

# Boost operators in Coulomb gauge QCD: The pion form factor and Fock expansions in $\phi$ radiative decays

M. Gómez Rocha<sup>1</sup>, F.J. Llanes-Estrada<sup>2,a</sup>, D. Schütte<sup>3</sup>, and S. Villalba-Chávez<sup>1</sup>

<sup>1</sup> Institut für Physik, Universität Graz, Universitätsplatz 5, A-8010 Graz, Austria

<sup>2</sup> Departamento de Física Teórica I, Universidad Complutense, 28040 Madrid, Spain

<sup>3</sup> Universität Bonn, Institut für Theoretische Kernphysik, Nussallee 14-16, D-53115 Bonn, Germany

Received: 23 December 2009 / Revised: 11 February 2010

Published online: 28 April 2010 – © Società Italiana di Fisica / Springer-Verlag 2010

Communicated by V. Vento

**Abstract.** In this article we re-derive the Boost operators in Coulomb Gauge Yang-Mills theory employing the path integral formalism and write down the complete operators for QCD. We immediately apply them to note that what are usually called the pion square, quartic ... charge radii, defined from derivatives of the pion form factor at zero squared momentum transfer, are completely blurred out by relativistic and interaction corrections, so that it is not clear at all how to interpret these quantities in terms of the pion charge distribution. The form factor therefore measures matrix elements of powers of the QCD boost and Möller operators, weighted by the charge density in the target's rest frame. In addition we remark that the decomposition of the  $\eta'$  wave function in quarkonium, gluonium, ..., components attempted by the KLOE Collaboration combining data from  $\phi$  radiative decays, requires corrections due to the velocity of the final-state meson recoiling against a photon. This will be especially important if such decompositions are to be attempted with data from  $J/\psi$  decays.

## 1 Introduction

### 1.1 The pion form factor

Form factors traditionally encode the structure of a composite target as accessible from elastic reactions. For a scalar target, we can denote the elastic form factor simply as  $F(Q^2)$ , where  $Q^2 = -q^2$ ,  $q = p - p'$ . It is conventionally normalized as  $F(0) = \mathcal{Q}$ , the particle's charge.

For a non-relativistic particle with unit charge ( $\mathcal{Q} = 1$ ,  $q^0 \simeq 0$ ), an expansion around zero momentum transfer allows for a physical interpretation of the form factor in terms of its rest frame charge density  $\rho(r)$ , given by

$$F(Q^2) = 1 - \frac{1}{3!} \langle r^2 \rangle_\rho Q^2 + \frac{1}{5!} \langle r^4 \rangle_\rho Q^4 + \mathcal{O}(Q^6). \quad (1)$$

Here we used the notation of ref. [1]. In hadron physics one often quotes also the curvature of the form factor [2] via

$$F(Q^2) = 1 - \frac{1}{6} \langle r^2 \rangle Q^2 + \mathcal{C}_V^\pi Q^4 + \mathcal{O}(Q^6). \quad (2)$$

Comparing with eq. (1) we see that  $\mathcal{C}_V^\pi = \frac{1}{5!} \langle r^4 \rangle$ .

Beyond the charge normalization  $\int d^3r \rho(r) = \langle 1 \rangle_\rho = 1$ , the derivative at the origin provides, in non-relativistic

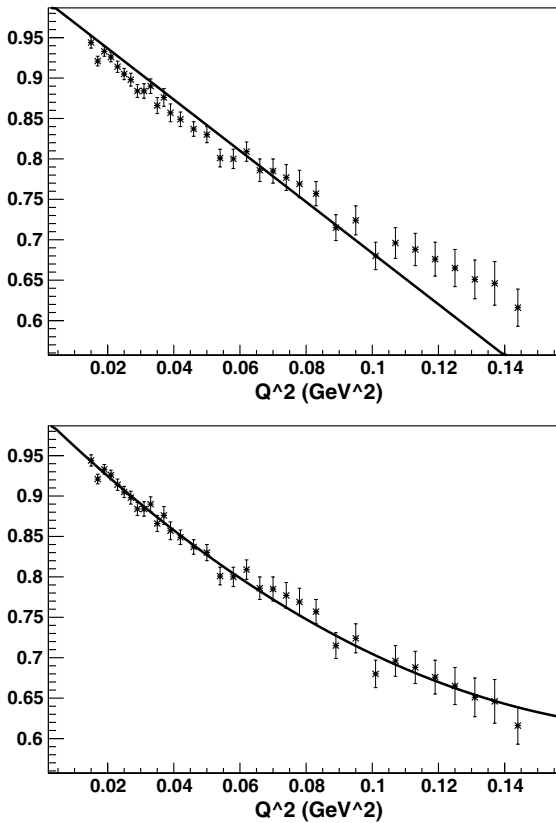
quantum mechanics, the target charge radius

