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ON SEXTIC CURVES WITH BIG MILNOR NUMBER

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Abstract. In this work we present an exhaustive description, up to projective isomorphism, of all irreducible sextic curves in $\mathbb{P}^2$ having a singular point of type $\mathbb{A}_n$, $n \geq 15$, only rational singularities and global Milnor number at least 18. Moreover, we have developed a method for an explicit construction of sextic curves with at least eight –possibly infinitely near– double points. This method allows one to express such sextic curves in terms of arrangements of curves with lower degrees and it provides a geometric picture of possible deformations. Because of the large number of cases, we have chosen to carry out only a few to give some insights into the general situation.

1. Introduction

A classification of sextic curves with only rational singularities has been given by Yang in [11], where all possibilities are listed. Nevertheless, for any given configuration of singular points, Yang does not provide further information about the number or the dimension of the families occurring with such combinatorics.

In this paper we define a general method to construct sextic curves with at least eight –possibly infinitely near– double points. This method is based on the construction of a pencil of cubics associated to these eight points. The sextic curve to be constructed will turn out to be symmetric with respect to the natural involution of the elliptic fibration defined by the pencil –obtained by taking the opposite in the group law of each fiber. After taking the quotient of this fibration by the involution, the existence and properties of the sextic curve can be derived from the existence of certain arrangements of plane projective curves involving lines, conics or cubics, and total degree at least four. It should be highlighted the essential role that the group structure of cubics plays in these ideas turning into approachable a problem of a

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very complicated computational nature. An application of this method appears in the work of S.Y. Orevkov and E.I. Shustin [8].

Our method allows us to explicitly (and completely) describe different families of sextics. Also, it allows us to have a geometric picture of the deformations of the sextics by means of the deformations of arrangements of curves with smaller degree, in terms of how they intersect. Along the way we find several cases of non-connected families of curves with a given configuration of singularities. Some of them provide Zariski pairs, which may be viewed as degenerations of the classical Zariski pair (sextics with six ordinary cusps) and the property of being Zariski pairs is detected either by the Alexander polynomial (as in [14], [6], [1], [4] or [3]) or by the fundamental group. The latter is the case for the new example of Zariski pair of irreducible rational curves found in this work, see [3] and [10]. In other cases, we do not know whether or not generic members are topologically equivalent. The main difficulty, as it was already mentioned in [2], is that such families are associated with Galois transformations in some number fields. In such a situation where conjugated curves in a number field come into play, none of the known invariants of curves has been shown to be effective.

Another asset of this method is that it allows one to compute the fundamental group of the sextic curves using fundamental groups of the arrangements of curves of lower degree involved in the aforementioned construction. Another trick to compute fundamental groups comes from symmetries: in some cases there are special members of the equisingular families possessing a richer symmetry which makes the computation of the fundamental group easier.

2. General method

Let $\mathcal{C} \subset \mathbb{P}^2$ be an irreducible sextic curve with at least eight double points (possibly infinitely near). We recall that a point Q is said to be infinitely near to another point P if Q appears as a result of a sequence of blow-ups in a neighborhood of P. By convention, we shall say that P is infinitely near to itself. Let us choose eight such double points with the following convention: if Q is infinitely near to P and Q is chosen, then P has to be chosen as well.

Let us list these eight points as

$$\mathcal{L} := \bigcup_{j=1}^{r} \{P_0^j, P_1^j \dots, P_{n_j}^j\}, \quad \sum_{j=1}^{r} (n_j + 1) = 8, \tag{1}$$

where for each $j = 1, \dots, r$, the ordered sequence

$$\mathcal{L}^j := \{P_0^j, P_1^j, \dots, P_{n_j}^j\}$$

of n_j+1 distinct points satisfies: if $\mathcal{T}^j$ denotes the multiplicity tree with minimal element P_0^j, $\mathcal{L}^j$ may be viewed as a connected linear subtree with minimal element P_0^j.

Let (T, P_0^j) be a germ of a curve at P_0^j. The multiplicity of T at P_i^j is the multiplicity of the strict transform of T at P_i^j. Let $h := \frac{f}{g}$, $f, g \in \mathcal{O}_{\mathbb{P}^2, P_0^j}$, be a germ of meromorphic function. We can define the multiplicity $\nu_{P_i^j}(h)$ of h at P_i^j as the difference of multiplicities of the germs defined by f and g. We will say that a divisor $\mathcal{D} \subset \mathbb{P}^2$ *passes through a point* $P_i^j \in \mathcal{L}$ if $\nu_{P_i^j}(D) > 0$, where D is a local equation for $\mathcal{D}$ at P_0^j. Thus, the following defines an ideal sheaf in $\mathbb{P}^2$ supported on $\mathcal{L}_0 := \{P_0^1, ..., P_0^r\}$:

$$(\mathcal{I}_{\mathcal{L}})_{P_i^j} := \{g \in \mathcal{O}_{\mathbb{P}^2, P_i^j} \mid g \text{ passes through } P_i^j\}.$$

Since $\dim H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}/\mathcal{I}_{\mathcal{L}}) = 8$ and

$$0 \to \mathcal{I}_{\mathcal{L}}(3) \to \mathcal{O}_{\mathbb{P}^2}(3) \to \mathcal{O}_{\mathbb{P}^2}/\mathcal{I}_{\mathcal{L}} \to 0$$

is an exact sequence, there is a pencil of cubics $\Gamma \subset \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{I}_{\mathbb{P}^2}(3)))$. Since the sextic $\mathcal{C}$ is irreducible and by Bezout Theorem, the set of base points of Γ is finite and hence the pencil Γ is base component free. Therefore the singular points of a generic member of Γ are base points. Hence, by a simple argument of multiplicities of intersection, the generic cubic in Γ is smooth. Let $\sigma : X \to \mathbb{P}^2$ be the resolution of the pencil obtained by blowing up the nine base points of Γ, that is, the eight points of $\mathcal{L}$ and an extra one, which may be infinitely near to some points in $\mathcal{L}$. This extra point will be denoted G in this paper.

Convention 2.1. After blowing up a point, we use the same notation for the exceptional divisor as we did for the point. The same convention will be followed for strict transforms. When necessary, we shall specify the surface where the divisor is being considered. For example, the multiplicity of intersection of $\mathcal{D}_1$ and $\mathcal{D}_2$ considered as divisors in $\mathbb{P}^2$ shall be denoted by $(\mathcal{D}_1 \cdot \mathcal{D}_2)^{\mathbb{P}^2}$, whereas the multiplicity of intersection of their strict transforms in X shall be denoted by $(\mathcal{D}_1 \cdot \mathcal{D}_2)^X$.

Note that the surface X is rational. Therefore the group of divisors algebraically equivalent to zero module linear equivalence is trivial, and hence the Neron-Severi group of X, denoted by $NS(X)$, coincides with its Picard group $\mathrm{Pic}(X)$. It is well known that $\mathrm{Pic}(X)$ is freely generated by H (the strict transform of a generic line in $\mathbb{P}^2$), $P_0^j, P_1^j, \ldots, P_{n_j}^j$, $j = 1, \ldots, r$, and G.

On the other hand, the linear system $\sigma^*\Gamma$ has a divisor of base components, say

$$B := \sum_{j=1}^{r} \sum_{k=0}^{n_j} p_k^j P_k^j + gG,$$

for certain $p_k^j, g \in \mathbb{N}^+$. Hence $\sigma^*\Gamma - B$ defines an elliptic fibration $\pi : X \to \mathbb{P}^1$ of which G is a section. Let us consider the Mordell-Weil group $MW(X)$ of the sections of π with zero element G. We will always make this choice in this paper. According to [9] one can define an epimorphism

$$\varphi : NS(X) \to MW(X)$$

as follows. Let $\mathcal{D} \in \operatorname{Div}(X)$ be a divisor in X. On each smooth fiber X_ζ, $\mathcal{D}$ can be restricted to a divisor $\mathcal{D}_\zeta$. Using the group law on X_ζ, any divisor $\mathcal{D}_\zeta$ defines, by adding its points, a point $s_\zeta \in X_\zeta$. Since the closure of these elements gives rise to a section and degree zero divisors are sent to G (by Abel's Theorem), this map factorizes through $NS(X) = \operatorname{Pic}(X)$.

It is well known that the kernel of φ is freely generated by G, a generic fiber F and the irreducible components of the singular fibers of φ which do not intersect G. After fixing a singular fiber, one can replace F by the irreducible component of this fiber intersecting G.

Note that there are two (non-disjoint) kinds of reducible fibers of π: fibers containing exceptional components of σ and fibers coming from reducible elements of the pencil.

Following the construction above, let us denote by $\mu : X \to X$ the involution determined by taking the opposite on each fiber X_ζ. Then, one has the following

Lemma 2.2. *The curve $\mathcal{C}$ does not intersect G and $\mu(\mathcal{C}) = \mathcal{C}$.*

Proof. Note that the divisor $\mathcal{C}$ in X –see the convention above– is linearly equivalent to

$$6H - \sum_{j=1}^{r}\sum_{k=0}^{n_j} 2(k+1)P_k^j - \alpha G \tag{2}$$

for a certain non-negative integer α. Analogously, the strict transform of a generic fiber F in X is linearly equivalent to

$$3H - \sum_{j=1}^{r}\sum_{k=0}^{n_j} (k+1)P_k^j - \beta G$$

for a certain positive integer β. Hence $\mathcal{C}$ is a linear combination of F and G. Since both F and G are in $\ker\varphi$, also $\mathcal{C} \in \ker\varphi$.

By Bezout Theorem $(\mathcal{C}\cdot F)^{\mathbb{P}^2} = 18$. Since the intersection multiplicity at the base points of the pencil Γ is equal to $16+\varepsilon$, where $\varepsilon = 0, 1$ or 2, one has $(\mathcal{C}\cdot F)^X = 2-\varepsilon$. By the previous paragraph $\mathcal{C} \in \ker\varphi$, then the points of $\mathcal{C}$ in any fiber add up to zero. Hence, if $\varepsilon = 1$ then $\mathcal{C}$ intersects generically F at G. Therefore G has to be a component of C, which is impossible since $\mathcal{C}$ is irreducible. If $\varepsilon = 2$ then G is infinitely near a point of $\mathcal{C}$ in $\mathbb{P}^2$. Therefore $(\mathcal{C}\cdot G)^X$ is a positive integer and hence there exists a fiber F_0 passing through $\mathcal{C}\cap G$. If so, then $(F_0\cdot\mathcal{C})^{\mathbb{P}^2} > 18$

which is also impossible by Bezout Theorem. Hence, $\varepsilon = 0$, which implies the first part of the lemma. Therefore also $\alpha = 0$ in (2) and the two points of intersection of $\mathcal{C}$ with F are opposite each other. This implies $\mu(\mathcal{C}) = \mathcal{C}$. □

Lemma 2.3. *Let $\tau : X \to Y = X/\mu$ be the quotient induced by μ from the above construction. Then the resulting surface Y is smooth and rational.*

Proof. Y is rational since X is and τ is onto. Moreover, since the fixed points of μ are a disjoint union of smooth irreducible curves – see figures 13-22 at the end of the paper – Y is also smooth. □

Corollary 2.4. *Let $\tilde{\mathcal{C}} \subset X$ be a reducible curve such that:*

- $\mathcal{C} \subset \tilde{\mathcal{C}}$;
- $\mu(\tilde{\mathcal{C}}) = \tilde{\mathcal{C}}$;
- $\tau_| : X \setminus \tilde{\mathcal{C}} \to Y \setminus \tau(\tilde{\mathcal{C}})$ *is an unramified 2-fold covering and*
- *there exists a sequence of blowing-downs $\kappa : Y \to Z \cong \mathbb{P}^2$ which is an isomorphism on $Y \setminus \tau(\tilde{\mathcal{C}})$.*

Then, there exists a curve $\mathcal{C}$ with the chosen eight double points (and may be some added conditions) if and only if the (reducible) curve $\kappa(\tau(\tilde{\mathcal{C}})) \subset Z$ exists.

