Phase-shifting interferometry corrupted by white and non-white additive noise

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Abstract: The standard tool to estimate the phase of a sequence of phase-shifted interferograms is the Phase Shifting Algorithm (PSA). The performance of PSAs to a sequence of interferograms corrupted by *non-white* additive noise has *not* been reported before. In this paper we use the Frequency Transfer Function (FTF) of a PSA to generalize previous white additive noise analysis to non-white additive noisy interferograms. That is, we find the ensemble average and the variance of the estimated phase in a general PSA when interferograms corrupted by non-white additive noise are available. Moreover, for the special case of additive white-noise, and using the Parseval's theorem, we show (for the first time in the PSA literature) a useful relationship of the PSA's noise robustness; in terms of its FTF spectrum, and in terms of its coefficients. In other words, we find the PSA's estimated phase variance, in the spectral space as well as in the PSA's coefficients space.

OCIS codes: (120.3180) Interferometry; (120.2650) Fringe analysis.

References and links

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1. Introduction

The use of Phase-Shifting Algorithms (PSAs) is the most common way to estimate the modulating phase in Phase Shifting Interferometry (PSI). Bruning et al. [1] published the first all digital phase demodulation method in interferometry. Since then, dozens of PSAs have been published and an encyclopedic book containing most of them is available [2].

A PSA is an arctangent ratio of two linear combinations of temporal interferograms used to estimate their modulating phase. But the PSA alone do not give, in an explicit way, the demodulation characteristics of this process. People are not only interested in using a given PSA, but also wants to know how good this phase estimation is when non-ideal conditions in the interferometric data arise. For example, it is important to know, how sensitive a given PSA is to: Interferogram's detuning, harmonics, noisy data, detector's non-linearities, etc [2]. By simply looking at the arctangent ratio of a PSA one cannot know all these limitations. In particular, to obtain extremely low-noise wavefront's measurements, the sensitivity of PSAs to noise is paramount.

In 2009 Servin et al. [3,4] used the spectra of the PSAs to analyze their sensitivity to white additive noise. Non-spectral methods for noise analysis in PSI have been reported before [5–8] using: Taylor expansion of the fringe irradiance [5]; joint statistical distribution of the noise [6]; PSA's x-characteristic polynomial [7]; the derivative of the PSA's arctangent ratio [8]. All these works assume white-additive corrupting noise, and ends-up with formulas for the variance of the PSA's phase-noise using the algorithm's coefficients.

As in our previous works [3,4], here we use spectral (FTF) techniques. But now we extend previous analysis [3–8] to phase-shifting interferometry corrupted by *no-white* (pink) additive noise. Moreover using Parseval's theorem we unify in a single theory our spectral approach [3,4] with previous non-spectral methods [5–8] which give the estimated phase variance as a formula depending upon the coefficients of the phase-shifting algorithm.

2. Analysis of the noise rejection of Phase-Shifting Algorithms

For decades in Electrical Engineering, the Frequency Transfer Function (FTF) which is the Fourier transform ($F[\bullet]$) of the system's impulse response $h(t)$ ($H(\omega) = F[h(t)]$), has been used as an efficient analyzing tool for their linear systems [9]. Here we use the PSA's FTF to find the estimated phase variance due to a sequence of interferograms corrupted by *non-white* additive noise. Moreover, for the special case of white additive noise, the equivalence between the noise analysis based on the PSA's spectrum [3,4], and on its coefficients [5–8] is also shown, and finally two interesting examples of its application is given.

Probably the best way to fully understand this (fairly compressed) section is to review our previous work on the FTF applied to PSA [4]. Let us start by displaying the standard mathematical model for a set of N phase-shifted interferograms corrupted by additive noise with carrier ω_0 (radians/interferogram) as,

$$I(x, y, k) = a(x, y) + b(x, y)\cos[\phi(x, y) + \omega_0 k] + n(x, y, k); \quad k = 0, ..., N-1. \quad (1)$$

The function $I(x,y,k)$ represents the noisy interferograms at site (x,y,k); $a(x,y)$ is the illumination background, and $b(x,y)$ the fringe contrast. The interferogram's additive noise $n(x,y,k)$ is assumed to be *non-white*, ergodic, zero-mean, and Gaussian. The *non-flat* power spectral density of the non-white noise is $F[E\{n(t)n(t + \tau)\}] = \eta(\omega)$ [9]. Finally $\varphi(x,y)$ is the phase that we want to estimate.