$$\langle r^2 \rangle = -6 \left. \frac{dF}{dQ^2} \right|_{Q=0}. \quad (3)$$

It is known from nuclear physics, but sometimes ignored in the hadron physics community, that the non-relativistic interpretation should be modified as effects of boosting the pion wave function begin to appear. Since the pion is such a light target,  $1/m_\pi^2$  corrections are even larger than the actual  $\langle r^2 \rangle$  measured experimentally or computed on a lattice.

The situation is even worse for the curvature, that has been the object of our recent focus. Using both chiral perturbation theory and dispersion relations, we have found a reliable value for the curvature of  $4.0 \pm 0.5 \text{ GeV}^{-4}$ . Many other authors quoted in [3] have also dedicated time to extracting this curvature. It behooves us to examine whether this number —and the more common  $\langle r^2 \rangle$ , have any interpretation in terms of the target wave functions. We show, indeed, that this is far from trivial. In fig. 1 we re-plot the form factor data from the NA7 experiment at CERN [4], omitting the few points at higher momentum transfer (as they have large error bars that might still be underestimated). One can then obtain a direct fit that is sensitive to the “square radius” and curvature. From the parabolic fit, one can derive  $F'(0) = -2.017 \pm 0.047 \text{ GeV}^{-2}$ , or in units

<sup>a</sup> e-mail: fllanes@fis.ucm.es



**Fig. 1.** Linear and parabolic fit to the elastic pion form factor at small momentum transfer. The linear fit has  $\chi^2/\text{dof} \simeq 77/32$ . The parabolic fit has  $\chi^2/\text{dof} \simeq 22/32$ .

of square radius,  $\langle r^2 \rangle = 0.48 \pm 0.01 \text{ fm}^2$ . The curvature is less well determined,  $\mathcal{C}_V^\pi \simeq 8 \pm 1 \text{ GeV}^{-4}$ . Adding a small systematic error to the data the uncertainty in the curvature could be as high as  $2 \text{ GeV}^{-4}$ , still somewhat higher than the result obtained from the Omnès representation quoted above.

Several works in atomic and nuclear physics have pointed out the necessity of  $1/m$  corrections when talking about the charge radius of a composite target as extracted from electron scattering or from the isotope shift [5], for example the so-called Darwin-Foldy terms [6]. In general much of the literature features idle discussion about where the various effects are to be ascribed, to the radius, or to its extraction from electron scattering, or to final-state rescattering, etc. In the end, a lot of the discussion is linguistic. In nuclear physics one is not quite yet in possession of a full Hamiltonian for the few-nucleon problem, so the corrections are worked out on a semiempirical basis [7] and in atomic physics the corrections due to both relativity and interactions are small, and the first-order ones can be accessed by means of simple quantum-mechanical reasoning, although some works exist providing first-principles computations [8].

In hadron physics, however, the situation is radically different. Corrections are large, since the energies involved match particle masses (the most salient example being the pion) and interactions are strong. In addition one is in

possession of the Hamiltonian of the fundamental theory, QCD, and in this article we seek the first brick in the direction of gaining understanding of form factors in terms of the canonical quantization of the theory in Coulomb gauge.

Much is known already in light front [9] and point form quantization [10,11], with recent insight from AdS-QCD [12]. However, a good discussion of boost operators of QCD in a phenomenological context is nowhere to be found in the literature for canonical, Coulomb gauge quantization. In related quark models there has been indeed interesting work within the Bakamjian-Thomas construction [13]. Here we attempt to continue stimulating the discussion and bring it a step closer to QCD.

## 1.2 $\phi$ radiative decays and Fock space expansion

The KLOE Collaboration [14,15] has proposed to employ their excellent data set of radiative  $\phi$  decays to pseudoscalar mesons,  $\phi \rightarrow \gamma\eta$ ,  $\phi \rightarrow \gamma\eta'$ , to measure the gluonium content of the  $\eta'$  as well as the pseudoscalar mixing angle. The theoretical assumptions behind the analysis are simple considerations about the flavor structure of the couplings of gluonium, quarkonium, etc.

