THE WHITEHEAD LINK, THE BORROMEAN RINGS AND THE KNOT 9_{46} ARE UNIVERSAL

by

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1. INTRODUCTION.

A link L in S^3 is *universal* if every, closed, orientable 3-manifold is a covering of S^3 branched over L. Thurston [1] proved that universal links exist and he asked if there is a universal knot, and also if the Whitehead link and the Figure-eight knot are universal. In [2], [3] we answered the first question by constructing a universal knot. The purpose of this paper is to prove that the Whitehead link and the Borromean rings, among others, are universal. The question about the Figure-eight knot remains open, but we show that the ribbon knot 9_{46} is universal.

2. THE CONSTRUCTIONS.

The starting point is that every closed, orientable 3-manifold is a 3-fold covering of S^3 branched over a negative closed braid L corresponding to a representation ω: π_1 ($S^3 \quad$ L) $\rightarrow S_3$ onto the symmetric group S_3 of the numbers $\{1, 2, 3\}$, which sends meridians of L to transpositions in such a way that the three meridians intervening in a crossing are endowed with different transpositions (see [3]). In the Figures of this article we will endow overpasses of links with transpositions, meaning that the corresponding meridians are sent to these transpositions.

We now modify this closed braid L by applying in various places the move of Fig. 1, which do not modify the covering manifold [4], [5], [6]. Each crossing

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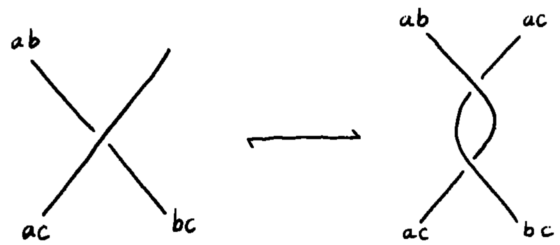

Fig. 1

of L can be modified as depicted in Fig. 2, thus obtaining a link L_1 as the one diagrammaticaly depicted in Fig. 3.

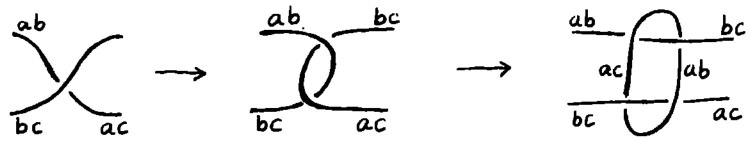

Fig. 2

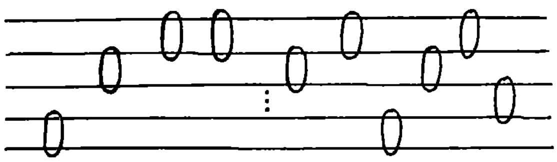

Fig. 3

The closed braid L_1 is naturally decomposed in vertical pieces, each one containing just one little circle as depicted in Fig. 4. We modify this pieces as shown in Fig. 4, and we do this successively.

Once this has been done, we use again the move of Fig. 1 to modify the crossings of Fig. 4 marked with stars. We can obtain an infinite number of different patterns as the two shown in Figures 5 and 6. Each pattern will give rise to a different universal link. We have modelled the two patterns that we will use in this article as to obtain the Whitehead link and the Borromean rings. It is essential to note that the permutations intervening in the crossings with stars of Fig. 4 are all different, thus allowing the application of the move of Fig. 1.