Because of the large number of cases, in this paper we restrict our attention to the case where $\mathcal{L} = \{P_0^1, \dots, P_7^1\}$ in (1), i.e., $\mathcal{C}$ has a singular point of type $\mathbb{A}_n$ with $n \geq 15$. We will also ask $\mathcal{C}$ to have some other singular points of type $\mathbb{A}_k$ so that its global Milnor number is greater than or equal to 18.

3. Settings and preliminaries

Given $\mathcal{D}_1, \mathcal{D}_2 \in \mathrm{Div}(X)$ two divisors on the rational surface X, we will use the following notation:

- $\mathcal{D}_1 = \mathcal{D}_2$ if they are equal as divisors,
- $\mathcal{D}_1 \sim \mathcal{D}_2$ if they are linearly equivalent,
- $\mathcal{D}_1 \overset{MW}{=} \mathcal{D}_2$ if they are equal in the Mordell-Weil group.

Let us recall some definitions of transforms of curves under rational morphisms. Let X and X' be smooth surfaces and $f : X \to X'$ a birational morphism. By the universal property of blowing-up we know that f decomposes in a finite number of blowing-ups. Let $E_1, \dots, E_s$ be the exceptional components of f on X and E^f its image. We will denote $f^*\mathcal{D}$ the pull-back of a divisor $\mathcal{D}$ under f and it is called *total transform* of $\mathcal{D}$.

Remark 3.1. Note that this definition also applies to morphisms $g : X \to X''$. For the rational mapping $h := g \circ f^{-1} : X' \dashrightarrow X''$, the *total transform* of a divisor $\mathcal{D}$ is the image by g of the total transform of $\mathcal{D}$ by f.

One can also define the *strict transform* $\tilde{f}\mathcal{D}$ *of an irreducible effective divisor* $\mathcal{D}$ as the Zariski closure of $f^{-1}((\mathcal{D})\setminus E^f)$, where $(\mathcal{D})$ is the zero locus of $\mathcal{D}$. Suppose a divisor $\mathcal{D}$ admits a decomposition of the form $\mathcal{D} = \sum_{i=1}^r m_i\mathcal{D}_i$ in irreducible divisors. Then it is well known that there exist $n_1,\dots,n_s \in \mathbb{N}$ non-negative integers such that

$$f^*\mathcal{D} = \sum_{i=1}^r m_i\tilde{f}\mathcal{D}_i + \sum_{j=1}^s n_jE_j.$$

The *strict transform of a general effective divisor* $\mathcal{D}$ is defined as

$$\sum_{i=1}^r m_i\tilde{f}\mathcal{D}_i = f^*\mathcal{D} - \sum_{j=1}^s n_jE_j.$$

Let us consider a pencil Γ in X' without base components, whose resolution is given by f. Let us denote by $f^*\Gamma$ its pull-back. Then the base locus of the linear system $f^*\Gamma$ is an effective divisor given by a linear combination of the exceptional components $E_1,\dots,E_s$, say $B = \sum_{j=1}^s b_jE_j$. The *virtual transform of* $\mathcal{D}$ *with respect to* Γ can be defined as

$$\check{\mathcal{D}} := \sum_{i=1}^r m_i\mathcal{D}_i + \sum_{j=1}^k (n_j - b_j)E_j = f^*\mathcal{D} - B.$$

Note that effectiveness of $\mathcal{D}$ does not assure effectiveness of $\check{\mathcal{D}}$.

Proposition 3.2. *Let* Γ *be a base component free pencil in* $\mathbb{P}^2$ *and* $\sigma: X \to \mathbb{P}^2$ *a resolution of* γ *obtained by blowing-up its base points. Let* $\mathcal{C} \in \operatorname{Div}(\mathbb{P}^2)$ *be an effective divisor and let* F *be a general member of a pencil* Γ *as above. Suppose that* $|F - \mathcal{C}|$ *is non-empty and that* $\check{\mathcal{C}}$ *is effective. Then* $\mathcal{C}$ *is a member of* Γ.

Proof. The result is a consequence of the following facts:

i) $\check{\mathcal{C}}^2 = \mathcal{C}^2 - F^2$,
ii) $F^2 \geq \mathcal{C}^2$,
iii) $\check{F}\cdot\check{\mathcal{C}} \geq 0$, since $\check{\mathcal{C}}$ is effective.

Using *iii*) and the fact that $\check{F}^2 = 0$ one has

$$0 \leq (F - \mathcal{C})^2 = (\check{F} - B - \check{\mathcal{C}} + B)^2 = \check{\mathcal{C}}^2 - 2\check{F}\cdot\check{\mathcal{C}}.$$

Hence *i*) and *ii*) imply that $\check{\mathcal{C}}^2 = 0$ and therefore $\check{\mathcal{C}} \sim k\check{F}$. Note, however, that $F^2 = \mathcal{C}^2 = k^2F^2$ and that $\mathcal{C}$ is effective, hence $k = 1$. □

Suppose the linear system Γ considered above defines an elliptic fibration $\pi: X \to \mathbb{P}^1$, that is, we assume the general fiber F is a smooth elliptic curve. Such a fibration can only have a certain type of special fibers, classified by Kodaira in [5]. The figures 13-22 describe the possible fibers and the action of the involution μ –see Lemma 2.2– on such fibers. Thick curves or dots mean fixed points by μ. The divisor

S_0 represents the 0-section and the added divisors that do not belong to the fiber represent the 2-torsion curves.

As mentioned at the end of the previous section we will only deal with the case where $\mathcal{L} = \mathcal{L}^1$ in (1), that is, where the eight points chosen are infinitely near to the first one. In order to simplify notation we denote by $E = E_1$ the chosen point in $\mathbb{P}^2$ and by $E_2, \ldots, E_8$ the remaining points in $\mathcal{L}$. The ninth base point in the pencil of cubics passing through $\mathcal{L}$ is denoted by G. The following two divisors will be frequently used:

- the tangent line to $\mathcal{C}$ at E, denoted by T
- and the conic of highest contact with $\mathcal{C}$ at E, denoted by Q.

We will distinguish several cases, depending on the values of $(\mathcal{C} \cdot T)_E$ and $(\mathcal{C} \cdot Q)_E$ and on the position of G. In such cases the starting point will always be the pencil Γ of cubics passing through $\mathcal{L}$ as defined in section 1 and we will reproduce the conditions of Corollary 2.4. Also, once and for all,

- the map $\sigma : X \to \mathbb{P}^2$ denotes the resolution of Γ;
- the map $\pi : X \to \mathbb{P}^1$ denotes the resulting elliptic fibration;
- μ denotes the involution considered above;
- Y the quotient of X by μ;
- τ the projection from X onto Y;
- Z the projective plane obtained after a sequence of blowing-downs from Y and
- R the divisor of X containing the points of order 2, that is, points not in G, fixed by μ.

We introduce also the following notation. Let $\Sigma(T_1, \ldots, T_r)$ be the space of all irreducible projective plane curves of degree 6 with r singular points of type $T_1, \ldots, T_r$. Let $\mathscr{M}(T_1, \ldots, T_r)$ be the quotient of $\Sigma(T_1, \ldots, T_r)$ by the action of the projective group. Note that fundamental group is an invariant of each connected component of these moduli spaces, see [13].

4. Case $(\mathcal{C} \cdot T)_E = 6$

This is the most special and simple case and it will help to understand the general situation. It is easily seen that $Q = 2T$. The virtual transform of $3T$ with respect to Γ is not only an effective divisor, but in fact passes through $\mathcal{L}$. In particular, by Proposition (3.2), $3T$ is a member of Γ. Let F be a generic member of Γ, then $(3T \cdot F)_E = 3(T \cdot F)_E = 9$ and hence, the ninth base point of Γ, G, is also infinitely near to E. The fiber F_0 of π containing T is the virtual transform of $3T$ by σ with respect to Γ. One has

$$F_0 \sim 3T + 2\sum_{j=1}^{3} jE_j + \sum_{j=4}^{8}(9-j)E_j$$

and F_0 is a singular fiber of π of type II^*. Note that F_0 is the only reducible fiber of π since $E_1, \ldots, E_8$ are contained in F_0 and $3T$ is the only reducible cubic of the pencil. Let us consider the group $MW(X)$ of sections of π with zero element G. The kernel of the epimorphism φ is generated by G, $E_1, \ldots, E_8$ and T, where $T \sim H - E_1 - 2E_2 - 3\sum_{j=1}^{9} E_j$. Thus, $MW(X)$ is trivial.

The irreducible component of F_0 intersecting G is E_8. According to Figure 18, R intersects F_0 only at T (and transversally). Note also that R intersects the generic fiber in X at three points. From these data one can deduce that R is the strict transform of a line not containing E.

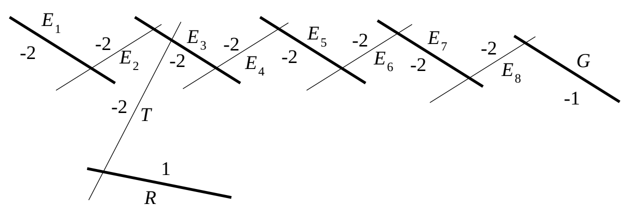

FIGURE 1. Curves in X.

Note that $\tilde{\mathcal{C}} := E_1 \cup \cdots \cup E_8 \cup G \cup T \cup R \cup \mathcal{C}$ verifies the conditions in Corollary (2.4). Figure 2 shows the images of $\tilde{\mathcal{C}} \setminus \mathcal{C}$ in Y.

