# Chiral perturbation theory and the $f_2(1270)$ resonance

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Within chiral perturbation theory, we study elastic pion scattering in the $I=0,J=2$, channel, whose main features are the $f(1270)$ resonance and the vanishing of the lowest order. By means of a chiral model that includes an explicit resonance coupled to pions, we describe the data and calculate the resonance contribution to the $O(p^4)$ and $O(p^6)$ chiral parameters. We also generalize the inverse amplitude method to higher orders, which allows us to study channels with vanishing lowest order. In particular, we apply it to the $I=0,J=2$ case, finding a good description of the $f_2(1270)$ resonance, as a pole in the second Riemann sheet.

Chiral perturbation theory (ChPT) [1,2] is a powerful tool to describe low energy hadronic interactions. ChPT is based on the identification of pions, kaons and the eta as the Goldstone bosons associated with the spontaneous chiral symmetry breaking of QCD (pseudo Goldstone bosons indeed, since the three lightest quarks have small mass). The ChPT Lagrangian is then built as a derivative and mass expansion over the symmetry breaking scale $4\pi F \simeq 1.2$ GeV, compatible with the symmetry constraints. The calculations are renormalizable order by order and depend on just a finite set of parameters at each order, which can be determined from a few experiments and then used to obtain predictions for other processes. These parameters contain information on heavier states not included explicitly in the Lagrangian [1,3,4].

In this work, and within the context of $SU(2)$ ChPT (the $u$ and $d$ quark sector), we first study the contribution from the $I=0,J=2$ lightest resonance to the chiral parameters, using a resonant model that describes well the data on that channel. Next, we study how, by means of unitarization methods, it is possible to generate a resonance from the ChPT expansion in the $I=0,J=2$ channel. Although these techniques, and particularly the inverse amplitude method, have been extensively applied in the literature, obtaining remarkable descriptions of meson-meson scattering, they had never been applied to this channel. The reasons were that the lowest chiral order vanishes, so that the formalism has to be generalized, and that the first contribution to the imaginary part of the amplitude appears at three loops, where there are no calculations available. We conclude by showing and discussing our numerical results confronted with data.

The most remarkable feature of the $D$ wave isoscalar pion scattering channel is the $f_2(1270)$. Therefore a possible phenomenological approach to this channel is a model where a $J=2,I=0$ resonance is introduced explicitly in a chiral invariant way. In a first approximation we will neglect the kaons since their branching ratio from the $f_2(1270)$ is about 5%. Thus we consider the $SU(2)$ chiral symmetry framework, where the pions are grouped in $U(x)=\exp(i\tau^a\pi^a(x)/F)$, $\tau^a$ being the Pauli matrices. The $f_2$ is described by a symmetric real tensor field $f_{\mu\nu}$ with perturbative mass $M$. Its chiral global invariant interaction to pions at lowest order in derivatives is [4] $\mathcal{L}_{int}=gf_{\mu\nu}\operatorname{tr}\partial^\mu U\partial^\nu U^\dagger$, where $g$ is a coupling with dimension of energy. Note that the resonance only couples to an even number of pions, and

in particular the interaction with two pions is $\mathcal{L}_{int}=(2g/F^2)f_{\mu\nu}\partial^\mu\pi^a\partial^\nu\pi^a$. In order to obtain the Feynman rules, the resonance in the initial or final states is described by a plane wave $\Phi_{\mu\nu}(\vec{k},\lambda)$, where $\vec{k}$ is the momentum and $\lambda$ is the helicity, satisfying $(k^2-M^2)\Phi_{\mu\nu}(\vec{k},\lambda)=0$. The sum over polarizations is given by [5]

$$\sum_\lambda \Phi_{\mu\nu}(\vec{k},\lambda)\Phi_{\rho\sigma}(\vec{k},\lambda)^\dagger=X_{\mu\nu,\rho\sigma},$$