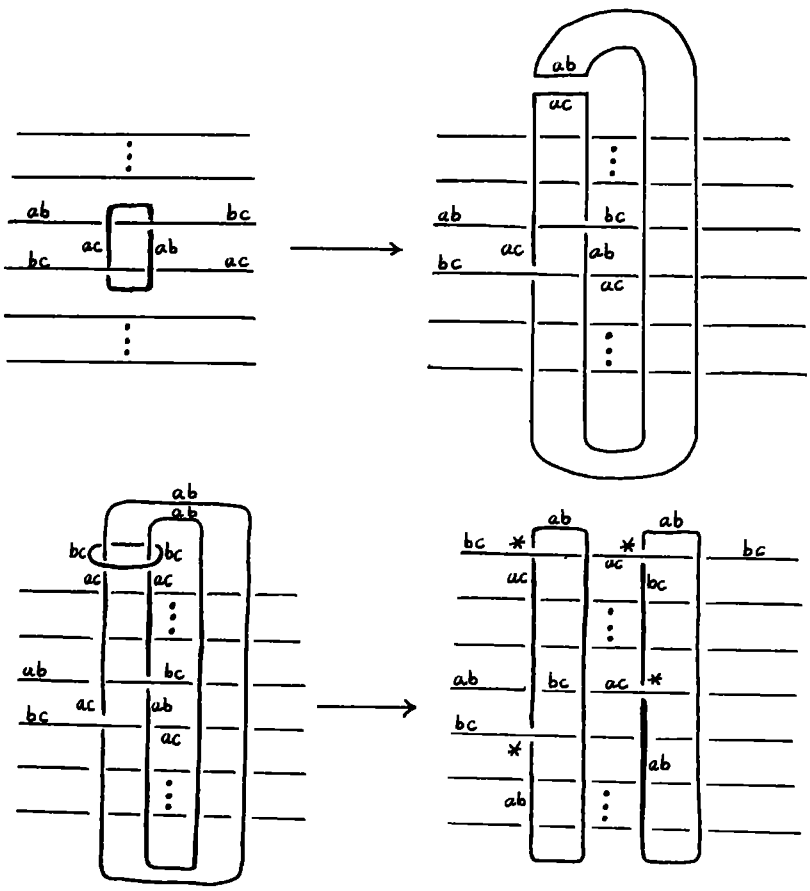

Fig. 4

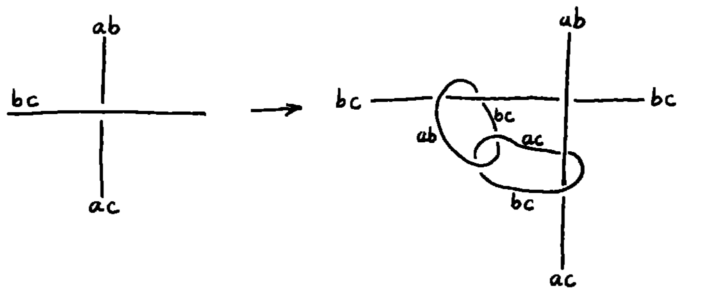

Fig. 5

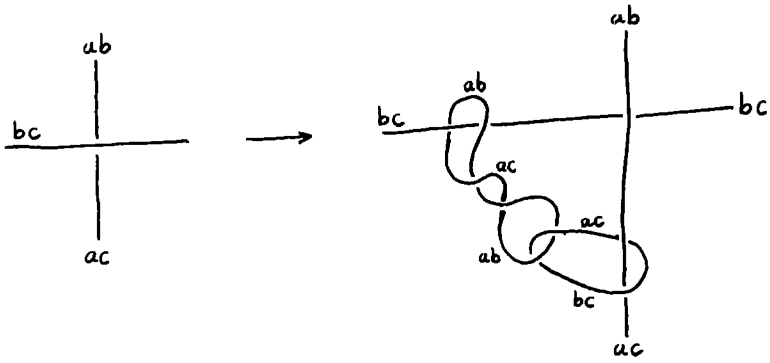

Fig. 6

Finally, after adding some new components, and placing them in the proper locations, we obtain a link as the one of Fig. 7. The circles of Fig. 7 are assumed

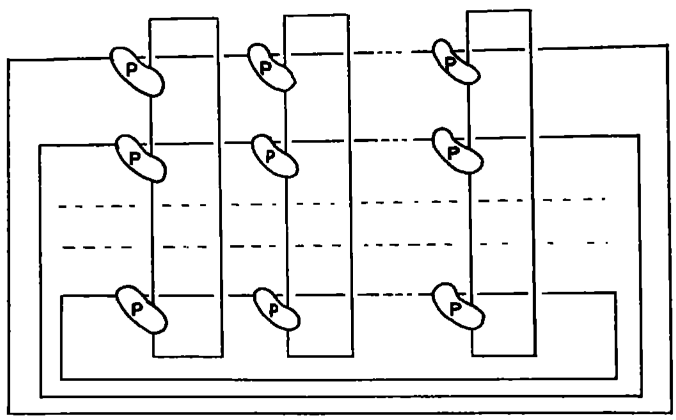

Fig. 7

to be all filled up with one pattern P. We will use the patterns shown in Figures 5 and 6, but an infinite number of others can be used, thus providing infinitely many universal links.