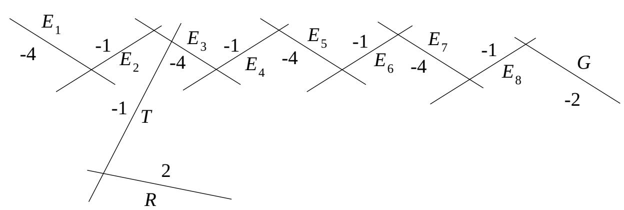

FIGURE 2. Curves in Y.

Since we have blown up 9 times $\mathbb{P}^2$, the Euler characteristic of X is $\chi(X) = 12$. The mapping μ is unramified outside six disjoint rational curves (on which it is an isomorphism). Then, $\chi(Y) = 12$. Hence, blowing down the following 9 curves $T, E_2, E_4, E_6, E_8, G, E_3, E_5$ and E_1 will produce a surface Z isomorphic to $\mathbb{P}^2$. The image of R is a cuspidal cubic and the image of E_7 is the tangent line to R at the cusp. The contracted components are sent to two points.

In order to decide whether such a configuration exists, we need to study what the curve $\mathcal{C}$ is transformed into. Note from the sequence of blowing-ups, that $\mathcal{C}$ should have intersection number 2 with E_8 (and 0 with $E_1, \ldots, E_7$). By Lemma (2.2) $\mathcal{C}$ cannot pass through the last base point G. According to Figure 18, the action of μ on E_8 fixes only two points which are $E_8 \cap E_7$ and $E_8 \cap G$. Since $\mathcal{C}$ is invariant and does not pass through these points, $\mathcal{C}$ must intersect E_8 in X at *generic*

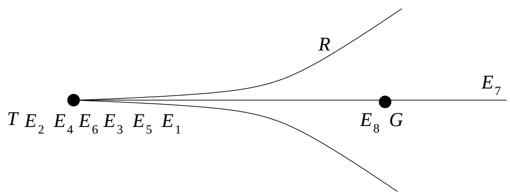

FIGURE 3. Curves in Z.

points and then $\mathcal{C}$. In particular, this fact implies that $\mathcal{C} \subset \mathbb{P}^2$ has an $\mathbb{A}_{15}$-singularity at E.

By Bezout Theorem, $\mathcal{C}$ has to intersect R outside E with multiplicity 6. Hence the image of $\mathcal{C}$ in Z (denoted again by $\mathcal{C}$) is a conic tangent to E_7 (not at the cusp of R).

Let $\{Q_1, \dots, Q_k\} = \mathcal{C} \cap R \subset Z$ the intersection points of $\mathcal{C}$ and R in Z. Let $m_i = (\mathcal{C} \cdot R)_{Q_i}$ $i = 1, \dots, k$ and suppose we order $m_1, \dots, m_k$ so that $m_1 \geq \dots \geq m_k$ (note that $m_1 + \dots + m_k = 6$). Such a situation will be denoted by $[m_1, ..., m_k]$. Note that the curve $\mathcal{C}$ is irreducible if and only if there are at least two odd m_j's and that the inverse image of Q_j is a singular point on $\mathcal{C}$ of type $\mathbb{A}_{m_j-1}$.

These are the possible configurations of singular points for a curve with total Milnor number 18 or greater: $[5, 1]$, $(\mathbb{A}_{15} + \mathbb{A}_4)$; $[3, 3]$, $(\mathbb{A}_{15} + 2\mathbb{A}_2)$; $[4, 1, 1]$, $(\mathbb{A}_{15} + \mathbb{A}_3)$; $[3, 2, 1]$, $(\mathbb{A}_{15} + \mathbb{A}_2 + \mathbb{A}_1)$.

In the next step we will give suitable equations to the components of $\tilde{\mathcal{C}}$ in Z and decide which configuration of singular points does exist. After a change of coordinates in Z, one can assume that R has equation $x^2z - y^3 = 0$ and that the line E_7 has equation $x = 0$. Note that the parametrization $\mathbb{C} \to \operatorname{Reg}(R)$, $t \mapsto [1 : t : t^3]$ is an isomorphism of groups, where $\mathbb{C}$ has the additive group structure and $\operatorname{Reg}(R)$ has the geometric group structure with zero element the inflection point of R, i.e, the sum of three elements is equal to zero if and only if they lie in the same line.

Let us denote by t_0 the parameter of the point of highest contact (at least three) between R and $\mathcal{C}$. It is an easy computation to show that $t_0 = 0$ (the inflection point) does not produce an irreducible curve with the desired Milnor number and by a projective change of coordinates we can assume $t_0 = 1$. Since $\mathcal{C}$ does not pass through the singular point $[0 : 0 : 1]$ of R, we can choose an equation of $\mathcal{C}$ such that the coefficient of z^2 is equal to 1. We impose also $\mathcal{C}$ to be tangent to $x = 0$ and, hence, the general equation for $\mathcal{C}$ is:

$$a_{2,0}x^2 + a_{1,1}xy + \frac{1}{4}a_{0,1}^2y^2 + a_{1,0}xz + a_{0,1}yz + z^2 = 0.$$

In order to understand $\mathcal{C} \cap R$, we replace $(x, y, z) \mapsto (1, t, t^3)$, obtaining a monic polynomial in t of degree 6, such that the coefficient of t^5 vanishes by the geometric group structure. Therefore one has the following equation:

$$(3)\qquad (t-1)^3 p_1(t) = t^6 + a_{0,1}t^4 + a_{1,0}t^3 + \frac{1}{4}a_{0,1}^2 t^2 + a_{1,1}t + a_{0,0},$$

with $p_1(t) := t^3 + 3t^2 + bt + c$. Solving the equations we obtain an equation for $\mathcal{C}$ depending on one parameter b such that

$$p_1(t) = t^3 + 3t^2 + bt - \frac{b^2}{12} + 2b - 4.$$

One can check that there is no solution for cases $(\mathbb{A}_{15}+\mathbb{A}_4)$ and $(\mathbb{A}_{15}+2\mathbb{A}_2)$. The solutions $b=36$, for $(\mathbb{A}_{15}+\mathbb{A}_3)$ and $b=\frac{8}{3}$, for $(\mathbb{A}_{15}+\mathbb{A}_2+\mathbb{A}_1)$, are the unique possible ones.

Remark 4.1. One can also calculate the rational map $\rho = \kappa \circ \tau \circ \sigma^{-1} : \mathbb{P}^2 \dashrightarrow Z$ as follows: the set $\{\rho_1(x,y,z)=0\}$ is the inverse image of the curve $\{x=0\}$, that is, E_7. Following the sequence of blowing-ups and blowing-downs of ρ, one can check that $\rho^* E_7 = 3T$. Analogously, $\{\rho_2(x,y,z)=0\}$ is the inverse of $L_2 = \{y=0\}$, which is a line intersecting R at the cusp and an inflection point. One can check that $\rho^* L_2 = 2T + \tilde{L}_2$, where $\tilde{L}_2$ is a line. Finally, $\{\rho_3(x,y,z)=0\}$ is the inverse of $L_3=\{z=0\}$, which is the line tangent to R at the inflection point $L_2 \cap R$. One has that $\rho^* L_3$ is a cubic with a cusp at $R \cap \tilde{L}_2$ and an inflection point at $T \cap \tilde{L}_2$ having T as tangent line. Using suitable coordinates on $\mathbb{P}^2$ such that $T=\{x=0\}$, $\tilde{L}_2=\{y=0\}$ and $R=\{z=0\}$ one can write ρ as

$$\rho(x,y,z) = [x^3 : x^2y : xz^2 + y^3].$$

The equation of the curve $\mathbb{A}_{15}+\mathbb{A}_3$ is

$$-156\,x^5\,y - 140\,x^4z^2 - 140\,x^3y^3 + 225\,x^4y^2 + \\ + 40\,x^6 + x^2z^4 + 2\,xz^2y^3 + y^6 + 30\,x^3yz^2 + 30\,x^2y^4 = 0$$

where $\mathbb{A}_{15} = [0:0:1]$ and $\mathbb{A}_3 = [1:1:0]$. Finally, the equation of the curve $\mathbb{A}_{15}+\mathbb{A}_2+\mathbb{A}_1$ is

$$-12\,x^5y + 20\,x^4z^2 + 20\,x^3y^3 + 75\,x^4y^2 - 20\,x^6 + \\ + 27\,x^2z^4 + 54\,xz^2y^3 + 27\,y^6 - 90\,x^3yz^2 - 90\,x^2y^4$$

where $\mathbb{A}_{15} = [0:0:1]$, $\mathbb{A}_2 = [1:1:0]$ and $\mathbb{A}_1 = [5:-3:0]$.

Proposition 4.2. *The fundamental groups for the curves with equations above in $\mathscr{M}(\mathbb{A}_{15},\mathbb{A}_3)$ and $\mathscr{M}(\mathbb{A}_{15},\mathbb{A}_2,\mathbb{A}_1)$ are abelian*

Proof. The steps to compute the fundamental group are the following:

- Compute the fundamental group G_1 of the the curve $\mathcal{C} \cup E_7 \cup R \subset Z$.
- The covering τ allows to compute the fundamental group G_2 of $\tilde{\mathcal{C}} \subset \mathbb{P}^2$ as an index 2 subgroup of G_1.

- Find the meridians of the non-exceptional components of $\tilde{\mathcal{C}}$ different from $\mathcal{C}$ –in this case R and T- as elements of G_2.
- The group G of $\mathcal{C} \subset \mathbb{P}^2$ results from factoring G_2 by the subgroup generated by such meridians.

In the case $\mathbb{A}_{15} + \mathbb{A}_2 + \mathbb{A}_1$ the first step is very simple. If we send E_7 to the line at infinity, the real figure is very simple in both cases: we have the curve $z - y^3 = 0$ and a parabola. It is easy to check that G_1 is abelian. It forces G_3 to be abelian.

For the case $\mathbb{A}_{15} + \mathbb{A}_3$, it is easier to use the equations above which depend only on x, y, z^2. □

5. The General Case

The purpose of this section is to provide equations –up to projective change of coordinates– for both the divisor of 2-torsion R, the birational map $\rho = \kappa \circ \tau \circ \sigma^{-1} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ and the families of curves in the most general situation.

The birational map ρ can be described geometrically as follows. We assume the eight infinitely near points to be generic, that is, no line passes through the first three, no conic through the first six and the ninth base point is not infinitely close to E. The first two conditions are essential; the third one is only technical.

The pencil Γ contains exactly one nodal cubic S whose node lies at E. Such a cubic satisfies that one branch is transversal and the other has order of contact seven with F, a general member of Γ. Its virtual transform produces a special fiber

$$F_0 = S + \sum_{j=1}^{7} E_j$$

of type I_8. There are no other reducible fibers. After the resolution of the pencil, $\mathrm{Pic}(X)$ is freely generated by $E_1, \ldots, E_8, G, H$. We recall that $\ker \varphi$ is freely generated by $G, S, E_1, \ldots, E_7$, where

$$S \sim 3H - \sum_{j=1}^{7} (j+1)E_j - 8E_8 - G.$$

Thus $MW(X)$ is generated by H and E_8 with the relation $3H \overset{MW}{=} 8E_8$, that is, $MW(X)$ is isomorphic to $\mathbb{Z}$.