with $X_{\mu\nu,\rho\sigma}=\frac{1}{2}(X_{\mu\rho}X_{\nu\sigma}+X_{\mu\sigma}X_{\nu\rho})-\frac{1}{3}(X_{\mu\nu}X_{\rho\sigma})$ and $X_{\mu\nu}=g_{\mu\nu}-k_\mu k_\nu/M^2$. Thus, the $f_2\to\pi\pi$ decay amplitude is $T(f\to\pi\pi)=-(4g/F^2)k_1^\mu k_2^\nu\Phi_{\mu\nu}$, and its partial width is $\Gamma(f\to\pi\pi)=(g^2/80\pi M^2 F^4)(M^2-4M_\pi^2)^{5/2}$. Using [6] $M_{f_2}=1270$ MeV$\simeq M$, $\Gamma(f\to\pi\pi)=158$ MeV, $F=92.4$ MeV and $M_\pi=139.57$ MeV, we find $g\simeq 40$ MeV. This is just a naive estimate from a first order calculation. It is also straightforward to obtain the pion scattering amplitude from this model, which, due to crossing and chiral symmetry, has the following form:

$$T_{abcd}(s,t,u)=A(s,t,u)\delta_{ab}\delta_{cd}+A(t,s,u)\delta_{ac}\delta_{bd}$$
$$+A(u,t,s)\delta_{ad}\delta_{cb},$$

where $a$, $b$, $c$ and $d$ are the isospins of the pions and $s$, $t$ and $u$ are the Mandelstam variables. In our model the three terms correspond to the $f_2$ exchange in the $s$, $t$ and $u$ channels, respectively, with its propagator given by $D_{\mu\nu,\rho\sigma}(k)=iX_{\mu\nu,\rho\sigma}/(k^2-M^2)$, so that

$$A_f(s,t,u)=\frac{g^2}{F^4(M^2-s)}\{2[(2M_\pi^2-t)^2+(2M_\pi^2-u)^2]$$
$$-\frac{4}{3}(s-2M_\pi^2)^2-(2s^2/3M^2)(s+4M_\pi^2)$$
$$+\frac{2}{3}(s^2/M^2)^2\}. \tag{1}$$

In order to compare with ChPT we calculate the lowest order in the momenta and the pion mass, and we find

$$A_f(s,t,u) = \frac{g^2}{F^4 M^2} \{ 2[(2M_\pi^2 - t)^2 + (2M_\pi^2 - u)^2]$$

$$- \tfrac{4}{3}(s - 2M_\pi^2)^2 \} + O(p^6).$$

As expected the $O(p^2)$ vanishes. By comparing with the ChPT scattering amplitude [1], we obtain the $f_2$ contribution to the chiral parameters $\bar{l}_1$ and $\bar{l}_2$: $\Delta \bar{l}_2 = -\tfrac{3}{2} \Delta \bar{l}_1 = 96\pi^2 g^2 / M_f^2$, in agreement with the calculation in [4] performed with a different notation and in the chiral limit. Thus $\Delta \bar{l}_1 \simeq -0.65$ and $\Delta \bar{l}_2 \simeq 0.95$. There is no contribution to $\bar{l}_3$ and $\bar{l}_4$. Let us now compare with the dominant $\rho(770)$ contribution [1] $\Delta \bar{l}_1 = -2\Delta \bar{l}_2 = -96\pi^2 f^2 / M_\rho^2$, so that $\Delta \bar{l}_1 = -7.6$ and $\Delta \bar{l}_2 = 3.8$, since the $\rho\pi\pi$ chiral invariant coupling is $f \simeq 69$ MeV [1]. Nevertheless, the $f_2$ contributions are comparable to those of scalar resonances [3,4]. At $O(p^6)$, $\pi\pi$ scattering can be parametrized with six constants $b_i$ [8]. The first four are dominated by the $\bar{l}_i$, and only $b_5$ and $b_6$ are genuinely $O(p^6)$, whose $f_2$ contribution is

$$\Delta b_5 = -\Delta b_6 = -F^2 g^2 / M_f^4 \simeq -6.8 \times 10^{-6},$$

whereas that of the $\rho$ is $\Delta b_5 = \tfrac{1}{3} \Delta b_6 = F^2 f^2 / 4M_\rho^4 \simeq 3 \times 10^{-5}$. When obtaining the $f_2$ contribution to the chiral parameters from our the model, we may wonder how well it describes the $J=2, I=0$ data. To that end we have to evaluate the partial wave

$$a_{02}(s) = \frac{1}{64\pi} \int_{-1}^{1} d(\cos\theta) T_{I=0}(s,t,u) P_{J=2}(\cos\theta),$$