Now, the link of Fig. 7 is the preimage of the solid part of Fig. 8 in some cyclic covering of S^3 branched over the doted part of the same Figure. Also the link of Fig. 8 is the preimage of the solid part of Fig. 9 in some cyclic covering of S^3 branched over the doted part of the same Figure.

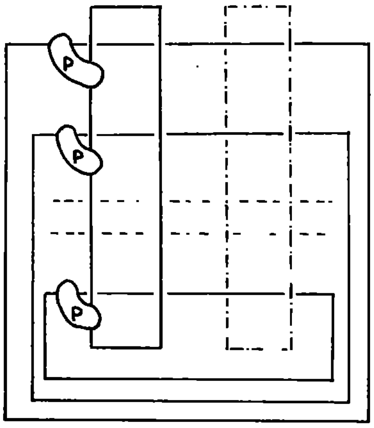

Fig. 8

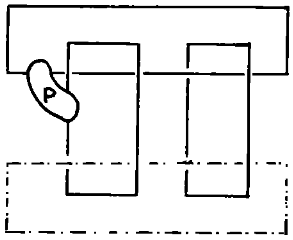

Fig. 9

Using the pattern P of Fig. 5 we obtain the following universal link (Fig. 10).

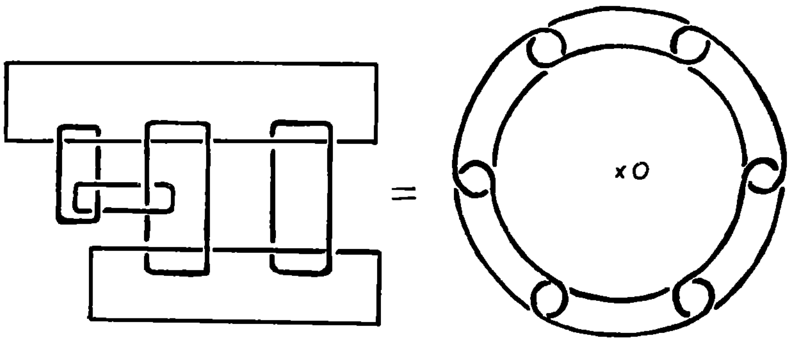

Fig. 10

Using the pattern P of Fig. 6 we obtain the universal link of Fig. 11.

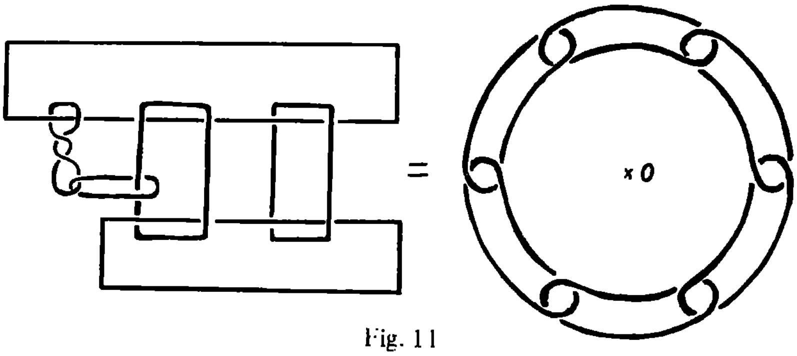

Fig. 11

Projecting now the link of Fig. 10 (resp. Fig. 11) under the cyclic covering over S^3 determined by the obvious 3-fold (resp. 6-fold) rotation around an axis through the point 0 we obtain the Borromean rings and the Whitehead link, respectively. Thus these links are universal (Fig. 12).