A general N-steps linear PSA may be written as,

$$\tan[\hat{\varphi}(x, y)] = \frac{\sum_{k=0}^{N-1} c_k I(k)\sin(\omega_0 \, k)}{\sum_{k=0}^{N-1} c_k I(k)\cos(\omega_0 \, k)}. \quad (2)$$

The real numbers c_k are the PSA's coefficients. The estimated phase $\hat{\varphi}$ may differ a bit from φ which is the true modulating phase. This PSA is tuned at $\omega = \omega_0$ (radians/sample). The estimated phase $\hat{\phi}$ is the angle of the following complex number,

$$S = \sum_{k=0}^{N-1} c_k I(k) \cos(\omega_0 k) + i\left[\sum_{k=0}^{N-1} c_k I(k) \sin(\omega_0 k)\right] = \sum_{k=0}^{N-1} c_k I(k) e^{i\omega_0 k}. \qquad (3)$$

This complex number may be seen as the convolution product of a *N*-step digital-filter $h(t)$ (the PSI algorithm) and the data $I(t)$ evaluated at the middle-point $t_m = N$-1. At this middle-point both $I(t)$ and $h(t)$ fully overlap (see Fig. 1). The full convolution product $S(t) = I(t)*h(t)$ is,

$$S(t) = [I(t)*h(t)] = \left\{\sum_{k=0}^{N-1} I(k)\delta(t-k)\right\} * \left\{\sum_{k=0}^{N-1} c_k \delta(t-k) e^{i\omega_0 k}\right\}. \qquad (4)$$

The symbol * represents the convolution product over *t*.
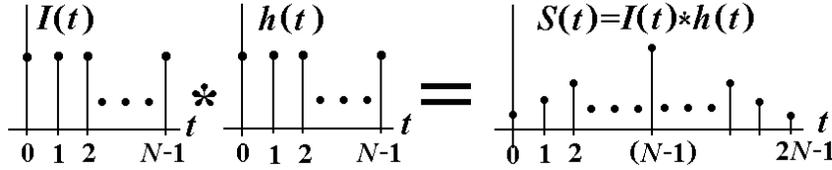

Fig. 1. Convolution product $S(t)$ of a sequence of *N* interferograms $I(t)$ and a digital filter (the PSA) $h(t)$ having also *N* samples. The middle-point occurs at $t_m = N$-1.

This convolution product may be Fourier transformed to,

$$I(\omega)H(\omega) = \left\{a\delta(\omega) + \frac{b}{2}\delta(\omega-\omega_0)e^{i\phi} + \frac{b}{2}\delta(\omega+\omega_0)e^{-i\phi} + \eta(\omega)\right\}\left\{\sum_{k=0}^{N-1} c_k e^{ik(\omega-\omega_0)}\right\}, \quad (5)$$

where $H(\omega) = \sum_k c_k \exp[ik(\omega-\omega_0)]$ is the FTF of the PSA. To find the estimated phase, we need to remove $a(x,y)$ and one of the two complex $b(x,y)/2$ terms. Arbitrarily choosing to keep the term at $+\omega_0$, $H(\omega)$ must have at least two zeroes: one at $H(0) = 0$, and the other at $H(-\omega_0) = 0$. After convolution we have,

$$S = [I(t)*h(t)]\Big|_{t_m=N-1} = \frac{b}{2}H(\omega_0)e^{i[\phi+\omega_0 t_m]} + n_H(t_m). \qquad (6)$$

This complex number contains an irrelevant piston $\omega_0 t_m$ which may be set to zero. Note that the amplitude of our input signal $(b/2)\exp(i\varphi)$ is now multiplied by the PSA filter's response $H(\omega_0)$ at $+\omega_0$. The PSA's response $H(\omega_0)$ at ω_0 is a very important piece of information that has not been taken explicitly into account before [5–8]. Also note that $n_H(t_m)$ is the output noise *after filtering*, which is now *complex and non-white* with variance [9],

$$E\{n_H^2(t_m)\} = E\{S^2\} = \sigma_S^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\eta(\omega)|^2 |H(\omega)|^2 \, d\omega. \qquad (7)$$