By assumption, the conic Q has five infinitely near points in common with $\mathcal{C}$ at E. Therefore $(Q \cdot F)_E = 5$, where F is a general member of the pencil Γ, and hence, it is a section of π satisfying $Q \overset{MW}{=} 2H - 5E_8$. Analogously, since $(T \cdot F)_E = 2$, T is a section and $T \overset{MW}{=} H - 2E_8$. By a standard argument of determinants one can see that Q generates $MW(X)$, in fact

$$E_8 \overset{MW}{=} 3Q, \quad H \overset{MW}{=} 8Q, \quad \text{and} \quad T \overset{MW}{=} 2Q.$$

According to Figure 14, $\mu(E_j) = E_{8-j}$, $j = 1, \ldots, 7$ and $\mu(S) = S$, that is, S and E_4 are globally fixed. Let $E_8' = \mu(E_8)$ be the image of E_8 by μ. Since $E_8 \cap S = E_8 \cap G = E_8 \cap E_1 = \cdots = E_8 \cap E_6 = \emptyset$ and E_8 intersects E_7 transversally at one point, so behaves E_8' with respect to the corresponding images by μ. Note also that, since E_8 is a section and rational, so is E_8'.

Notation 5.1. A curve of degree d is of type $E(m_1, \ldots, m_8)$ if the multiplicity sequence of the strict transforms of the curve at the base points over E is $(m_1, \ldots, m_8)$.

Let us denote by a the multiplicity of intersection of E_8' with E_8 and by d the degree of $E_8' \subset \mathbb{P}^2$. With this notation $E_8' \subset \mathbb{P}^2$ is of type $E(a+1, a, \ldots, a)$. Since $E_8' \subset X$ is a section and $E_8' \cap G = \emptyset$, one has

$$(E_8' \cdot F)_E = (a+1) + 7a = 3d - 1. \tag{4}$$

Moreover, since $E_8' \subset X$ is rational and smooth, the genus formula on $\mathbb{P}^2$ gives:

$$(d-1)(d-2) = (a+1)a + 7a(a-1). \tag{5}$$

The only possible solution of (4) and (5) is $d = 6$, $a = 2$.

From Figure 14, the curve R intersects E_4 transversally at two points and S at one point. Note that $E_8' \cap E_8$ consists generically of two distinct points fixed by μ. Hence R passes through $E_8' \cap E_8$ transversally to E_8' and E_8. Therefore R is of type $E(4, 4, 4, 4, 2, 2, 2, 2)$ and hence has degree 9. Note that R is irreducible since otherwise there would be a 2-torsion section. Therefore $\pi_{|R}$ is three to one.

Let us denote $\mu(Q)$ by Q'. Note that $(E_8' \cdot Q)_E^{\mathbb{P}^2} = 11$ and hence $(E_8' \cdot Q)^X = 1$. Applying μ, $(E_8 \cdot Q')^X = 1$. The strict transform Q' intersects $\mu(E_5) = E_3$ and $\mu(E_8') = E_8$. Therefore Q' is of type $E(2, 2, 2, 1, \ldots, 1)$ and its self-intersection is -1. Thus $Q' \subset \mathbb{P}^2$ is a rational quartic and Q and Q' are disjoint in X.

Let us consider the curve

$$\tilde{\mathcal{C}} := \mathcal{C} \cup S \cup \bigcup_{j=1}^{8} E_j \cup E_8' \cup R \cup G \cup Q \cup Q'$$

satisfying the conditions of Corollary (2.4). The curves in $\tilde{\mathcal{C}} \setminus \mathcal{C}$ are shown in Figure 4 and their images in Y appear in Figure 5.
By Lemma (2.3), Y is a smooth rational surface and, by a straightforward computation, its Euler characteristic is 8. So, after blowing down the following curves: S, Q, E_1, E_2 and E_4 one obtains a surface Z isomorphic to $\mathbb{P}^2$. The image of R is a nodal cubic, where E_4 is its singular point and S an inflection point. The curve G on Z is the tangent line to R at S. The image of E_3 is the line joining S and E_4. This line also contains the point Q. Finally, the image of E_8 is a smooth conic

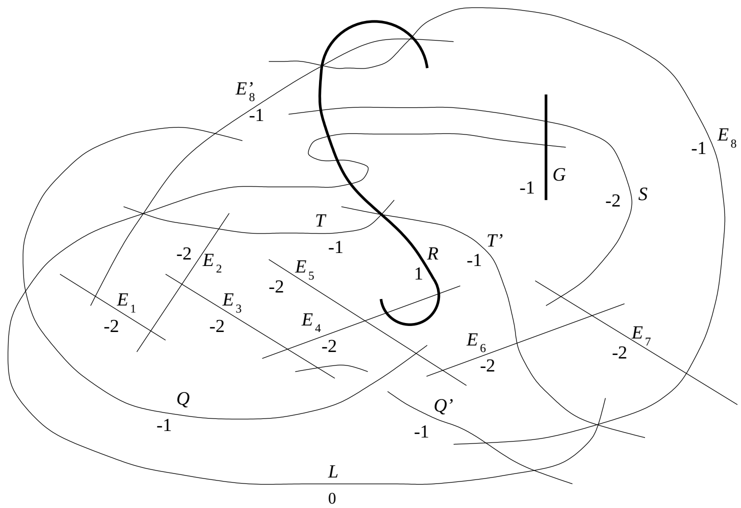

FIGURE 4. Curves in X.

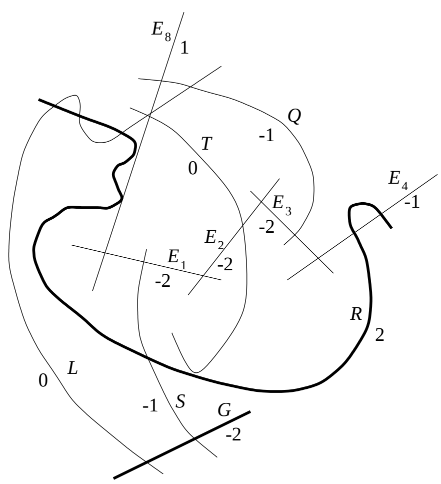

FIGURE 5. Curves in Y.

tangent to G at S passing through Q and tangent to R at two points. The pencil of cubics is sent to a pencil of lines through Q.

Let us consider the space of sextics Σ with the same singularity types as $\mathcal{C}$ and let $\widetilde{\Sigma}$ be the Zariski open set of curves $\mathcal{C}'$ such that the first seven infinitely near points of $\mathcal{L}'$ are *generic* as in this section. Note that for any $\mathcal{C}' \in \widetilde{\Sigma}$ there is a projective automorphism that sends the first seven points of the list $\mathcal{L}'$ onto the first seven points of $\mathcal{L}$. We denote by $\widetilde{\mathscr{M}}$ the quotient of $\widetilde{\Sigma}$ by the group of projective automorphisms. Its dimension is reduced by exactly eight units.

This strategy has been extensively used by I. Luengo. By a straight forward calculation one can check that the isotropy group of these seven infinitely near points is cyclic of order 3. If we denote by $\mathscr{N}$ the set of

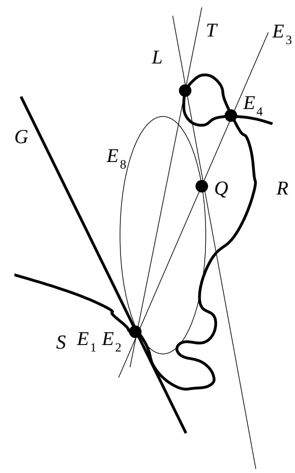

FIGURE 6. Curves in Z.

sextics in $\widetilde{\Sigma}$ having the first seven infinitely near double points in common with $\mathcal{C}$. The mapping $\mathscr{N} \to \widetilde{\mathscr{M}}$ is surjective and $3:1$ as orbifolds. Note that $\widetilde{\mathscr{M}}$ might exhibit singularities due to isotropy that are not present in $\mathscr{N}$ –see [7]. For our purpose of proving the existence of families of sextics, finding their dimension and possible degenerations, this method of factoring out will be very useful. Once the existence is proved, one can compose with some suitable projective transformations in order to eliminate the isotropy at least for components in dimension 0.

Remark 5.2. We will see that the closure of $\widetilde{\mathscr{M}}$ is a connected component of $\mathscr{M}$ obtained just by adding a finite number of points from §4 and §7.

We are going to fix equations for the curves in $\mathbb{P}^2$ and Z. We begin by fixing $E_1, \ldots, E_7$.

- the point $E = E_1$, as $[0:0:1]$, (two conditions)
- the point E_2 by fixing, for instance, the line T, $\ell = \{x = 0\}$, as the line passing through P_1, that is, the common tangent direction of Γ at P (one more condition).
- the points E_3, E_4 and E_5 by fixing the conic Q of maximal contact with Γ. Note that such a conic has to pass through E_1 and E_2, for instance $f_Q = xz - y^2 = 0$ (three conditions).
- the remaining points E_6 and E_7 (two conditions), chosen such that they are the infinitely near points of $x - y^2 - y^5 = 0$ (with affine coordinates).
- The equation for the nodal cubic S is $f_S : x^3 + y^3 - xyz = 0$.

Note that the ideal $\mathcal{I} \subset \mathbb{C}\{x, y\}$ defining the germs that pass through the points $E_1, ..., E_7$ is generated by $(x - y^2 - y^5, y^7)$. Therefore, the complex number $u \in \mathbb{C}$ in $\varphi_u \equiv x - y^2 - y^5 - uy^7$ parametrizes the last infinitely close point E_8 by means of the intersection of the strict transform of φ_u and E_7. Also it is readily seen that the multiplication by a cubic root of unity realizes the isotropy (hence only u^3 may be considered as a projective invariant).

On the other hand, by a suitable selection of coordinates on Z, one can define R as the set of zeroes of the nodal cubic $xyz + x^3 - y^3 = 0$, where $[0 : 0 : 1]$ is the nodal point and, for instance, $[1 : 1 : 0]$ is the inflection point S. Hence

$$G = \{3\,(x - y) + z = 0\} \quad \text{and} \quad E_3 = \{x - y = 0\},$$

where G is the tangent line to R at S and E is the line joining S and the singular point of R. Note that the parametrization

$$\begin{array}{cccc} \varphi: & \mathbb{C} & \to & R \\ & t & \mapsto & [t : t^2 : t^3 - 1]. \end{array}$$

induces a group isomorphism, by restriction to $\mathbb{C}^*$, between the multiplicative group $\mathbb{C}^*$ and the set of non-singular points of R endowed with the geometric group structure of zero element S. In fact, this condition determines φ up to the automorphism $t \to t^{-1}$ which corresponds to the permutation of the two branches of R at the nodal point.