While there is no objection to the model analysis of the pseudoscalar mixing angle [16], we believe that a point has been missed by the community in what regards the gluonium content of  $\eta'$ . This is the fact that the Fock space expansion for either of the  $\eta$  or  $\eta'$

$$|\eta\rangle = c_1|q\bar{q}\rangle + c_2|g\bar{g}\rangle + c_3|q\bar{q}g\rangle + \dots \quad (4)$$

is dependent on the reference frame. When one makes an assumption about the gluonium ( $g\bar{g}$  and larger number of gluons) content of a meson, one is presumably referring to the rest frame of the meson, since the concept of quantum-mechanical state requires a quantization surface that is presumably taken to be  $t = 0$  in the rest frame. But, although the  $\eta'$ -meson is produced non-relativistically in  $\phi$  radiative decays, the analysis hinges also on decays to the  $\eta$ -meson, and this is produced with a velocity  $v \simeq 0.55$ .

As we will show, even if there was no gluonium at all in either of the two mesons, one should expect a gluonium content of the  $\eta$ -meson boosted to such reference frame of order  $\sin(\phi_P)v^2/2 \simeq 0.1$ , given that the pseudoscalar mixing angle seems to be close to 38 degrees. This is the well-known effect that the boost operators involve interactions and change the particle content.

This is not necessarily a fatal flaw of the analysis, but a call of attention that, while effective hadron Lagrangians are explicitly Lorentz invariant, such Fock space decompositions are tied to a reference frame where canonical QCD quantization is carried out, and this needs to be specified and consistently handled.

The necessity of specifying a reference frame will become more acute as the natural next step in the analysis [17–19] will be to employ the data basis for radiative  $J/\psi$  decays accumulated by BES and others, and given the mass of 3097 MeV of the charmonium ground state,

the velocity of the produced mesons will now be decidedly relativistic. For  $J/\psi \rightarrow \gamma\eta$ ,  $v_\eta = 0.94$ , and for  $J/\psi \rightarrow \gamma\eta'$ ,  $v_{\eta'} = 0.83$ , that are indeed significant.

The rest of this paper is organized as follows. In sect. 2 we motivate the necessity of constructing the Boost operators with a careful setup of the form factor for a scalar target. Then in sect. 2.1 we particularize to the case of the pion, the lightest hadron where relativistic corrections are most significant. Section 2.2 is then dedicated to dispelling some misconception about the charge radius interpreted in the Breit frame.

Although pieces thereof can be found in the literature, it is convenient to write down the boost operators of QCD in Coulomb gauge for future reference, which we do in sect. 3. There we also present a relatively easy derivation based on modern functional methods in sects. 3.1, 3.2 and 3.3.

Finally, sect. 4 presents the application of these boost operators to  $\phi \rightarrow \eta\gamma$  (or similar decays) where it has been until now ignored, and sect. 5 presents our final remarks.

## 2 Rigorous interpretation of the form factor in canonically quantized field theory

The form factor can be expressed, in non-relativistic normalization for the charge density, as

$$\text{out}\langle p' | j^\mu | p \rangle_{\text{in}} = \frac{(p + p')^\mu}{2\sqrt{EE'}} F(q^2). \quad (5)$$

The energy momentum  $p^\mu = (m_\pi, \mathbf{0})$  corresponds to a pion at rest and time  $t = -\infty$  before the momentum transfer by the virtual photon,  $q^2$ , boosts it to a frame with  $p'$ .

The charge density is taken at  $t = 0$ , that we employ as quantization surface to define the equal-time commutation and anticommutation relations for field operators. To undertake any computation one needs to propagate the initial state  $|p\rangle_{\text{in}}$  to  $t = 0$ . The propagators that accomplish this operation are usually called Moeller operators in the interaction picture,  $\Omega_- = U(t = -\infty, 0)$ . Likewise the propagation of the final state to the quantization surface is carried out by  $\Omega_+^\dagger = U^\dagger(t = 0, \infty)$ . The product of both operators would reconstruct the  $S$ -matrix,  $\Omega_+^\dagger \Omega_- = S$ , but here an insertion of the current operator occurs at time  $t = 0$ , see eq. (7) below.