The variance of the output signal σ_S^2 is equal to the variance of the noise *after being filtered* by the PSA. The ensemble average of the output $E\{S\}$ and its variance $E\{S^2\}$ (Eq. (6)) are,

$$E\{S\} = \frac{b}{2}H(\omega_0)e^{i\varphi}, \qquad E\{S^2\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\eta(\omega)|^2 |H(\omega)|^2 \, d\omega. \qquad (8)$$

Note that, the expected angle of $E\{S\}$ is the searched noiseless phase, *i.e. angle*$[E\{S\}] = \phi$. A phasorial representation of Eq. (6), is shown in Fig. 2. From the triangle having as "base" σ_S and as "height" $H(\omega_0)b/2$, the standard deviation of the estimated phase is,

$$\sqrt{E\{\hat{\phi}^2\}} = \sigma_{\hat{\phi}} = 2\tan^{-1}\left(\frac{\sigma_S/2}{(b/2)H(\omega_0)}\right) \approx \frac{\sigma_S}{H(\omega_0)(b/2)}, \qquad (9)$$

where the approximation for small additive noise, $\tan(x){\approx}x$ was made. On the other hand, combining Eq. (8) and Eq, (9) we find its variance as,
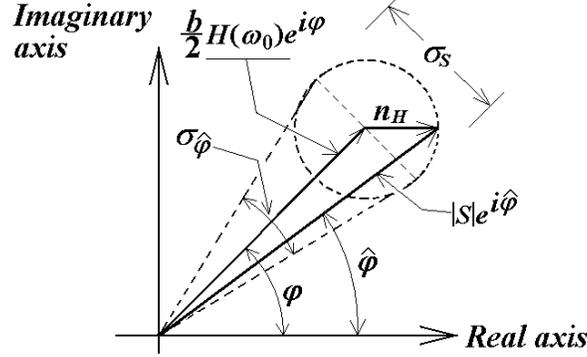

Fig. 2. Phasor representation of the output signal S and its corrupting noise n_H. The noiseless signal $(b/2)H(\omega_0)\exp[i\varphi]$ is inclined an angle φ. The output-noise n_H has a variance of $\sigma_S^2 = E\{n_H^2\}$ and its random angle is uniformly distributed within $[0,2\pi]$, so its phasor may point anywhere within the circle shown. The noisy estimated phase is φ-hat.

$$E\{\hat{\phi}\} = \phi, \text{ and } \quad E\{\hat{\phi}^2\} = \frac{1}{(b/2)^2}\frac{1}{|H(\omega_0)|^2}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}|\eta(\omega)|^2 |H(\omega)|^2 d\omega\right]. \quad (10)$$

These two formulas (Eq. (10)) are presented for the first time in the PSA's literature for *non-white* (pink) noise having a *non-flat* spectrum $\eta(\omega)$; *and this is our first main result*.

Now let us discuss the important case of *white-noise*, having a flat spectral density $\eta(\omega) = \eta/2$ (Watts/Hertz). Additionally, using Parseval's theorem [9], and the response of $h(t)$ at ω_0, or $H(\omega_0)$ [4], the variance of the estimated phase in Eq. (10) may be re-written as,

$$\sigma_{\hat{\phi}}^2 = \frac{\sigma_n^2/(b/2)^2}{|H(\omega_0)|^2}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}|H(\omega)|^2 d\omega\right] = \frac{\sigma_n^2/(b/2)^2}{\left|\sum_{k=0}^{N-1}c_k\right|^2}\left[\sum_{k=0}^{N-1}|c_k|^2\right]. \qquad (11)$$

Equation (11) shows (for the first time in optical interferometry literature) the two sides of the coin, namely; the PSA's phase variance based on its algorithm's spectrum $H(\omega)$ [3,4], and on its coefficients c_k [5–8] for the white-noise case. The noise-to-signal input's ratio $4\sigma_n^2/b^2$ depends only on the interferogram' data; the rest depends, on the spectrum, or on the coefficients of the PSA. The compact and useful formula using the PSA's coefficients in Eq. (11) has not been stated in this handy form before [5–8]. Equation (11) permits to easily compare among several PSAs and choose the one with the best signal to noise power-ratio.

