigcirc^{\pm 1})$. This means that $W^4 \# k(\pm \mathbb{C}P^2) \cong m(\pm\mathbb{C}P^2)$. Thus, Kirby's calculus cannot work with framed links in $\lambda \# S^1 \times S^2$, if $\lambda > 0$.

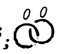
VIII. EXAMPLE: HEEGAARD DIAGRAMS FOR $2W^4$

Let $(\lambda \# S^1 \times S^2;w)$ be a pair where w is a framed link of μ components. Let W^4 be the orientable 4-manifold with handle

presentation $H^0 \cup \lambda H^1 \cup \mu H^2$, where μH^2 is attached along the framed w.

LEMMA 1. Let n be a system of framed meridians for w, then $(\lambda \# S^1 \times S^2;\ w \cup n)$ is a Heegaard diagram for $2W^4$.

Proof. $2W^4$ has a handle presentation $H^0 \cup \lambda H^1 \cup \mu H^2 \cup \mu H'^2 \cup \lambda H^3 \cup H^4$, obtained by doubling the presentation $W^4 = H^0 \cup \lambda H^1 \cup \mu H^2$. The cocore of H^2_i is the attaching sphere of the corresponding H'^2_i, and is framed by the product structure of H^2_i. Isotoping this framed cocore out of H^2_i it becomes the framed meridian n_i.

Corollary 3. $(S^3;$  $)$ *represents* $2(S^2 \times D^2) \cong S^2 \times S^2$, *and* $(S^3;$ $)$ *represents* $2(\mathbb{C}P^2\text{-ball}) \cong \mathbb{C}P^2 \# -\mathbb{C}P^2 \cong S^2 \tilde{\times} S^2$. *(see footnote 4).*

The main idea for studying $2W^4$ in a purely four dimensional setting is illustrated by the first two examples.

Examples:

a) *Mazur manifolds* - $S^1 \times S^2 \cong \partial(S^1 \times B^3)$ is illustrated in Figure 3 by $(R^3+\infty)$ - Int $B_1 \cup B_2$, identifying ∂B_1 with ∂B_2 by reflection in the mediatrix plane of the centers of B_1, B_2. The curve w is Mazur curve [6] (with arbitrary framing), and n is a framed meridian for w. By Lemma 1 $(S^1 \times S^2;w,n)$ is a Heegaard diagram for $2W^4$. Mazur proved $2W^4$ to be S^4 by looking at $2W^4$ as the boundary of $W^4 \times I$. We prove this in Figure 3 by using Heegaard moves, because the Heegaard diagrams $(S^1 \times S^2;\ w,n)$ and $(S^3;\phi)$, being equivalent, represent the same closed 4-manifold S^4.

b) In the example a) we have avoided Mazur phenomenon without passing to $W^4 \times I$. In this example we see that the framing

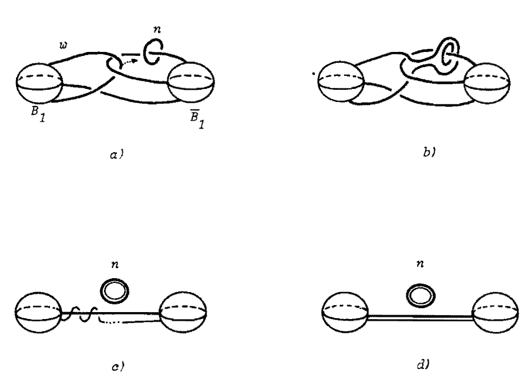


Fig. 3.

of a curve w with associated meridian n can be modified mod 2 (see Fig. 4.) This illustrates why framings in $\partial\,(\#\, S^1 \times B^4) \cong \partial(S^1 \times B^3 \times I)$ are in $\mathbb{Z}_2$.

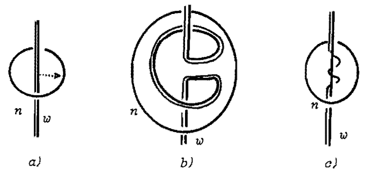


Fig. 4.

c) *Heegaard diagrams for* $F_g \times S^2$. Let F_g be a closed, orientable surface of genus g. Since $F_g \times S^2 = \partial(F_g \times D^2)$ we can apply Lemma 1, after getting a diagram (see footnote 2) for $F_g \times D^2$.

A handle presentation $F_g = H^0 \cup 2g\, H^1 \cup H^2$ is shown in Figure 5a (for g=2). Hence Figure 5b is a diagram $(2g\ \#\ S^1 \times S^2; w)$ for $F_g \times D^2$. Thus, according to Lemma 1, $(2g\ \#\ S^1 \times S^2; w, n)$ is a Heegaard diagram for $F_g \times S^2$.

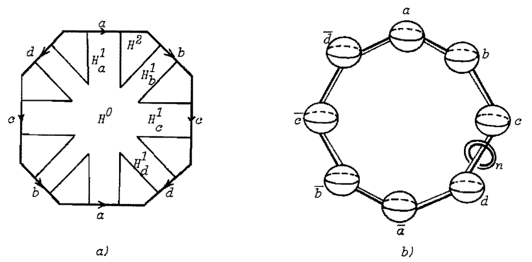


Fig. 5.