where $T_0(s,t,u) = 3A(s,t,u) + A(t,s,u) + A(u,t,s)$. However the amplitude in Eq. (1) is not appropriate since it is a perturbative amplitude where the resonance appears with zero width (indeed, it is singular at $s = M^2$). Frequently, this problem is solved introducing by hand the width in the resonant propagator. Such an amplitude behaves as a Breit-Wigner around the resonance position, but this method does not provide the proper analytic structure. In addition, it usually breaks chiral symmetry and in particular spoils the Weinberg low energy theorems. Therefore we will consider here a different method with better properties. Instead of the tree level partial wave $a_{02}$ obtained from Eq. (1) we will use

$$\tilde{a}_{02}(s) = \frac{a_{02}(s)}{1 - J(s)a_{02}(s)}, \qquad (2)$$

where $J(s)$ is the Mandelstam two body function

$$J(s) = \frac{\sigma(s)}{\pi} \log \frac{\sigma(s) - 1}{\sigma(s) + 1}$$

where

$$\sigma(s) = \sqrt{1 - 4M_\pi^2 / s}.$$

Note that Eq. (2) is nothing but the resummation of a geometric series generated by the tree level amplitude and the
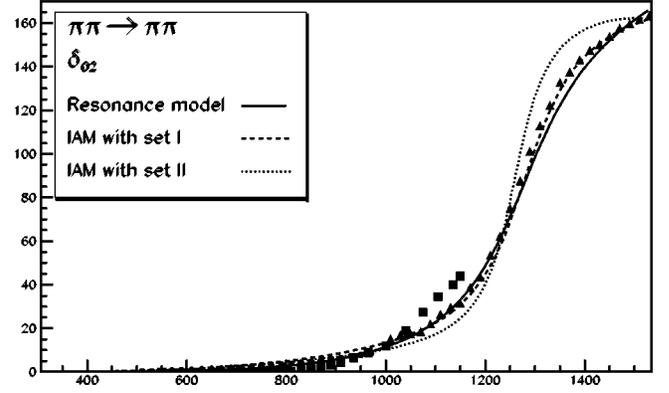


FIG. 1. Experimental data ( [12] squares, [13] circles, [14] triangles) on the $I=0, J=2$ $\pi\pi$ scattering phase shift and the results of our unitarized chiral resonance model (continuous line) of Eq. (8) with the parameters in set I (dashed line), and with those in set II (dotted line).

two pion $s$-channel loop. Thus it is not perturbative in the $g$ coupling. Its main properties are the following: it has a right cut starting at $s = 4M_\pi^2$ and there is a pole in the second Riemann sheet associated to the resonance, which provides the usual Breit-Wigner shape. Similar approaches with other resonances have been successfully applied in [7]. As usual, the physical mass and the total width of the resonance can be obtained approximately from the position of the pole in the complex plane. Finally $\tilde{a}_{02}$ satisfies elastic unitarity, i.e.

$$\text{Im}\, \tilde{a}_{IJ} = \sigma |\tilde{a}_{IJ}|^2 \Rightarrow \text{Im}\, \tilde{a}_{IJ}^{-1} = -\sigma. \qquad (3)$$

In Fig. 1, we show a fit to the data of the $\tilde{a}_{02}$ phase, with parameters $M = 1157$ MeV and $g = 44$ MeV, in good agreement with the estimations presented above. Up to here we have obtained information on the ChPT parameters from an explicit resonance. Let us now see how we can also generate a resonance from ChPT.

Within ChPT the amplitude is an expansion

$$a_{ChPT}(s) = a_2(s) + a_4(s) + a_6(s) + a_8(s) + \cdots \qquad (4)$$

where $a_k(s)$ stands for the $O(p^k)$ contribution, or more precisely, the $O(1/F^k)$ term. In the elastic region, $a_{ChPT}$ only satisfies unitarity, Eq. (3), in a perturbative sense:

$$\text{Im}\, a_2 = 0, \quad \text{Im}\, a_6 = a_2\sigma a_4^* + a_4\sigma a_2, \qquad (5)$$

$$\text{Im}\, a_4 = a_2\sigma a_2, \quad \text{Im}\, a_8 = a_2\sigma a_6^* + a_6\sigma a_2 + a_4\sigma a_4^*, \quad \ldots.$$