The equation of E_8 is given by a homogeneous polynomial of degree 2, say $f_{E_8}(x, y, z)$. Since the nodal point $[0 : 0 : 1] \notin E_8$, one can assume that the coefficient of z^2 in f_{E_8} is 1. The composition $p(t) := (f \circ \varphi)(t)$ is, hence, a one-variable monic polynomial of degree 6. Moreover, p has the following factorization

$$p(t) = (t - 1)^2(t - t_1)^2(t - t_2)^2$$

where the values 1, t_1 and t_2 correspond to the three points of contact two. According to the group structure of $\mathrm{Reg}(R)$ one has $(t_1t_2)^2 = 1$. The case $t_1t_2 = 1$ forces E_8 to be a double line, thus $t_1t_2 = -1$. Hence

$$p(t) = (t - 1)^2(t - t_1)^2(t + \frac{1}{t_1})^2, \quad t_1 \neq 1, \tag{6}$$

or

$$p(t) = (t - 1)^2(t^2 - wt - 1)^2, \quad w := t_1 - \frac{1}{t_1} \neq 0, \tag{7}$$

depending on whether or not we want to single out a solution.
This condition determines f_{E_8} as

$$\begin{aligned} &z^2 + (2 - 2w)zx + (-2 - 2w)zy + (-6w + w^2 - 3)x^2 + \\ &+ (-2w^2 + 6)xy + (-3 + 6w + w^2)y^2 = 0. \end{aligned}$$

Therefore Q is the point $[1 : 1 : 4w]$. Let L be the tangent line to E_8 at Q. The image of $T : x - y - z = 0$ is a line through S and tangent to R at the point $[-1:1:-2]$ with parameter $t=-1$. Note that L also passes through this point and its equation is:

$$(2w+1)x+(2w-1)y-z=0.$$

As usual we also denote by $L\subset\mathbb{P}^2$ the cubic of Γ obtained as preimage of $L\subset Z$. The total transforms by ρ (i.e., the images by σ of the total transforms by $\kappa\circ\tau$) are:

$$\begin{array}{lcl}\rho^*E_3 &=& Q+Q'+S\\ \rho^*L &=& Q+Q'+L\\ \rho^*G &=& 3S\\ \rho^*E_8 &=& Q+Q'+E_8'+2S\\ \rho^*R &=& 2R+3S\end{array}\tag{8}$$

With these data one can in principle calculate equations for the map ρ.

One technical difficulty appears. The equations of the curves in $\mathbb{P}^2$ depend on a parameter u (but only u^3 is well-defined) whereas the equations of the curves in Z depend on w (but only w^2 is well-defined). One can verify that the parameters u and w are linked by the equation $16u^3+w^2=0$. Using a new parameter v such that $u=-v^2$ and $w=4v^3$ one can give all the equations in terms of v. With some effort and Maple we can compute the equation of the quartic Q':

$$f_{Q'}: v^2z^2x^2+zx^2y-2v^2zxy^2-x^4-xy^3+v^2y^4=0.$$

The preimage R of the 2-torsion is:

$$\begin{aligned}f_R: v^2z^5x^4&-v^4z^4x^5+z^4x^4y-4v^2z^4x^3y^2-z^3x^6-\\ &-4v^2z^3x^5y+4v^4z^3x^4y^2-4z^3x^3y^3+6v^2z^3x^2y^4+\\ &+3v^2z^2x^7+2z^2x^5y^2+12v^2z^2x^4y^3-6v^4z^2x^3y^4+\\ &+6z^2x^2y^5-4v^2z^2xy^6+2zx^7y-6v^2zx^6y^2-zx^4y^4-\\ &-12v^2zx^3y^5+4v^4zx^2y^6-4zxy^7+v^2zy^8-x^9-\\ &-2x^6y^3+3v^2x^5y^4+4v^2x^2y^7-v^4xy^8+y^9=0.\end{aligned}$$

Finally the equation for the cubic L is

$$f_L: y^3+v^4y^2x+v^2y^2z+v^2yx^2-yxz+x^3-v^4x^2z-v^2xz^2=0.$$

In order to set the equations we proceed as follows. First, we consider an automorphism ρ_1 such that the total transform by ρ_1 of the coordinate axis is sent to the lines E_3, L, G:

$$\rho_1([x:y:z])=[8v^3x+y-4x+z:y-4x-8v^3x+z:16v^3(z-3x)].$$

Second, we construct ρ_2 taking into account the equations in (8). These computations define ρ_2 up to a diagonal transformation. Finally constants are properly adjusted:

$$\rho_2([x:y:z])=[f_Qf_{Q'}f_S:4f_Qf_{Q'}f_L:-4f_S^3].$$

All it remains is to find all possible irreducible cubics $\mathcal{C} \subset Z$ with the following two general features:

(1) $\mathcal{C}$ has a double point at $Q = E_8 \cap E_3 \subset Z$.
(2) $S \in Z$ is an inflection point of $\mathcal{C}$ and G is the tangent line to $\mathcal{C}$ at S.

Before stating a strategy to find such curves, we will describe how the position of $\mathcal{C} \subset Z$ with respect to E_8 and R affects the singularity types of $\mathcal{C} \subset \mathbb{P}^2$.

Notation 5.3. A lot of cases may be considered in principle. Some of them come from the particular contraction from Y to Z and are not essential. For example, let us consider the intersection of $\mathcal{C}$ and E_8. We deduce geometrically that $(\mathcal{C} \cdot E_8)^Z_S$ is exactly 2. The other fixed intersection point Q verifies $(\mathcal{C} \cdot E_8)^Z_Q \geq 2$ and what is relevant is the excess of intersection at Q. We will say that $\mathcal{C}$ and E_8 have *two relevant intersection points* if either $\#(\mathcal{C} \cap E_8) \setminus \{S, Q\} = 2$ or $\#(\mathcal{C} \cap E_8) \setminus \{S, Q\} = 1$ and $(\mathcal{C} \cdot E_8)^Z_Q = 3$, where $\#A$ denotes the cardinal of A. Analogously, $\mathcal{C}$ and E_8 have *one relevant intersection point* if either $\#(\mathcal{C} \cap E_8) \setminus \{S, Q\} = 1$ or $(\mathcal{C} \cdot E_8)^Z_Q = 4$. In the latter case, we will say that this relevant point is *special* if it also belongs to R. Note that *being special* only applies if the number of relevant intersection points is 1.

Proposition 5.4. *The singularities of $\mathcal{C} \subset \mathbb{P}^2$ depend on $\mathcal{C} \subset Z$ as follows:*

- *If $\mathcal{C}$ and E_8 has two relevant intersection points, then $\mathcal{C}$ has an $\mathbb{A}_{15}$-singularity*
- *If $\mathcal{C}$ and E_8 has one relevant intersection point, say P, then $\mathcal{C}$ has an $\mathbb{A}_{16+s}$-singularity, where $s := (\mathcal{C} \cdot R)^Z_P$.*

We know $(\mathcal{C} \cdot R)^Z_S \geq 3$. Then $\mathcal{C} \cap R = \{S, P_1, \ldots, P_r\}$ with multiplicities $3 + m_0, m_1, \ldots, m_r$, where $m_0 \geq 0$, $1 = m_1 = \cdots = m_s < m_{s+1} \leq \cdots \leq m_r$ and $m_0 + m_1 + \cdots + m_r = 6$. If $m_0 > 1$, then S produces a singular point of type $\mathbb{A}_{m_0 - 1}$. The points $P_1, \ldots, P_s$ do not produce singular points. For $j = s + 1, \ldots, r$, if $P_j \notin E_8$, then it produces a singular point of type $\mathbb{A}_{m_j - 1}$.

Remark 5.5. With this picture in mind, it is easy to see how deformations arise. If $\mathcal{C}^t$ deforms into $\mathcal{C}^0$, one of the following kinds of deformation occur:

- Two intersection points of $\mathcal{C}^t \cap R$ with intersection multiplicities m, n degenerate into an intersection point of $\mathcal{C}^0 \cap R$ of multiplicity $m + n$. Hence, $\mathbb{A}_{m-1} + \mathbb{A}_{n-1} \Rightarrow \mathbb{A}_{n+m-1}$.
- An intersection point of $\mathcal{C}^t \cap R$ with intersection multiplicity $m > 1$ degenerates into an intersection point of $\mathcal{C}^0 \cap R \cap E_8$, different from S. Hence, $\mathbb{A}_{m-1} + \mathbb{A}_{15} \Rightarrow \mathbb{A}_{15+m}$.

Construction 5.6. We will describe a strategy to collect all the suitable cubics. Note that a direct attack toward finding $\mathcal{C}$ is a very expensive task in computational terms. The use of pencils of cubics and the geometrical group structure on $\mathcal{C}$ will be essential in our approach.

One starts by considering the pencil of cubics generated by $\mathcal{C}$ and R. It is easily seen that $G+B$ is a member of such pencil, where B is a conic (maybe reducible) which intersects R in the same manner as $\mathcal{C}$ does, except that $(R\cdot B)_S^Z=(R\cdot\mathcal{C})_S^Z-3$. Namely, let $f_{\mathcal{C}}(x,y,z)=0$ be the equation of $\mathcal{C}$ monic in z^3 and let $f_B(x,y,z)=0$ be the equation of B monic in z^2. If $g(t):=f_B(t,t^2,t^3-1)$, then

$$f_{\mathcal{C}}(t,t^2,t^3-1)=g(t)(t-1)^3.$$

This property determiners B completely. Hence, the pencil is also generated by R and $G{+}B$, and we can consider $\mathcal{C}$ as the element passing through Q. There is an additional condition in this construction: $\mathcal{C}$ must have a double point at Q, which will impose extra conditions on the parameters involved. Now we fix the singularity types for $\mathcal{C}$ and we apply the conditions in proposition 5.4 to find specific curves. We translate the conditions of 5.4 in a quantitative way.

We need also to consider a parametrization $[t:s]$ of the conic E_8. By convenience we may suppose that S is the point for $t=\infty$ and Q is the point for $t=0$. The root $t=\infty$ has multiplicity exactly 2, and $t=0$ is a root of multiplicity at least 2. Hence, replacing this parametrization in the equation of $\mathcal{C}$, will result in a polynomial in t of degree 4 divisible by t^2. We denote by $e_8(t)$ the quotient of degree 2.

