Role of the filter phase in phase sampling interferometry

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Abstract: Any linear phase sampling algorithm can be described as a linear filter characterized by its frequency response. In traditional phase sampling interferometry the phase of the frequency response has been ignored because the impulse responses can be made real selecting the correct sample offset. However least squares methods and recursive filters can have a complex frequency response. In this paper, we derive the quadrature equations for a general phase sampling algorithm and describe the role of the filter phase.

References

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1 Quadrature phase detection

Phase sampling interferometry (PSI) [1] is an experimental technique for phase measurement based on the introduction of a linear phase shift between a set of interferograms. In the temporal case, if we describe the interferogram as a rectangular 2D matrix, the measured intensity at every pixel can be described as a set of temporal samples given by

$$g(t) = b + m\cos(\varphi(t) + \omega_0 t) \quad t = 1...N, \tag{1}$$

where b is the background or DC term, m the modulation or AC term, and ω_0 the carrier frequency. In PSI, the main objective is the computation of the modulating phase $\phi(t)$ for every sample.

All linear phase sampling algorithms (PSAs) can be described as linear quadrature filters characterized by their impulse response $h(t)$ or their frequency response $H(\omega)$ [2]. In our case, the temporal interferogram can be considered as an aperiodic signal and the impulse and frequency responses are related by the discrete time Fourier transform and its inverse given by

$$H(\omega) = FT(h) = \sum_{t=-\infty}^{\infty} h(t)e^{-it\omega} \quad \text{and} \quad h(t) = FT^{-1}(H) = \int_{-\pi}^{-\pi} H(\omega)e^{it\omega} d\omega. \tag{2}$$

The interferogram spectrum is given by

$$G(\omega) = FT(g(t)) = b\delta(\omega) + \frac{1}{2}m\left[C(\omega - \omega_0) + C(\omega + \omega_0)^*\right], \tag{3}$$

where $C(\omega) = FT(e^{i\varphi(t)})$. If $C(\omega)$ is narrowband, $C(\omega - \omega_0)$ will be a side-lobe contained in an interval $\Delta\omega_0$ with $\omega_0 \in \Delta\omega_0$. The objective of any PSA is the filtering of one of the lobes of $G(\omega)$ so that the output signal spectrum is given by

$$Q(\omega) = H(\omega)G(\omega) = \frac{1}{2}mH(\omega)C(\omega - \omega_0). \tag{4}$$

Transforming back Eq. (4) in the temporal domain, we obtain the analytical signal associated with g

$$q(t) = h(t) * g(t) = \frac{1}{2}mH(\omega_0)e^{-i(\varphi(t) + \omega_0 t)}. \tag{5}$$

Finally, from the angle of $q(t)$ we can obtain the modulating phase $\phi(t)$. From this analysis the minimal conditions that a quadrature filter must fulfil are [2]

$$H(-\Delta\omega_0) = H(0) = 0, \; H(\Delta\omega_0) \neq 0, \tag{6}$$

where $-\Delta\omega_0$ represents the interval $\Delta\omega$ for negative frequencies. In Eq. (6) by $H(\Delta\omega_0) \neq 0$ we indicate that the frequency response is different form cero for the frequency interval $\Delta\omega_0$, and we use the same notation for $H(-\Delta\omega_0) = 0$. The condition $H(0) = 0$ is not properly a quadrature condition but it is necessary to filter the DC term. Equations (5) and (6) present, in terms of linear systems, the classical results of PSI demodulation. However, as we will show in the next section, they are only valid in the case of a real frequency response and do not take into account the effects of the PSA filter phase. Obviously we have been using linear PSAs for the last thirty years without taking into account any filter phase effects. One applies the PSA formula (the typical arctan calculation) and obtains the phase. This has been possible because all classical linear PSAs have symmetrical impulse responses and therefore their frequency responses can be always expressed as a real function. However, recently there have appeared more sophisticated PSAs with complex frequency responses, like the least squares [3] [4] and recursive techniques [2]. These kinds of PSAs have much more interesting properties than classical ones with respect to noise rejection, missing samples and border processing. With complex frequency responses the filter phase has to be taken into account to fully understand how the corresponding PSA works. Additionally, we are going to clarify an apparently trivial question about classical PSAs: at which sample do we measure the phase when we use a PSA?