In that region, all the contributions to $\text{Im}\, a_{ChPT}$ come from the two pion loop function $J(s)$. Indeed, by analyzing the different Feynman diagrams contributing to Eq. (4) it is possible to write $a_4 = a_2 J a_2 + a_{4L}$, where $a_{4L}$ contains the polynomial and the left cut contribution to $a_4$ but is real in the elastic region. This is due to the fact that, in that region, we can only get the imaginary part from one loop [one $J(s)$] with two vertices of $O(p^2)$ (the two $a_2$ factors), and the rest has to be real. Similarly

$$a_6 = a_2 J a_2 J a_2 + 2 a_{4L} J a_2 + a_{6L},$$

$$a_8 = a_2 J a_2 J a_2 J a_2 + 3 a_{4L} J a_2 J a_2 + 2 a_{6L} J a_2$$
$$+ a_{4L} J a_{4L} + a_{8L}, \tag{6}$$

where the $a_{kL}$, which include the corresponding left cut and polynomial contributions, are renormalization scale independent and real on the right cut. Note that $\operatorname{Im} J(s) = \sigma(s)$ for $s > 4 M_\pi^2$ so that Eqs. (5) follow immediately.

A case of special interest for this work occurs when $a_2 = 0$. Then, in the elastic region, $a_{4L} = a_4$ and $\operatorname{Im} a_4 = 0$. The same happens with $a_6$ and the first right cut contribution comes from $a_8$, which satisfies

$$a_8 = a_4 J a_4 + a_{8L}, \quad \operatorname{Im} a_8 = a_4 \sigma a_4. \tag{7}$$

Actually, this occurs for $I=0, J=2$ where the chiral expansion starts at $O(p^4)$, i.e., $a = a_4 + a_6 + a_8 + \cdots$.

In order to improve the unitary behavior of the chiral expansion one of the most widely used techniques is the inverse amplitude method (IAM). Its name is due to the fact that $a_{IJ}^{-1}$ has the same analytic structure as $a_{IJ}$ (apart from eventual new poles coming from the amplitude zeros). In particular, it should also have a right cut on the elastic region, where $\operatorname{Im} a^{-1} = -\sigma$ due to Eq. (3).

The IAM can be derived with the help of an auxiliary function $G \equiv a_2^2 a^{-1}$ which satisfies a dispersion relation with exactly the same right cut contribution as $a_2 - a_4$, since $a_2$ is real and $\operatorname{Im} a^{-1} = -\sigma$. Indeed, if the left cut contribution and the polynomial part of $G$ are evaluated perturbatively we arrive to $G \simeq a_2 - a_4$. Hence we find a unitarized amplitude $\tilde{a} = a_2^2 / (a_2 - a_4)$. The details of this derivation can be found in [9]. Let us simply recall that this simple formula has been applied to the $\pi\pi$ and $\pi K$ elastic scattering [9], and it generates the $\sigma$, $\rho$ and $K^*$ resonances from the corresponding $O(p^4)$ amplitudes. A similar equation in matrix form, although without a justification from dispersion theory, has also been applied within a coupled channel formalism, describing successfully all the meson-meson interactions below 1200 MeV and generating seven light resonances [10]. In addition, this method has also been generalized to $O(p^6)$ calculations for the lowest spin channels where $a_2 \neq 0$ [9,11].

However, the IAM has not been derived or applied when $a_2 = 0$. In what follows, we will present a generalization of the IAM equation and its derivation for the case when $a_2 = 0$. In particular we will apply the method to the $I=0, J=2$ channel. In this case it is possible to write a dispersion relation for the chiral expansion up to $O(p^8)$ (with five subtractions to ensure convergence). As discussed before, the first non-vanishing contribution to $\operatorname{Im} a$ on the right cut comes from $a_8$, Eq. (7). Thus the right cut contribution to this dispersion relation is

$$a_{8R}(s) = \frac{(s-s_0)^5}{\pi} \int_{4M_\pi^2}^\infty \frac{a_4(s')\sigma(s')a_4(s')ds'}{(s'-s_0)^5(s'-s-i\epsilon)},$$

where we have used the second relation in Eq. (7) and $s_0$ is a subtraction point. This strongly suggests the use of the

auxiliary function $G \equiv a_4^2 a^{-1}$, since $\operatorname{Im} G = -\operatorname{Im} a_8$ on the right cut. Writing another dispersion relation (with five subtractions) for $G$, its right cut contribution will be precisely $-a_{8R}$. Neglecting the possible pole contribution and evaluating the left cut and the polynomial contributions perturbatively it is not hard to find

$$G \simeq a_4 - a_6 - a_8 + a_6^2/a_4,$$

$$\Rightarrow \tilde{a} = \frac{a_4}{1 - a_6/a_4 - a_8/a_4 + a_6^2/a_4^2}. \tag{8}$$

This is the generalized expression for the IAM, which satisfies exactly elastic unitary and has the correct low energy ChPT expansion, i.e., Eq. (4) with $a_2 = 0$. Strictly speaking, this method is only justified in the elastic region, but whenever the inelasticity of a given channel is small we expect it to be a good approximation.