We list the possibilities for the singularities of the sextic curve with global Milnor number at least 18. We need to take into account:

- The multiplicity list of the roots of the monic polynomial $g(t)$ (which must have roots of odd multiplicity in order to have irreducible curves). Such a list will be denoted by $(m_1,\dots,m_r)$.
- The number $\beta\in\{1,2\}$ of different roots of $e_8(t)$, i.e., the number of relevant intersection points.
- The existence of special relevant points. We will define the following invariant η as empty if no multiple roots of g $(\neq 1)$ lies on E_8, and as the multiplicity otherwise.

In table 1 can be found all the possibilities for $\mu=18$ and in table 2 for $\mu=19$ with the possible degenerations from $\mu=18$.

Theorem 5.7. *Let us fix $\widetilde{\mathscr{M}}$ for one of the configurations in table 1. Then $\widetilde{\mathscr{M}}$ is non-empty, connected and has dimension 1. Moreover, the curves appearing in §4 (and later in §7) are in the closure of some $\widetilde{\mathscr{M}}$.*

Proof. The statement can be proved computationally and the authors can provide the Maple files upon request. Note that the expected dimension of $\mathscr{M}$ for these singularity types is 1. ∎

Case	Singularities	Root list of g	β	η
1	$\mathbb{A}_{15}+\mathbb{A}_2+\mathbb{A}_1$	$(1,2,3)$	2	
2	$\mathbb{A}_{15}+\mathbb{A}_3$	$(1,1,4)$	2	
3	$\mathbb{A}_{16}+2\mathbb{A}_1$	$(1,1,2,2)$	1	
4	$\mathbb{A}_{16}+\mathbb{A}_2$	$(1,1,1,3)$	1	
5	$\mathbb{A}_{17}+\mathbb{A}_1$	$(1,1,2,2)$	1	2
6	$\mathbb{A}_{18}$	$(1,1,1,3)$	1	3

Table 1

Singularities	Root list of g	β	η	Degeneration
$\mathbb{A}_{15}+2\mathbb{A}_2$	$(3,3)$	2		1
$\mathbb{A}_{15}+\mathbb{A}_4$	$(1,5)$	2		1,2
$\mathbb{A}_{16}+\mathbb{A}_2+\mathbb{A}_1$	$(1,2,3)$	1		1,3,4
$\mathbb{A}_{16}+\mathbb{A}_3$	$(1,1,4)$	1		2,3,4
$\mathbb{A}_{17}+\mathbb{A}_2$	$(1,1,2,3)$	1	2	1,4,5
$\mathbb{A}_{18}+\mathbb{A}_1$	$(1,2,3)$	1	3	1,3,5,6
$\mathbb{A}_{19}$	$(1,1,4)$	1	4	1,2,3,4,5,6

Table 2

Theorem 5.8. *All the curves in table 2 but $\mathbb{A}_{17}+\mathbb{A}_2$ and $\mathbb{A}_{15}+2\mathbb{A}_2$ exist and all possible degenerations also exist. The spaces $\mathscr{M}=\widetilde{\mathscr{M}}$ are discrete and verify:*

(i) *$\mathscr{M}(\mathbb{A}_{15},\mathbb{A}_4)$ consists of two points. Such curves can be defined by equations conjugated in $\mathbb{Q}(i)$.*

(ii) *$\mathscr{M}(\mathbb{A}_{16},\mathbb{A}_2,\mathbb{A}_1)$ consists of three points. Such curves can be defined by equations conjugated in the number field $\mathbb{Q}(\alpha)$, where α is a root of $17s^3-18s^2-228s+536=0$.*

(iii) *$\mathscr{M}(\mathbb{A}_{16},\mathbb{A}_3)$ consists of two points. Such curves can be defined by equations conjugated in $\mathbb{Q}(\sqrt{17})$.*

(iv) *$\mathscr{M}(\mathbb{A}_{18},\mathbb{A}_1)$ consists of three points. Such curves can be defined by equations conjugated in $\mathbb{Q}(\beta)$, where β is a root of $19s^3+50s^2+36s+8=0$.*

(v) *$\mathscr{M}(\mathbb{A}_{19})$ consists of two points Such curves can be defined by equations conjugated in $\mathbb{Q}(\sqrt{5})$.*

In order to provide equations we use ρ. Note that $\rho^*(\mathcal{C})$ is a curve of degree 18. A direct factorization of this curve is in general complicated (equations for $\mu=18$ have coefficients in algebraic extensions of rational fields of transcendent degree at least one). Instead, one can use the fact that we already know a factorization of $\rho^*(\mathcal{C})$. For $\mu=19$, we generally find equations in very large number fields due to the isotropy. With some effort we are able to find the smallest number field where curves have equations. The following is an example of such equations:

$$f_{\mathbb{A}_{19}}: \left(\frac{37}{16}+\frac{7}{16}\,\alpha\right)y^6+(-5/8-1/8\,\alpha)\,y^5x+(-1/4\,\alpha+7/4)\,y^5z+$$
$$+\left(\frac{7}{32}\,\alpha+\frac{5}{32}\right)y^4x^2+\left(-\frac{81}{16}-\frac{15}{16}\,\alpha\right)y^4xz+y^4z^2+(1/2\,\alpha+3/4)\,y^3x^3+$$
$$+(3/2+1/2\,\alpha)\,y^3x^2z+(1/2\,\alpha-7/2)\,y^3xz^2+(-1/16\,\alpha-1/16)\,y^2x^4+$$
$$+(-3/16-1/16\,\alpha)\,y^2x^3z+\left(\frac{9}{16}\,\alpha+\frac{51}{16}\right)y^2x^2z^2-2\,y^2xz^3+$$
$$+(1/32\,\alpha+1/8)\,yx^5+(-3/4-1/2\,\alpha)\,yx^4z+\left(-3/8\,\alpha-\frac{7}{8}\right)yx^3z^2+$$
$$+(-1/4\,\alpha+7/4)\,yx^2z^3+\left(\frac{11}{64}+1/16\,\alpha\right)x^6+\left(\frac{3}{32}\,\alpha+\frac{5}{32}\right)x^5z+$$
$$+\left(1/32-\frac{5}{32}\,\alpha\right)x^4z^2+\left(-\frac{7}{16}-1/16\,\alpha\right)x^3z^3+x^2z^4=0,$$

where $\alpha^2=5$. This curve appeared in a work of Yoshihara [12] but with equations in a larger number field.

Remark 5.9. We have computed the fundamental group for the existing curves with $\mu=19$ using both the method in the proof of Proposition 4.2 and the program of J. Carmona. They all turn out to be abelian and hence it is also the case for $\mu=18$ (in this section).

6. Case $(\mathcal{C}\cdot T)_E=4$, $(\mathcal{C}\cdot Q)_E=12$

This is the other important case, since the curves we will find in will be degeneration of Zariski sextics with six cuspidal points in a conic.

By hypothesis, the cubic $Q+T$ passes exactly through eight base points of the pencil, hence $Q+T$ is a member of Γ and the ninth base point G belongs to T, but is not infinitely near to E. The virtual transform F_0 of $Q+T$ is

$$(Q+T)\check{}=F_0=Q+T+E_1+2\sum_{j=2}^{6}E_j+E_7,$$

hence F_0 is of type I_4^*. Note that F_0 is the only reducible fiber of π. The Picard group of X is freely generated by $H,E_1,\dots,E_8,G$ and one has:

$$T\sim H-G-E_1-2\sum_{j=1}^{8}E_j,\quad Q\sim 2H-\sum_{j=1}^{6}jE_j+6E_7+6E_8.$$

The free group $\ker\varphi$ is generated by G, Q,, T, $E_1,\dots,E_7$. Then $MW(X)$ is cyclic of order 2 with elements G,E_8.

According to Figure 16, all the irreducible components of F_0 are globally preserved by μ and E_6, E_4, E_2 are pointwise fixed. Also, the 2-torsion points of π form at least two irreducible components, one of them being E_8. Let us denote by R the remaining components, then R intersects the smooth part of F_0 at E_1 and Q. Checking multiplicities of intersection of the component of R passing through E_1 with Q, one concludes that R is in fact a line passing through E and hence, it verifies that $\pi_{|R}$ is $2:1$.

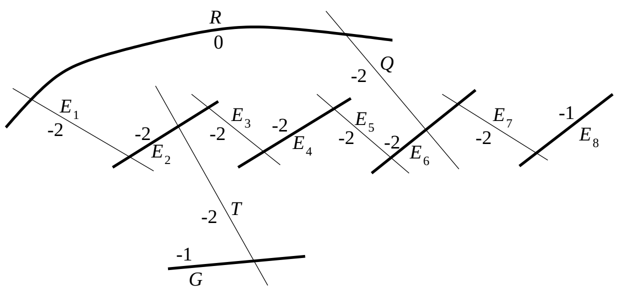

FIGURE 7. Curves in X.

Note that $\tilde{\mathcal{C}} := E_1 \cup \cdots \cup E_8 \cup G \cup Q \cup T \cup R \cup \mathcal{C}$ verifies the conditions in Corollary (2.2). Figure 14 shows the images of $\tilde{\mathcal{C}} \setminus \mathcal{C}$ in Y.

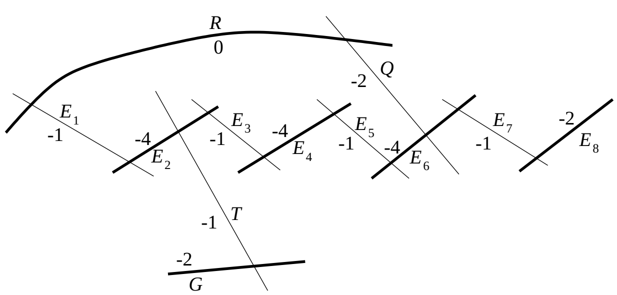

FIGURE 8. Curves in Y.

By Lemma (2.3), Y is a smooth rational surface with Euler characteristic equal to 12. So, blowing-down the following nine curves, $T, Q, E_1, E_3, E_5, E_7, G, E_6$ and E_4, will produce a surface Z isomorphic to $\mathbb{P}^2$. The image of R in Z is a smooth conic, the image of E_8 is a tangent line to R at a point and E_2 becomes a line through the point of tangency. The contracted components are sent to three points.

In order to decide whether such a configuration exists, one needs to determine the image of $\mathcal{C}$ on Z. Note that $(\mathcal{C} \cdot E_8)^Y = 2$, $(\mathcal{C} \cdot E_i)^Y = 0$ for $i = 1, \ldots, 7$ and $(\mathcal{C} \cdot F)^X_E = 4$ where F is a general cubic of the pencil. By Lemma (2.2), $\mathcal{C}$ does not intersect G. According to Figure 18, E_8 is pointwise fixed.

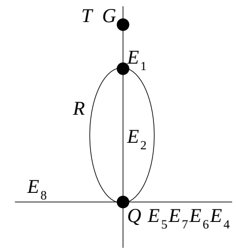

FIGURE 9. Curves in Z.