IX. HEEGAARD DIAGRAMS FOR MAPPING TORUS.

This construction is a generalization of [4].

A Heegaard diagram $(F_\lambda; v, w)$ for a closed, orientable 3-manifold M^3 is a scheme for a handlebody presentation $M^3 = H^0 \cup \lambda H^1 \cup \lambda H^2 \cup H^3$, where $\partial(H^0 \cup \lambda H^1) = F_\lambda$ and the system of belt-spheres (resp. attaching spheres) of λH^1 (resp. λH^2) is v (resp. w).

We can represent F_λ as the intersection of the (x,y)-plane with the model M_λ, for $\lambda\ \#\ S^1 \times S^2$, described in section 6.

The system v is represented by $\partial B \cap (x,y)$-plane, and w lies on F_λ. Figure 6 illustrates the case $M^3 = L(3,1)$.

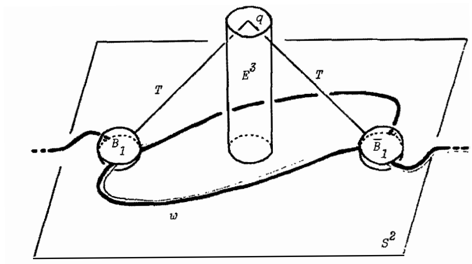


Fig. 6.

Let B^3 be a 3-ball in M^3. The punctured manifold $M_0^3 = M^3$ - int B^3 has the handlebody presentation $M_0^3 = H^0 \cup \lambda H^1 \cup \lambda H^2$. The following Lemma is then clear.

LEMMA 2. $(\lambda \# S^1 \times S^2; w')$, where w' is the system w of $(F_\lambda; v, w)$ framed by parallel curves lying on F_λ, is a diagram (see footnote 2) for $M_0^3 \times [0,1]$.

The boundary of $M_0 \times [0,1]$ is $M \# {-M}$, hence $(\lambda \# S^1 \times S^2; w)$ is also a surgery presentation for $M^3 \# {-M^3}$. The operation # is performed along a 2-sphere S^2 which is F_λ, cut along w' and closed with 2λ discs going by the λ 2-handles. The natural homeomorphism $k : M_0^3 \times \{0\} \to M_0^3 \times \{1\}$ is generated by reflection in the (x,y)-plane. Clearly (see Diagram 1) $M^3 \times [0,1]$ is obtained by attaching a 3-handle to $M_0^3 \times [0,1]$ along S^2.

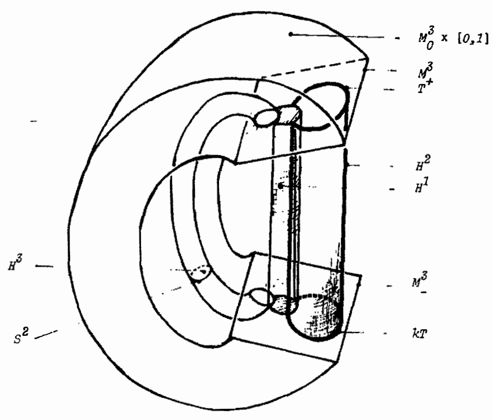


Diagram 1.

Let E^3 be a regular neighbourhood in $\{(x,y,z) | z \geq 0\}$ of a point in $F_\lambda - w$. Let T be the joint of $Q \in$ Int E^3 with the 2λ north poles of the system B in the model M_λ (see Figure 6). Thus T is a 1-spine of the handlebody $M_\lambda \cap \{(x,y,z) | z \geq 0\}$.

If an orientation preserving autohomeomorphism of M^3 is given, ~~we can~~ suppose that, up to isotopic deformation, h is an autohomeomorphism of M_0^3 which is the identity in E^3 and also that $hT \subset M_\lambda \cap \{(x,y,z) | z \geq 0\}$ and also that h permutes the handle structure of M^3 (*) (periodic f.i.)

We begin to build the mapping torus of h, $(M^3 \times [0,1])/(x = khx \; x \in M^3 \times \{0\})$, by taking a 1-handle in $M^3 \times [0,1]$ with attaching sphere $Q \cup kQ$, so that if $\alpha : \partial B^1 \times B^3 \to M^3 \times \{0,1\}$ is the attaching map, the composition

$$\alpha(\{0\} \times B^3) \xrightarrow{\alpha^{-1}} \{0\} \times B^3 \to \{1\} \times B^3 \xrightarrow{\alpha} \alpha(\{1\} \times B^3)$$

(*) that is needed to construct Diagram 1 the pasting of 2-handles and the 1 handle coherently.
This is the mistake in (1). (the h there is not periodic)

equals $k|\alpha(\{0\}xB^3)$. The effect of this in the diagram is to take off balls C^3 and kC^3, centered at Q and kQ, respectively, and to identify their boundaries by k (see Figure 7).