Alternatively, the unitarized amplitude above could be derived by considering the [2,2] Padé approximant of the chiral expansion in $1/F^2$, namely,

$$a^{[2,2]} = \frac{a_2 a_4^2 - a_2^2 a_6 + a_4^3 - 2 a_2 a_4 a_6 + a_2^2 a_8}{a_4^2 - a_2 a_6 + a_2 a_8 - a_4 a_6 + a_6^2 - a_4 a_8},$$

and then setting $a_2 = 0$.

In what follows we are going to confront Eq. (8) with the $I=0, J=2$ scattering data. However, at present there is no calculation of $a_8$ available, and probably it will remain unavailable for a long time. Nevertheless, from Eq. (6) we see that only $a_{8L}$ is unknown. Since we will be interested on the resonant region, $\sqrt{s} \simeq 1200$ MeV, we can expect that the left cut logarithmic contribution will be small. Concerning the $O(p^8)$ polynomial, we also expect its dominant term to be $a_{8L} \sim c\, s^4$, since any other polynomial term of that order will be suppressed by powers of $M_\pi^2/s$. Since ChPT is an expansion in powers of momenta over $4\pi F$ we get a crude estimate of $c \simeq (1/4\pi F)^8 \simeq 3 \times 10^{-25}$ MeV$^{-8}$.

Thus, in Fig. 1 (dashed line) we compare the $I=0, J=2$ phase shift data with the results of applying Eq. (8) to the ChPT amplitude with the parameters listed as set I in Table I. It is possible to get a remarkable description of the experiment, but it is not so good when comparing with the values given in the literature, listed in column two and three of the same table. Nevertheless, our parameters have the correct order of magnitude. Let us also remark that there are just six free parameters up to $O(p^6)$, with rather large uncertainties. However, such an accurate description of the data is somewhat unrealistic. The reason is that in our approach we are only taking into account the $\pi\pi$ state, whereas the actual $f_2$ resonance has 8% and 7% branching ratios to four pions and two kaons, respectively. Just with two pions in the intermediate state, we should expect to get a 15% narrower resonance. For illustrative purposes, we show in Fig. 1, as a dotted line, the result of applying Eq. (8) with set II in Table I, which is obtained if we allow for a narrower resonance. These parameters are much closer to those given in the lit-

TABLE I. Estimates of the $O(p^6)$ parameters are given in column two. In the third column we give values that, with the IAM up to $O(p^6)$, fit very well $\pi\pi$ scattering in the $(I,J) = (0,0)$, $(1,1)$ and $(2,0)$ channels. Set I with Eq. (8) describes remarkably well the $I = 0, J = 2$ data, but only agrees in the order of magnitude with previous values. Set II is closer to [11], but yields a narrower resonance (see Fig. 1), due to other coupled states not present in our approach.

| | $O(p^6)$ ChPT [8] | $O(p^6)$ IAM [11] | Set I | Set II |
|---|---|---|---|---|
| $10^2 b_1$ | $-9.2 \ldots -8.6$ | $-7.7 \pm 1.3$ | $-3$ | $-6.6$ |
| $10^2 b_2$ | $8.0 \ldots 8.9$ | $7.3 \pm 0.7$ | $4$ | $6.4$ |
| $10^3 b_3$ | $-4.3 \ldots -2.6$ | $-1.8 \pm 1.6$ | $3.8$ | $-3.6$ |
| $10^3 b_4$ | $4.8 \ldots 7.1$ | $4.8 \pm 0.1$ | $7$ | $6.7$ |
| $10^4 b_5$ | $-0.4 \ldots 2.3$ | $1.3 \pm 0.2$ | $8.7$ | $4.0$ |
| $10^4 b_6$ | $0.7 \ldots 1.5$ | $0.2 \pm 0.2$ | $1.6$ | $1.5$ |

erature, consistently with our expectations, since imposing the actual width of the $f_2$ with just two pions would force the parameters to get unnatural values. The introduction of the $K\bar{K}$ or the $4\pi$ coupled states is beyond the scope of this work. Nevertheless, the $O(p^4)$ generalization of the IAM to coupled two-particle states has been given with remarkable results in [10]. With the inclusion of many particle states, like four pions, the generalization of the unitary relation, like 3, becomes an integral equation, and we would not find simple algebraic solutions like the IAM, but would probably require a certain amount of numerical integration.