Since $\mathcal{C}$ is globally preserved, $\mathcal{C}$ must intersect E_8 (with multiplicity 2) in such a way that the local arrangement of curves may be globally preserved by the involution. A priori, $\mathcal{C}$ could intersect E_8 at a point of type $\mathbb{A}_r$ –by convention $\mathbb{A}_{-1}$ consists of two distinct points and $\mathbb{A}_0$ is a smooth tangent point. Such a point produces in $\mathbb{P}^2$ a singular point of type $\mathbb{A}_{16+r}$. By standard arguments, we see that $r \leq 3$. The condition on the involution excludes the cases $r = 0, 2$ (i.e. the cases $\mathbb{A}_{16}$ and $\mathbb{A}_{18}$). In principle we may have $r = -1$, giving $\mathbb{A}_{15}$, $r = 1$, giving $\mathbb{A}_{17}$ or $r = 3$, giving $\mathbb{A}_{19}$), this last case will be excluded later.

Note also that $(\mathcal{C} \cdot R)^Y = (\mathcal{C} \cdot R)^Z = 4$ and $(\mathcal{C} \cdot T)^Y = 2$. Since T is globally fixed, these two intersection points are opposite each other. The result in Z is very simple: $\mathcal{C}$ must be a conic tangent to E_2 at the image of G (or T). It could happen that $\mathcal{C}$ is tangent to E_8 (producing $\mathbb{A}_{17}$) or not (producing $\mathbb{A}_{15}$). As we can see, no chance is left for the case $\mathbb{A}_{19}$ by Bezout Theorem.

The first case will be denoted by $\varepsilon = 2$, whereas the second by $\varepsilon = 1$. The other singular points of $\mathcal{C} \subset \mathbb{P}^2$ should come from $\mathcal{C} \cap R \subset Z$. Let us denote such points by $P_1, \ldots, P_k$, where $(\mathcal{C} \cap R)^Z_{P_j} = m_j$, $m_1 \geq \ldots, \geq m_k$ and $\sum_{j=1}^k m_j = 4$. Such a configuration for $\mathcal{C}$ will be denoted by $[(\varepsilon), m_1, \ldots, m_k]$. Note that the inverse image of P_j is a singular point of $\mathcal{C}$ of type $\mathbb{A}_{m_j-1}$, $j = 1, \ldots, k$, where $\mathbb{A}_0$ means the type of a smooth point. These are the possible configurations of singular points for a curve with total Milnor number 18 or greater:

- $[(2), 2, 1, 1]$, giving $\mathbb{A}_{17} + \mathbb{A}_1$.
- $[(2), 3, 1]$, giving $\mathbb{A}_{17} + \mathbb{A}_2$.
- $[(1), 4]$, giving $\mathbb{A}_{15} + \mathbb{A}_3$.

Next, we give equations for the arrangement $\tilde{\mathcal{C}}$ in Z. By fixing three points in the projective plane Z, one can assume that

$$E_8 = \{x = 0\}, \quad E_2 = \{y = 0\}, \quad R = \{xz - y^2\} = 0.$$

Fixing a fourth point in Z, one can also assume that $[1 : 1 : 1] \in \mathcal{C} \cap R$ has the biggest contact order.

Let us consider the case $\mathbb{A}_{15} + \mathbb{A}_3$. We look for a conic tangent to E_2 and having a contact of order 4 with R at $[1 : 1 : 1]$. A straightforward computation gives the following equation for $\mathcal{C}$:

$$8\,y^2 - 4\,yx - 4\,yz + x^2 - 2\,xz + z^2 = 0.$$

This case is not essential since the curve obtained is in the closure of the corresponding family in §5.

Let us consider the cases $\mathbb{A}_{17} + \mathbb{A}_n$, $n = 1, 2$. We look for a conic tangent to E_2 and E_8 and having a contact of order $n+1$ with R at $[1 : 1 : 1]$. The case $n = 2$ is obtained by degeneration of $n = 1$. One has a one-parametric family of equations for $\mathcal{C}$ satisfying $n = 1$:

$$(s+3)^2\,y^2 - 2\left(s^2+s+2\right)yx - 2\,(s+3)\,yz + s^2x^2 - 2sxz + z^2 = 0,$$

where $s \in \mathbb{C} \setminus \{-1, 3\}$. The value $s = -1$ produces a reducible conic and $s = 3$ results in the case $n = 2$, $(\mathbb{A}_{17} + \mathbb{A}_2)$:

$$9\,x^2 - 28\,xy - 6\,xz + 36\,y^2 - 12\,yz + z^2 = 0.$$

Remark 6.1. In order to calculate the rational map $\rho = \kappa \circ \tau \circ \sigma^{-1} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ one can proceed as in the previous section. Note that $\rho^* E_8 = 2Q$, $\rho^* E_2 = Q + 2T$ and $\rho^* L_3 = \tilde{L}_3$ where $\tilde{L}_3$ is a quartic satisfying $(\tilde{L}_3 \cdot T)_E^{\mathbb{P}^2} = (\tilde{L}_3 \cdot R)_E^{\mathbb{P}^2} = 4$ and $(\tilde{L}_3 \cdot Q)_E^{\mathbb{P}^2} = 8$. One can fix three points on $\mathbb{P}^2$ so that $T = \{x = 0\}$, $R = \{y = 0\}$ and $Q = \{xz - y^2 = 0\}$. Hence

$$\rho(x, y, z) = [\alpha_1(xz - y^2)^2 : \alpha_2 x^2(xz - y^2) : \alpha_3 x^4 - \alpha_4 y^2(xz - y^2)].$$

Fixing a fourth point one can impose, for instance, $\rho(1, 1, 0) = [1 : -1 : 0]$ and therefore $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$. Thus

$$\rho(x, y, z) = [(xz - y^2)^2 : x^2(xz - y^2) : x^4 - y^2(xz - y^2)].$$

This allows us to give equations:

$$\left(x(y^2 - xz)(s+3) + y^2zs - xz^2s + x^3\right)^2 - 4(s+1)^2(xz - y^2)^3 = 0,$$

where $\mathbb{A}_{17} = [0 : 0 : 1]$, $\mathbb{A}_n = [1 : 0 : 1]$. Then the maximizing case $\mathbb{A}_{17} + \mathbb{A}_2$ is obtained for

$$C : \left(3\,y^2z - x^3 - 3\,z^2x\right)^2 - 64(xz - y^2)^3 = 0.$$

Remark 6.2. Let us remark that if we consider the family of curves in this section having a singular point of type $\mathbb{A}_{15}$, its dimension is exactly one less than the dimension of the whole space of curves of this type. This is rather natural since one condition is imposed: six infinitely near points of the curve in $\mathcal{C}$. We may look for all the curves with this property and a singular point of type $\mathbb{A}_{16}$. A priori, this space should have exactly one dimension less than the preceding one. But we have seen that this space is empty. If we impose the condition to have an $\mathbb{A}_{16}$ singularity, in fact we find a $\mathbb{A}_{17}$ singularity, and moreover, the dimension of this space is the expected one (despite of the a priori extra condition on the conic).

Theorem 6.3. *The fundamental group of the curves of type* $\mathbb{A}_{17} + \mathbb{A}_n$*,* $n = 1, 2$ *in this section are isomorphic to* $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$*.*

Proof. Let us consider the arrangement of curves in Z. It is easily seen that the conic $\mathcal{C}$ degenerates into a double line tangent to R preserving along the way the intersection multiplicities with R. If we consider the curve for the case $n = 2$, the intersection of the two conics consists of two points, with multiplicities 1 and 3. This arrangement of curves can be deformed in order to have these points close enough. The point of simple intersection produces a commutation relation for suitable meridians of the conics. The relation provided by the other point is then a consequence of this one. Thus, deforming the conic $\mathcal{C}$ in a suitable way, one can prove that the fundamental group of the arrangement of curves in Z is the same for $n = 0, 1, 2$. Since the fundamental group of $\mathbb{A}_{17} + \mathbb{A}_n$ is obtained from this one after factoring by some elements and taking an index 2 subgroup, we conclude that the groups in the statement (and also for $n = 0$) are isomorphic. Then, we can compute it directly for $\mathbb{A}_{17}$ with the method given in the proof of Proposition 4.2. The authors know that A. Degtyarev made the computations in an unpublished work. □

This corollary provides a new answer to Degtyarev's conjecture [3], answered by H. Tokunaga [10].

Corollary 6.4. *The space $\mathscr{M}(\mathbb{A}_{17}, \mathbb{A}_1)$ consists of two connected components which provide examples of irreducible rational Zariski pairs.*

7. Case $(\mathcal{C} \cdot T)_E = 4$, $(\mathcal{C} \cdot Q)_E = 10$ and G infinitely near to E

We present a short version of this case for the sake of completeness and because it provides curves with more symmetries than the general ones. Since there is an additional condition, we will not get here curves with Milnor number 19. In fact all the curves in this section are in the closure of the families obtained in §5. The exceptional components of X in this case are $E_1, \dots, E_8, E_9 = G$.

We consider the cubic with a node at S an belonging to the pencil: one branch is transversal and the other has order of contact eight with the general cubic F. Then

$$\check{S} = S + \sum_{j=1}^{8} E_j$$

and hence $\check{S}$ is the only reducible fiber of π. Its type is I_9. The group $MW(X)$ is in this case generated by $\operatorname{Pic}(X) = \langle E_1, \dots, E_8, G, H \rangle$. The kernel of φ is generated by $E_1, \dots, E_8, G$ and S, where

$$S \sim 3H - \sum_{j=1}^{8} (j+1) E_j - 9G.$$

Thus, $MW(X)$ is cyclic of order three and its elements are G, T, Q satisfying $G \overset{MW}{=} 0$ and $Q \overset{MW}{=} 2T \overset{MW}{=} -T$.

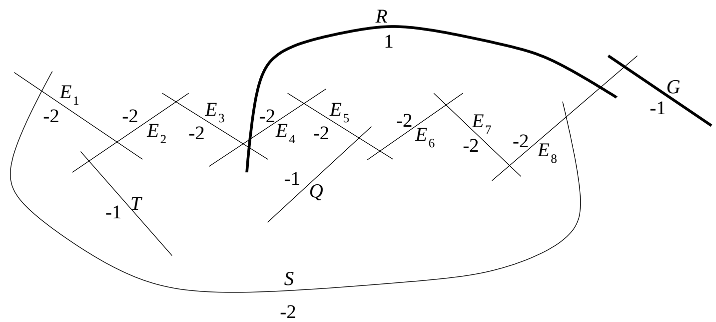

FIGURE 10. Curves in X.

According to Figure 13, one has $\mu(E_8) = E_8$, $\mu(S) = E_7$ and $\mu(E_j) = E_{7-j}$, $j = 1, 2, 3$. These components are globally fixed (not pointwise fixed). Let R be the curve of the 2-torsion points. Since MW has no 2-torsion, R is irreducible. The sequence of multiplicities of R at E is $(3_3, 2, 1_4, 0)$, where M_n represents an n-tuple of M's. Thus R is the strict transform of a sextic curve. Also note that $(R \cdot F)^X = 3$.