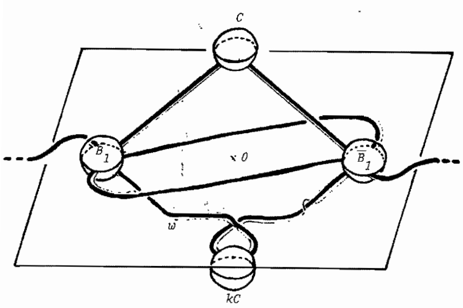


Fig. 7.

We continue the construction of the mapping torus by connecting T with khT, out of $C^3 \cup kC^3$, as suggested by Diagram 1, i.e. by adding λ 2-handles along the curves $(T \cup khT) \cap M_\lambda$, with framings invariants by kh. The effect in the diagram is illustrated in Figure 7.

Thus, the Heegaard diagram of the mapping torus is completed, because the result of connecting the rest of $M^3 x \{0\}$ with the rest of $M^3 x\{1\}$ is equivalent to take λ 3-handles and one 4-handle.

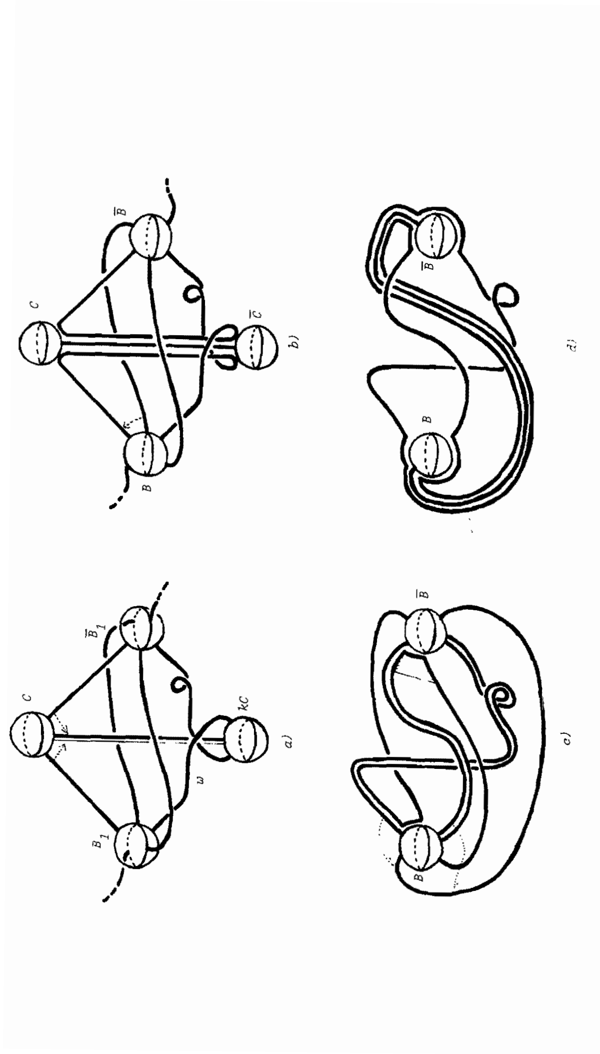
Example. Figure 7 represents a Heegaard diagram for the mapping torus of $h : L(3,1) \to L(3,1)$, h being the involution inducing the 2-fold cyclic covering $L(3,1) \to S^3$, branched over the trefoil knot (h is generated by a central reflection on the (x,y)-plane).

X. HEEGAARD DIAGRAMS FOR OPEN BOOKS WITH BINDING S^2

Let h be an orientation preserving autohomeomorphism of M_0^3 which is the identity in the boundary. Let V^4 be the mapping torus of h. Now, ∂V^4 has a natural product structure $\partial V^4 \cong S^1 x S^2$, and we consider the closed 4-manifold W^4 (resp. $\tilde{W}^4$) which is obtained by pasting the boundaries of V^4 and $B^2 x S^2$ together so that the product structures agree (resp. does not agree essentially). From the point of view of handle addition this can be done in two steps. In the first step, we add a 2-handle along the curve S^1 x point $\subset S^1 x\ S^2 \cong \partial V^4$, and afterwards we add a 4-handle to close the manifold. The difference between W^4 and $\tilde{W}^4$ depends on the framing of the handle addition.

Now the naturally framed S^1 x point $\subset \partial V^4$ appears in the diagram for the mapping torus of h as the joint of two points in ∂C^3 with their images by k (See Diagram 1). The inclusion of this framed curve in the diagram for the mapping torus of h gives a Heegaard diagram for W^4. The diagram for $\tilde{W}^4$ is obtained by changing the framing of S^1 x point by a complete twist.

Example. The diagram of Figure 8a corresponds to an open book with leave $L(3,1)_0$ and monodromy h as in the example of section 9. This manifold is S^4 [5] because the binding is the 2-twist spun knot of the trefoil knot. A different proof is given in Figures 8b to 8d, using Heegaard moves.



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