The final consistency check is the value of the $c$ parameter. As a matter of fact we have found many sets of parameters yielding results either like the dashed or the dotted lines in Fig. 1. For all of them, $c \simeq 10^{-25}$ to $10^{-24}$ MeV$^{-8}$, in good agreement with our expectations. The consistency of the whole picture is more remarkable taking into account the experimental errors, not given in the original references, but that could be roughly estimated by comparing the difference on the data points in the overlapping region of different experiments (see Fig. 1 between 1000 and 1200 MeV).

In summary, in this work we have studied a chiral model with a $I=0, J=2$ resonance which describes the $\pi\pi$ scattering data on that channel. We have then calculated the resonance contributions to the chiral parameters that govern $\pi\pi$ scattering at one and two loops, finding that, as expected, they are subdominant with respect to those of vector mesons (that is vector meson dominance), but comparable with the contributions from scalar resonances. We have also given a generalization of the inverse amplitude method to higher orders, which, in particular, is applicable to channels with vanishing lowest order. When applied to the $I=0, J=2$ channel, the IAM is able to generate a resonant behavior from the chiral expansion, in agreement with the data, taking into account that we are only considering the two pion state. This is an illustration of the power of this unitarization method which still gives qualitative results even close to its applicability limits.

[1] S. Weinberg, Physica A **96**, 327 (1979); J. Gasser and H. Leutwyler, Ann. Phys. (N.Y.) **158**, 142 (1984); J. Gasser and H. Leutwyler, Nucl. Phys. **B250**, 465 (1985); **B250**, 517 (1985); **B250**, 539 (1985).

[2] H. Leutwyler, hep-ph/0008124; A. Dobado, A. Gómez-Nicola, A.L. Maroto, and J.R. Peláez, *Effective Lagrangians for the Standard Model*, Texts and Monographs in Physics (Springer-Verlag, Berlin, 1997); A. Pich, Rep. Prog. Phys. **58**, 563 (1995); U.G. Meißner, *ibid.* **56**, 903 (1993).

[3] G. Ecker *et al.*, Nucl. Phys. **B321**, 311 (1989).

[4] J.F. Donoghue *et al.*, Phys. Rev. D **39**, 1947 (1989).

[5] Y. V. Novozhilov, *Introduction to Elementary Particle Physics* (Pergamon, New York, 1975).

[6] Particle Data Group, D. Groom *et al.*, Eur. Phys. J. C **15**, 1 (2000).

[7] J.A. Oller, E. Oset, and J.E. Palomar, Phys. Rev. D **63**, 114009 (2001).

[8] J. Bijnens *et al.*, Phys. Lett. B **374**, 210 (1996); Nucl. Phys. **B508**, 263 (1997).

[9] T.N. Truong, Phys. Rev. Lett. **61**, 2526 (1988); **67**, 2260 (1991); A. Dobado, M.J. Herrero, and T.N. Truong, Phys. Lett. B **235**, 134 (1990); A. Dobado and J.R. Peláez, Phys. Rev. D **47**, 4883 (1993); **56**, 3057 (1997).

[10] J.A. Oller, E. Oset, and J.R. Peláez, Phys. Rev. Lett. **80**, 3452 (1998); Phys. Rev. D **59**, 074001 (1999); **60**, 099906(E) (1999); F. Guerrero and J.A. Oller, Nucl. Phys. **B537**, 459 (1999); **B602**, 641(E) (2001); A. Gomez Nicola and J. Peláez, Phys. Rev. D **65**, 054009 (2002).

[11] J. Nieves and E. Ruiz-Arriola, Phys. Rev. D **65**, 036002 (2002).

[12] S.D. Protopopescu *et al.*, Phys. Rev. D **7**, 1279 (1973).

[13] P. Estabrooks and A.D. Martin, Nucl. Phys. **B79**, 301 (1974).

[14] C.D. Froggat and J.L. Petersen, Nucl. Phys. **B129**, 89 (1977).