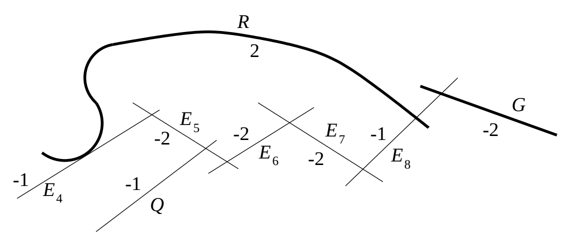

FIGURE 11. Curves in Y.

The Euler characteristic of the quotient Y is 8. Hence, after blowing-down the following 5 curves: $Q(= T), E_4, E_8, E_7, E_6$, a surface Z isomorphic to $\mathbb{P}^2$ is produced. The image of R in Z is a cuspidal cubic whose singular point is at E_4, G is the tangent line to the inflection point E_8 and E_5 is the line joining E_8 and E_4.

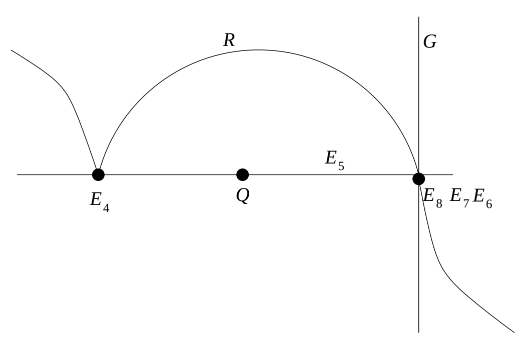

FIGURE 12. Curves in Z.

Let us study what the curve $\mathcal{C}$ is transformed into. Keeping track of its self-intersection accordingly to the sequence of blowing-ups and blowing-downs one can see that $\mathcal{C}$ becomes a cubic curve with a double point at Q. Also one can check that its intersection number with R at E_8 (an inflection of R) is at least 3 (thus E_8 is also an inflection point of $\mathcal{C}$). Let $E_8 = P_0, P_1, \ldots, P_k$ be the points in $\mathcal{C} \cap R \subset Z$ with contact orders $3+m_0$, $m_1 \geq \cdots \geq m_k$. Note that $\sum_{i=0}^k m_i = 6$. We will denote this situation by $[m_0, m_1, \ldots, m_k]$. Note that the inverse image of P_0 is a singular point of $\mathcal{C} \subset \mathbb{P}^2$ of type $\mathbb{A}_{15+m_0}$, while the inverse image of P_j, $j \geq 1$ is of type $\mathbb{A}_{m_j-1}$ (compare with proposition 5.4).

Let us explain the general procedure to obtain equations in Z. Consider the pencil of cubics generated by R and $\mathcal{C}$ in the projective plane Z. Let G' be a conic intersecting $\mathcal{C}$ at P_j with multiplicity m_j, $j = 0, \ldots, k$. By Proposition 3.2, $G + G'$ is an element of the pencil. In other words, $\mathcal{C}$ is an element of the pencil generated by R and $G + G'$ satisfying that $\mathcal{C}$ is singular at Q. Fixing three points in Z one can choose coordinates so that

$$R = \{x^2 z - y^3 = 0\}, \quad E_5 = \{y = 0\}, \quad G = \{z = 0\}.$$

Since $[1:1:1]$ is neither on E_5 nor on G, one can assume that $P_1 = [1:1:1]$. Let $\varphi : t \mapsto [1:t:t^3]$ be the parametrization of $\operatorname{Reg}(R)$ given in section 4 and $t_j := \varphi^{-1}(P_j)$, $j = 0, \ldots, k$. Note that $t_0 = 0$. Hence the equation of G' is of type $f_{G'}(x, y, z) = 0$, such that the coefficient of z^2 is equal to 1 and satisfies:

$$\varphi^{-1}(G') = \prod_{j=0}^{k} (t - t_j)^{m_j} \quad \text{and} \quad \sum_{j=1}^{k} m_j t_j = 0.$$

The following is a list of all existing configurations for $\mathcal{C} \subset Z$: $[0, 4, 1, 1]$, giving $(\mathbb{A}_{15} + \mathbb{A}_3)$ (one solution); $[1, 3, 1, 1]$, giving $(\mathbb{A}_{16} + \mathbb{A}_2)$ (two conjugate solutions in $\mathbb{Q}(\sqrt{13})$); $[1, 2, 2, 1]$, giving $(\mathbb{A}_{16} + \mathbb{A}_1 + \mathbb{A}_1)$ (one solution); $[2, 2, 1, 1]$, giving $(\mathbb{A}_{17} + \mathbb{A}_1)$ (one solution); $[3, 1, 1, 1]$, giving $(\mathbb{A}_{18})$ (one solution).

Remark 7.1. Using the procedure explained in the previous sections one can give suitable coordinates in $\mathbb{P}^2$ so that the rational map $\rho = \kappa \circ \tau \circ \sigma^{-1} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is defined by

$$\rho(x, y, z) = \\ = [4q(x, y, z) : 4vx(xz - y^2)(xyz + x^3 - y^3) : -v^3(xyz + x^3 - y^3)^2],$$

where $Q = [4 : 0 : v^3]$ and

$$q(x, y, z) := -y^6 + 3\,y^4xz + y^3x^3 - 3\,y^2x^2z^2 - yx^4z - x^6 + x^3z^3.$$

For example, we may obtain:

- $\mathbb{A}_{17} = [0 : 0 : 1]$, $\mathbb{A}_1 = [1 : 0 : -1]$:

$$x^2z^4 - 2\,z^3y^2x + y^4z^2 + 3\,y^2x^2z^2 + 6\,x^3z^2y - 6\,y^4xz - 10\,y^3x^2z + \\ + 6\,yx^4z + 4\,x^5z + 3\,y^6 + 4\,xy^5 - 6\,y^3x^3 - 3\,y^2x^4 + 3\,x^6 = 0.$$

- $\mathbb{A}_{18} = [0 : 0 : 1]$:

$$x^2z^4 - 2\,z^3y^2x + y^4z^2 + 6\,x^3z^2y - 10\,y^3x^2z + 4\,x^5z + 4\,xy^5 - 3\,y^2x^4 = 0.$$

The simplicity of these equations makes easier the direct computation of the fundamental groups. They turn out to be abelian.

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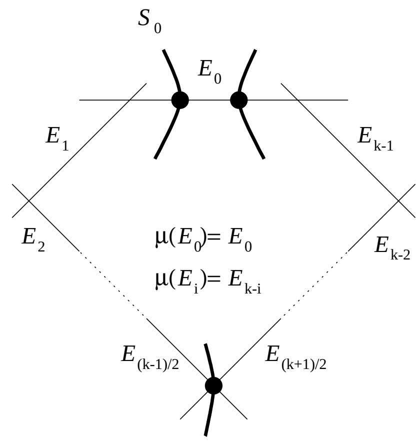

FIGURE 13. Fiber I_k for k odd.

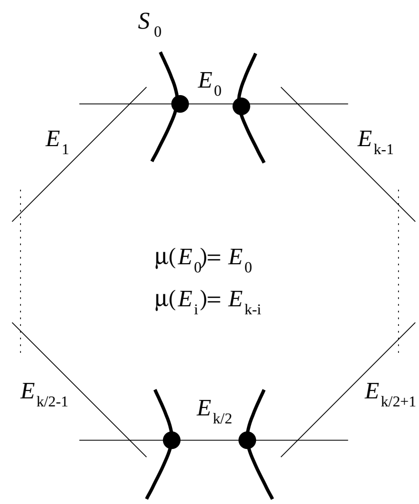

FIGURE 14. Fiber I_k for k even.

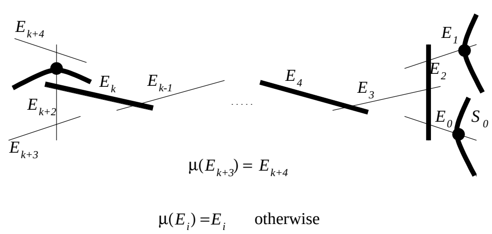

FIGURE 15. Fiber I_k^* for k odd.

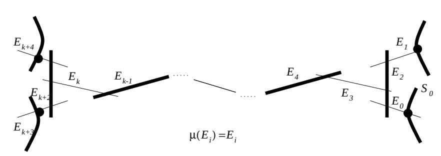

FIGURE 16. Fiber I_k^* for k even.

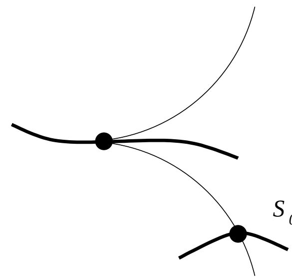

FIGURE 17. Fiber II.

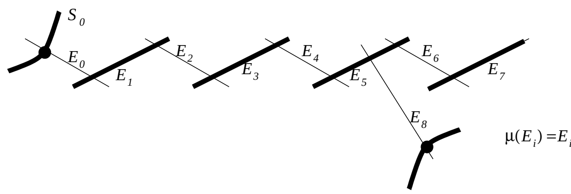

FIGURE 18. Fiber II^*.

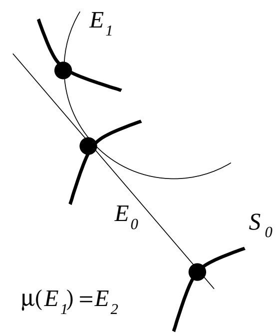

FIGURE 19. Fiber III.

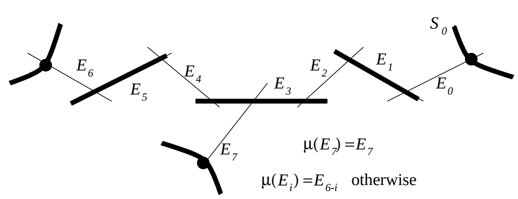

FIGURE 20. Fiber III^*.

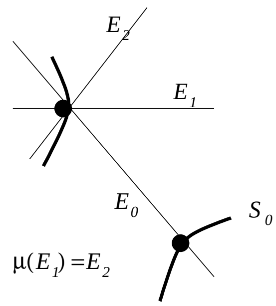

FIGURE 21. Fiber IV.

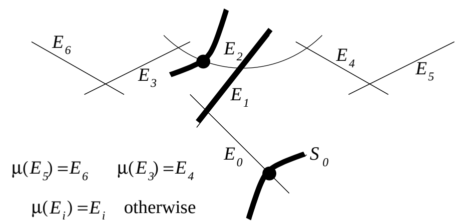

FIGURE 22. Fiber IV^*.