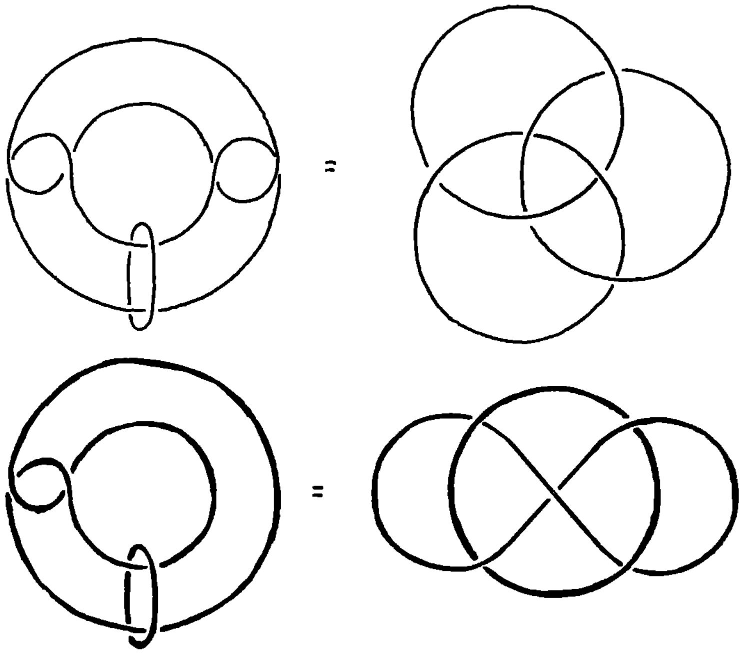

Fig. 12

Under the obvious 3-fold symmetry of the Borromean rings we obtain the universal link of two components shown in Fig. 13.

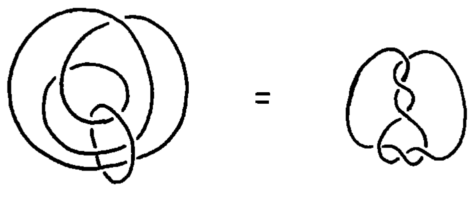

Fig. 13

Theorem 1. *The Whitehead link, the Borromean rings and the links of Figures 10, 11 and 13 are universal.*

Using the method, explained in [3], for obtaining a universal knot from a universal link, and applying it to the link of Fig. 14 b we obtain the knot of Fig. 15, which is universal.

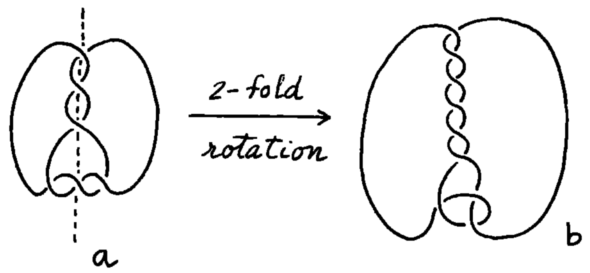

Fig. 14

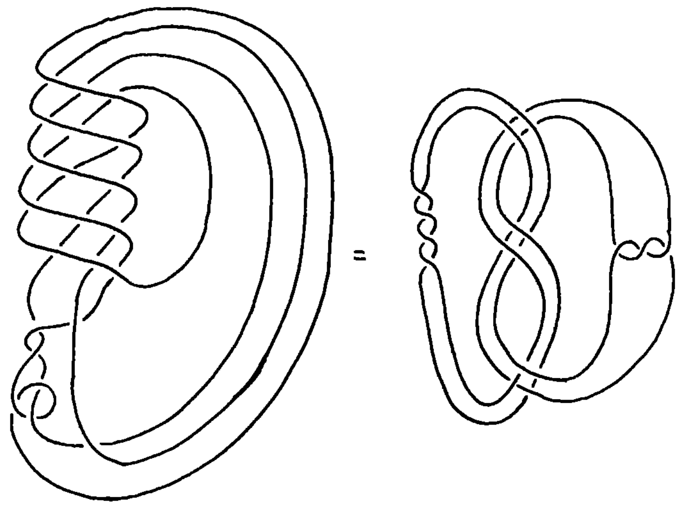

Fig. 15

Remarks.

1. The knot of Fig. 15 has a projection with 15 crossings.

2. Using other patterns to fill up P in Fig. 9, it is possible to obtain an infinite number of universal links and it is to be expected that the number of universal knots obtained from them by means of the method of [3] is also infinite. For instance the links L_n of Fig. 16 are all universal. Since one component of L_n is a [n/2]-double of the trivial knot, the family L_n contains infinitely many different links.