2 The role of the filter phase in quadrature phase detection: the group delay

We are going to start by analysing the effect of a PSA with non-linear phase tuned at frequency ω_0 on the next signal

$$s(t) = p(t)\cos(\omega_0 t). \tag{7}$$

In (7) we assume that the envelope $p(t)$ is narrowband. If we factorize the filter frequency response in terms of amplitude and phase, $H(\omega) = A(\omega)e^{i\Phi(\omega)}$, near ω_0 we can always approximate the filter phase as

$$\Phi(\omega) = \Phi_0 - \tau(\omega_0) \cdot (\omega - \omega_0), \tag{8}$$

with

$$\tau(\omega) = -\frac{d\Phi}{d\omega}\bigg|. \tag{9}$$

When we apply the PSA to $g(t)$, taking into account the quadrature conditions (6) and that the envelope $p(t)$ is narrow-band, we will obtain

$$F(\omega) = H(\omega)S(\omega) = \frac{1}{2}e^{i\Phi_0}A(\omega)P(\omega - \omega_0)e^{-i\tau(\omega - \omega_0)}, \tag{10}$$

where $P(\omega) = FT(p(t))$. Expression (10) can be rewritten as

$$F(\omega) = \frac{1}{2}e^{i\Phi_0}A(\omega)\left(P(\omega)e^{-i\tau\omega} * \delta(\omega - \omega_0)\right), \tag{11}$$

where * denotes the convolution product. And, using the convolution and modulation properties of the *FT* we obtain

$$f(t) = \frac{1}{2}e^{i\Phi_0}A(\omega_0)p(t-\tau)e^{-it\omega_0}. \tag{12}$$

That is, the filter phase introduces a time delay τ in the signal envelope. In signal processing, the time delay τ as given by (9) is denominated group delay [5]. The physical interpretation of τ is that the different packets of a signal will have different delays after passing through a linear system. For this reason, a desirable property for any digital filter is to have a linear phase for all ω so that the group delay is constant. In the case of quadrature phase detection, it is enough to have a linear phase around the tuning frequency ω_0 given that the interferogram is quasi-monochromatic.

Now we are ready to go forward and analyse the effect of the filter phase on a PSI temporal signal like (1). First, we must rewrite $g(t)$ as

$$g(t) = b + m\cos\varphi(t)\cos\omega_0 t - m\sin\varphi(t)\sin\omega_0 t. \tag{13}$$

We can see that $g(t)$ is composed by two narrowband signals like (7). Taking into account that $\cos(\varphi(t) + \pi/2) = -\sin\varphi(t)$, the application of the PSA will yield the next analytic signal

$$q(t) = \frac{1}{2}me^{i\Phi_0}A(\omega_0)e^{-i(\varphi(t-\tau) + \omega_0 t)}. \tag{14}$$

Again, we can compute the phase $\phi(t-\tau)$ from the angle of $q(t)$. Equation (14) is the generalization of the classical result given by Eq. (5) and it is the main result of this paper: the PSA filter phase determines the sample where we are measuring the modulating phase, and the time-shift is the group delay given in Eq. (9). Another interesting result is that if around the tuning frequency the phase is not linear, $\tau = \tau(\omega)$, the recovered phase will be distorted by the variable group delay.

In conclusion, the filter phase determines the temporal origin of the PSA and adds an additional requirement to the usual quadrature conditions. That is, the group delay must be constant around ω_0, $\tau(\Delta\omega_0) \approx \tau_0$.

3 Two illustrative examples

We are going to clarify the former discussion by analysing two PSAs. The first example is a filter with impulse response given by

$$h_1(t) = [-1, -2i, 2, 2i, -1].$$ (15)

Its frequency response is

$$H_1(\omega) = \sum_{t=1}^{5} h(t) e^{-it\omega} = (2 + 4\sin\omega - 2\cos 2\omega) e^{-i3\omega}.$$ (16)

In this case $H_1(-\pi/2) = H_1(0) = 0$ and $\tau = 3$. This means that h_1 represents a 5 samples PSA method tuned at $\omega_0 = \pi/2$ and constant group-delay of 3 samples. For this PSA the modulating phase will be obtained from

$$\tan\varphi(3) = \frac{\text{Im}(h_1 * g)}{\text{Re}(h_1 * g)}\bigg|_{t=0} = \frac{2(g(2) - g(4))}{2g(3) - g(1) - g(5)}.$$ (17)

The reader will recognise this equation as the standard form of the Hariharan 5 step PSA [6]. In general, in classical linear PSAs, if the filter length is K and the first sample start in $t = 1$ the group delay is $\tau = \text{round}((K+1)/2)$, where round(.) represents the rounding to the closest integer operation.