As far as we know, Eq. (11) showing this equivalence of the PSA's phase-variance, due to white additive noise, in the spectral and in the coefficients PSA's spaces is new, *and it is our second main result*.

3. Blindness of Phase Shifting Algorithms to its filter's output amplitude

Looking at the PSA (Eq. (2)), we realize that the magnitude of the output filtered signal $|S| = (b/2)H(\omega_0)$, is canceled-out by the PSA's ratio. In other words, the PSA' ratio is blind to the quadrature filter's output. Therefore, one may (erroneously) assume that the filter's output is (among other possibilities),

$$S = be^{i\varphi} + n_H . \qquad (12)$$

Assuming that Eq. (13) were correct (which is not) the variance for the estimated phase is,

$$E\{\hat{\phi}^2\} = \frac{E\{n^2\}}{b^2} \frac{1}{2\pi} \int_{-\pi}^{\pi} |H(\omega)|^2 \, d\omega = \left(\frac{o_n^2}{b^2}\right) \sum_{k=0}^{N-1} |c_k|^2 . \qquad (13)$$

From this equation, one may (erroneously) conclude that a high order PSA would give noisier phase estimation than a lower order one.

4. Two examples

4.1 The Least Squares Phase Shifting Algorithm

The (diagonal) Least-Squares (*LS*) *N*-steps PSA is [2,10],

$$\tan[\hat{\phi}_{LS}(x,y)] = \left[\sum_{k=0}^{N-1} I(k)\sin(\omega_0 \, k)\right] \Big/ \left[\sum_{k=0}^{N-1} I(k)\cos(\omega_0 \, k)\right], \quad \omega_0 = \frac{2\pi}{N}. \qquad (14)$$

All the coefficients of the *LS*-PSA are one ($c_k = 1$). The impulse response of this PSA is,

$$h_{LS}(t) = \sum_{k=0}^{N-1} e^{i\omega_0 k} \delta(t-k), \quad \omega_0 = \frac{2\pi}{N}, \qquad (15)$$

and its FTF is given by,

$$H_{LS}(\omega) = F[h_{LS}(t)] = \sum_{k=0}^{N-1} e^{i k(\omega - \omega_0)}, \quad \omega_0 = \frac{2\pi}{N}. \qquad (16)$$

Figure 3 shows graphically the coefficients c_k in $h_{LS}(t)$, and the spectrum $|H_{LS}(\omega)| = |F[h_{LS}(t)]|$ of this PSA. Figure 3 also shows that, the 6-steps *LS*-PSA is insensitive to at least the 2nd, 3rd, and 4th real-valued harmonics of the interferogram signal.
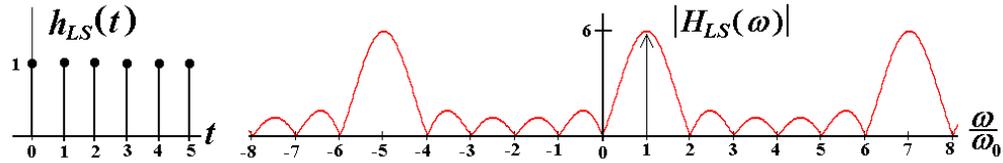

Fig. 3. Least-Squares PSA's impulse response $h_{LS}(t)$, and its spectral response $H_{LS}(\omega)$. The frequency axis is normalized to the carrier $\omega_0 = 2\pi/6$.

Equation (16) is a *N*-steps frequency-shifted mean-sampler. Equation (12) gives in this *LS* case,

$$E\{\hat{\phi}_{LS}^2\} = \frac{\sigma_n^2}{(b/2)^2} \left[\sum_{k=0}^{N-1} |c_k|^2\right] \Big/ \left|\sum_{k=0}^{N-1} c_k\right|^2 = \frac{\sigma_n^2}{(b/2)^2} \frac{N}{N^2} = \frac{1}{N} \frac{\sigma_n^2}{(b/2)^2} . \qquad (17)$$

This states the well known fact that, the sampled-mean variance of *N* observations corrupted by white-additive noise is reduced by a factor of $1/N$ [9]. For the case shown in Fig. 3, this factor is 1/6. This sampled-mean (the *LS*-PSA), is the Maximum Likelihood (ML) estimator of a parameter corrupted by white-additive Gaussian noise [9].