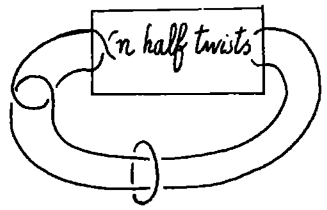

Fig. 16

3. THE UNIVERSAL KNOT 9_{46}.

The four hexagons of Fig. 17a form a disk D^2 which we assume embedded

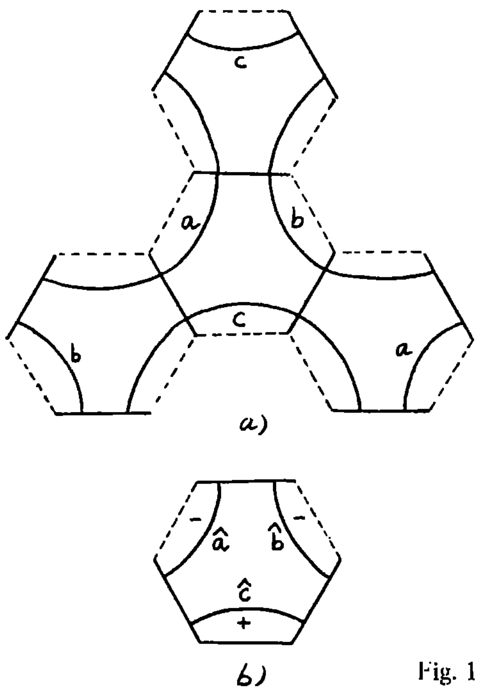

Fig. 17

in R^3. Take a small regular neighbourhood B^3 of D^2 in $\mathbb{R}^3$ and consider the folding map $p : B^3 \to \hat{B}^3$ along the solid sides of the central hexagon. Thus $p : B^3 \to \hat{B}^3$ is an irregular branched covering with 4 sheets. We are interested in the "double" $2p : 2B^3 \to 2\hat{B}^3$ of this covering. The preimages of the arcs $\hat{a}$, $\hat{b}$, $\hat{c}$ of Fig. 17 b are the arcs a, b, c of Fig. 17 a. We perform the move of Fig. 1 three times in regular neighbourhoods of the arcs $\hat{a}$, $\hat{b}$, $\hat{c}$ thus obtaining Fig. 18 b.

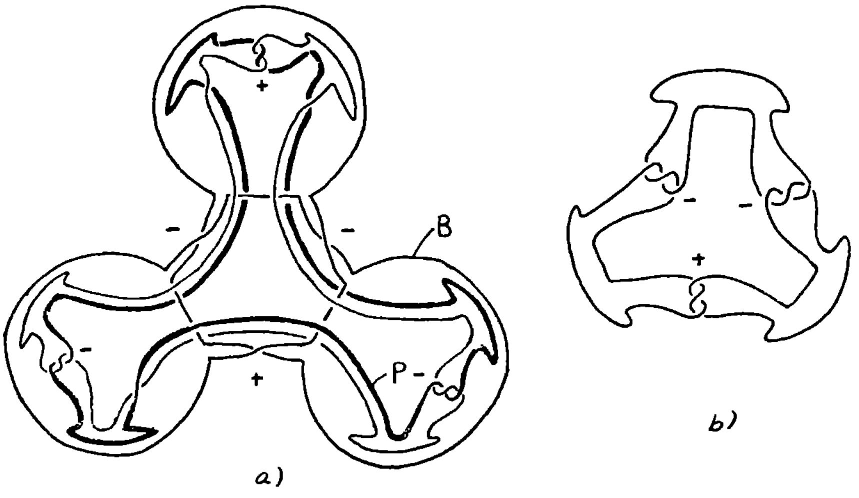

Fig. 18

In the covering this lifts to the modifications depicted in Fig. 18 a [5, page 65]. Now the union of the branching cover B and a component P of the pseudo-branching cover is the Whitehead link. Thus we have

Theorem 2. *The knot 9_{46} is universal.*

Remark. 9_{46} is a ribbon knot and it has genus one, (Fig. 19).

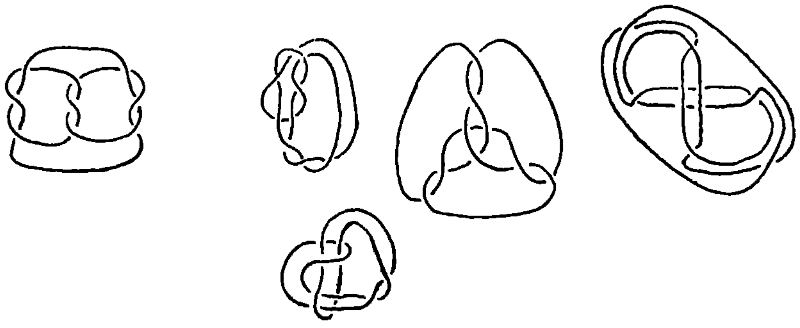

Fig. 19

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