The second example we present is more interesting from the point of view of the group delay. We are speaking of the first order recursive PSI filter presented in [2] given by

$$q(t) = \eta q(t-1) e^{-i\omega_0} + \left(g(t-1) e^{-i\omega_0} - 2g(t) + g(t+1) e^{i\omega_0}\right) * (\delta(t-1) - \delta(t+1)).$$ (18)

This filter is a combination of a five step PSA tuned at ω_0 (indeed, for $\omega_0 = \pi/2$ is the Hariharan PSA), with a first order low pass recursive filter with gain η. The practical interest of this recursive filter is the flexibility increasing in terms of noise rejection with respect to classical PSAs. In this recursive filter, the analytical signal at every sample, $q(t)$, is computed recursively from the previous sample and a filtered version of the interferogram g. The spectral response of this filter is [2]

$$H_2(\omega) = \frac{4i\sin(\omega)(\cos(\omega + \omega_0) - 1)}{1 - \eta\exp(-i(\omega - \omega_0))}.$$ (19)

It is easy to verify that $H_2(\omega)$ fulfills the quadrature conditions (6) and therefore represents a true PSA. As before, we measure the instantaneous phase determining the angle of the analytical signal $q(t)$. However, in this case the question is: at which sample are we measuring the phase? As we have shown, the answer depends on the group delay. In Fig. 1 we show the amplitude and the group delay of the PSA represented by $H_2(\omega)$ with $\eta = 0.75$ and $\omega_0 = \pi/2$ rad/sample. As can be seen, due to the non-linear phase, the group delay has a strong dependence with the frequency. As $\tau(\pi/2) \approx 4$ samples, if we apply the recursive PSI filter to a carrier interferogram with $\omega_0 \approx \pi/2$ we will obtain a demodulated phase with a delay of 4 samples with respect to the input signal, that is

$$\tan\varphi_r(t-4) = \frac{\text{Im}(H_2\{g(t)\})}{\text{Re}(H_2\{g(t)\})}.$$ (20)
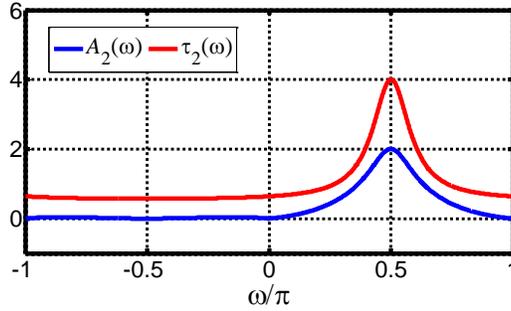
Fig. 1. Amplitude and phase delay for the recursive filter of Eq. (18) for $\eta = 0.75$ and $\omega_0 = \pi/2$ rad/sample.

To show the effect of this delay on the phase, Fig. 2 shows the demodulation of a temporal interferogram given by $g(t) = 100 + 50\cos\{2\cos(4\pi t / N) + 0.5\pi t\}$ with $N = 100$ samples, using the PSA represented in Fig. 1. Figure 2a shows a plot of the input temporal signal and Fig. 2b shows the demodulated phase, $\phi_r(t)$ together with the actual input phase. As expected there is a 4 samples delay between both phases.
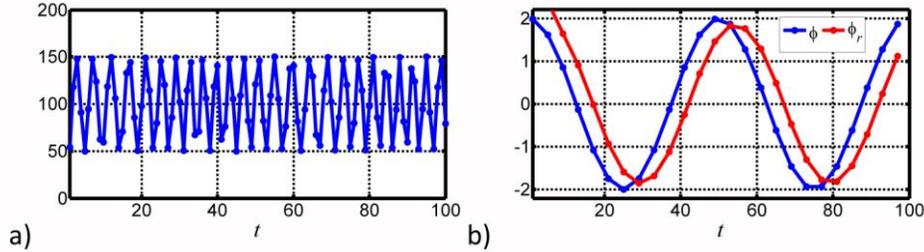

Fig. 2. Demodulation results using the recursive filter of Eq. (18). a) input signal, b) demodulation results (red) and actual phase (blue). In this figure there is a marker every four samples to make easier the observation of the 4 samples delay between both signals.