4.2 The Schwider-Hariharan 5-steps PSA

Here we apply Eq. (12) to the Schwider-Hariharan (*SH*) 5-steps PSA [11,12].
The *SH* 5-steps PSA is,

$$\tan[\hat{\varphi}_{SH}(x,y)] = \frac{2[I(t-1)-I(t-3)]}{I(t)-2I(t-2)+I(t-4)}. \tag{18}$$

Its Impulse response is $h_{SH}(t) = \delta(t) - 2\delta(t-2) + \delta(t-4) + 2i[\delta(t-1) - \delta(t-3)]$, or

$$h_{SH}(t) = \delta(t) + 2e^{i\omega_0}\delta(t-1) + 2e^{2i\omega_0}\delta(t-2) + 2e^{3i\omega_0}\delta(t-3) + e^{4i\omega_0}\delta(t-4). \tag{19}$$

So the coefficients of the *SH*-PSA are $c_0 = 1$, $c_1 = 2$, $c_2 = 2$, $c_3 = 2$, and $c_4 = 1$. The variance of the estimated phase is therefore,

$$E\{\hat{\varphi}_{SH}^2\} = \frac{\sigma_n^2}{(b/2)^2}\left[\sum_{k=0}^{4}|c_k|^2\right]\bigg/\left|\sum_{k=0}^{4}c_k\right|^2 = \frac{14}{64}\frac{\sigma_n^2}{(b/2)^2} = 0.91\left(\frac{1}{5}\frac{\sigma_n^2}{(b/2)^2}\right). \tag{20}$$

This is a remarkable 91% close to a 5-step Least-Squares PSA!. That is, the noise-rejection of a 5-step *SH*-PSA is just 9% below from the noise-rejection of a 5-step *LS* one.

The coefficients c_k in $h_{SH}(t)$, and the magnitude $|H_{SH}(\omega)|$ are shown in Fig. 4.
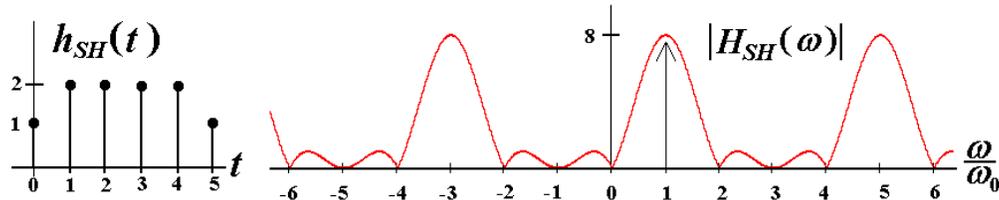

Fig. 4. Impulse response $h_{SH}(t)$ and its spectral plot $|H_{SH}(\omega)|$ of the 5-steps *SH*-PSA.

Note that $H_{SH}(\omega)$ touches tangentially the ω axis at $-\omega_0$. This second-order detuning robustness is well known and widely used in PSI [2,11,12]. Note also from Eq. (20) that, the *SH*-PSA penalizes 9% of noise robustness to gain second-order detuning robustness at $-\omega_0$; *i.e.* the *SH* algorithm has two spectral zeroes at $-\omega_0$. On the other hand (see Fig. 3), a *N*-steps *LS*-PSA has only single-order spectral zeroes.

5. Conclusions

We presented in Eq. (10) the sensitivity to *non-white* noise of a general PSA using the Frequency Transfer Function (FTF) or $H(\omega)$. Using the pink-noise spectrum of the interferometric data and the PSA's FTF the first, and the second-order ensemble averages of the estimated phase $\hat{\phi}$ were found (Eq. (10)). Additionally, for *white-noise*, and using Parseval's theorem, we have shown the equivalence (in Eq. (11)) between the spectral-based noise-variance [3,4] and the noise-variance based on the coefficients c_k of the PSA [5–8].

The noise sensitivity for a *N*-steps Least-Squares PSAs, and the noise analysis for the 5-steps Schwider-Hariharan PSA were presented as examples.

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