An undesired effect of the nonlinear filter phase is the distortion induced by the different group delays of the spectral components of the interferogram. Practically, one consequence is that the filter response to an edge is not symmetrical, and depends on the edge direction. Figure 3 shows the demodulation results for a 100 samples temporal interferogram with a rectangular pulse of 2 rad. Figure 3a shows the interferogram and Fig. 3b shows the demodulation results using the same filter represented in Fig. 1. As can be seen, there is clear asymmetry in the recovered phase. This is a well-known effect and can be suppressed if the interferogram can be stored. In this case, the solution is to process back the output of the recursive filter in reverse order: $q_T(t) = H_2\{q^*(-t)\}$. In digital signal processing this method is referred as bidirectional filtering. Mathematically, this is equivalent to demodulating back the signal with the same impulse response but time-inverted and complex conjugated. Therefore, if $h_2(t)$ is the impulse response of the recursive PSA, the total impulse response of the bidirectional filtering will be

$$h_T(t) = h_2(t) * h_2^*(-t), \tag{21}$$

from which it is easy to see that total frequency response is real and given by

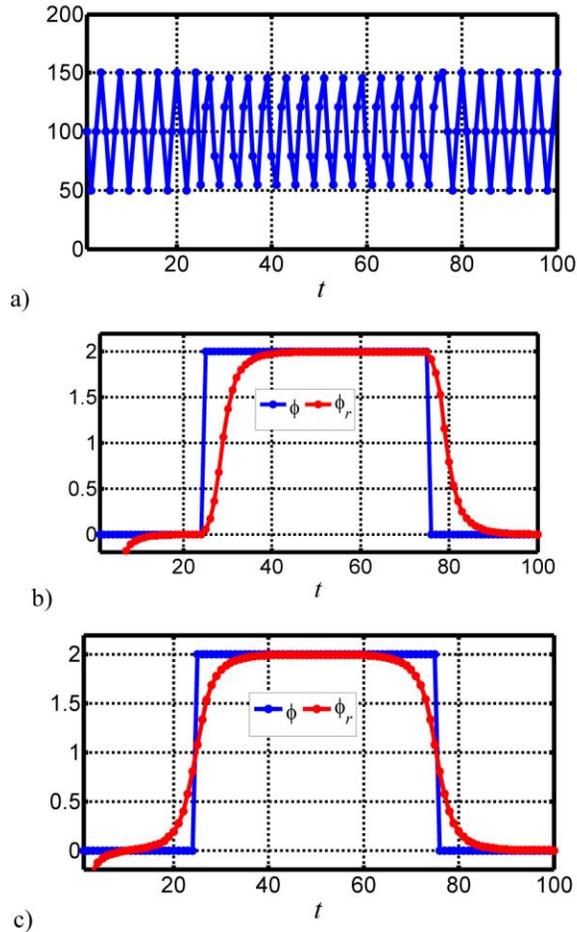$$H_T(t) = H(\omega)H^*(\omega) = |A(\omega)|^2. \tag{22}$$

Fig. 3. Recursive filter demodulation results with a discontinuous rectangular phase. a) temporal interferogram, b) demodulated phase in the forward-time pass, c) demodulated phase after the bidirectional filtering.

The resulting PSA has zero-phase as desired, and all the flexibility of a recursive filter. In Fig. 3c we show the demodulation results for the bidirectional filtering version of the recursive filter represented in Fig. 1. In Figs. 2 and 3 we have used $\eta = 0.75$. For smaller values, typically $\eta \leq 0.2$ we have practically a linear PSA with some noise rejection thanks to the recursive averaging. On the other side, for $\eta \geq 0.9$ we have a recursive filter with a highly non-linear phase. To test this behaviour, the MATLAB code to run all the examples presented in this paper can be downloaded from [7]

4 Conclusions

In this work, we have demonstrated the importance of the filter phase for the correct interpretation of the output of a general PSA. We have shown how the group delay determines the sample at which we are actually measuring the phase. We have analyzed under this new perspective a classical PSA and a more sophisticated recursive filter. Finally, in the case of a non-constant group delay we have shown how the bidirectional filtering can be used to compensate the filter phase effects.