The final-state pion is boosted, and to be able to use its wave function in the rest frame (where the charge density is defined) one needs to employ the boost  $\hat{\mathbf{K}}$  operator from QCD

$$\text{in}\langle p' | = \text{in}\langle p | e^{-i\hat{\mathbf{K}} \cdot \boldsymbol{\zeta}} \quad (6)$$

that we construct in sect. 3 below. In terms of these operators the form factor can be expressed via

$$\text{in}\langle p | e^{-i\hat{\mathbf{K}} \cdot \boldsymbol{\zeta}} \Omega_+^\dagger j^\mu(0) \Omega_- | p \rangle_{\text{in}} = \frac{(p + p')^\mu}{2\sqrt{EE'}} F(q^2) \quad (7)$$

(the Moeller operators introduce interactions in both sides of the current insertion). An alternative form of  $\Omega_+^\dagger j^\mu(0) \Omega_-$  is, for example,  $T(j^\mu(0) e^{i \int d^4x \mathcal{L}_I})$ .

The interpretation of the form factor in non-relativistic quantum mechanics (where it comes about in the Born approximation by expanding the Fourier transform of the potential causing the scattering)

$$F_{\text{NonRel}}(|\mathbf{q}|^2) = \mathcal{Q} - \frac{1}{3!} |\mathbf{q}|^2 \langle r^2 \rangle + \frac{1}{5!} |\mathbf{q}|^4 \langle r^4 \rangle \quad (8)$$

in terms of charge radii, is only a limiting case that can be recovered from eq. (7) and we will do it shortly.

Let us now obtain an equivalent  $q^2/m_\pi^2$  expansion of eq. (7) in powers of the momentum transfer to match eq. (8). The rapidity parameter in eq. (6) depends on the transferred momentum through

$$v = \tanh \zeta \Rightarrow \zeta = \frac{1}{2} \log \left( \frac{1+v}{1-v} \right). \quad (9)$$

From the above equation we can approximate

$$\zeta \simeq v + \frac{v^3}{3} + \dots O(v^5) \quad \text{with } v^2 \simeq -\frac{q^2}{m_\pi^2} - \frac{3}{4} \frac{q^4}{m_\pi^4}. \quad (10)$$

Expanding also  $E'$  in terms of  $q$ , and making  $\mathbf{q} \propto \mathbf{n}_z$  ( $\mathbf{n}_z = \mathbf{k}_z/|\mathbf{k}_z|$ ) infinitesimal so that one can truncate the Taylor expansion of the boost

$$\exp(i\hat{K}_z \zeta_z) = 1 + i\hat{K}_z \zeta_z - \frac{1}{2} \hat{K}_z^2 \zeta_z^2 - \frac{i}{3!} \hat{K}_z^3 \zeta_z^3 + \frac{1}{4!} \hat{K}_z^4 \zeta_z^4 + \dots \quad (11)$$

we obtain

$$\begin{aligned} F(q^2) &= \frac{2\sqrt{EE'}}{(p + p')^0} \langle p | e^{-i\hat{K}_z \zeta_z} \Omega_+^\dagger j^0(0) \Omega_- | p \rangle \\ &\simeq \left( 1 + \frac{q^2}{2m_\pi^2} + \frac{q^4}{32m_\pi^4} \right) \\ &\times \langle p | \left( 1 + \frac{\hat{K}_z^2}{2m_\pi^2} q^2 + \frac{\hat{K}_z^2}{24m_\pi^4} q^4 + \frac{\hat{K}_z^4}{4!m_\pi^4} q^4 \right) \\ &\times \Omega_+^\dagger j^0 \Omega_- | p \rangle \end{aligned} \quad (12)$$

(terms odd in  $\hat{K}_z$  are absent for a scalar target because of  $z \rightarrow -z$  reflection symmetry). We then have

$$F(q^2) \simeq F(0) + q^2 F'(0) + \frac{1}{2!} q^4 F''(0) + \dots, \quad (13)$$

where the derivatives are now

$$\begin{aligned} F(0) &= \langle p | \Omega_+^\dagger j^0 \Omega_- | p \rangle, \\ F'(0) &= \frac{1}{2m_\pi^2} F(0) + \frac{1}{2m_\pi^2} \langle p | \hat{K}_z^2 \Omega_+^\dagger j^0 \Omega_- | p \rangle, \\ F''(0) &= \frac{1}{12m_\pi^4} \langle p | \hat{K}_z^4 \Omega_+^\dagger j^0 \Omega_- | p \rangle - \frac{25}{48m_\pi^4} F(0) + \frac{7}{6m_\pi^2} F'(0). \end{aligned} \quad (14)$$

## 2.1 No square radius interpretation for the pion

To reproduce eq. (8), one first invokes the impulse approximation, by which the quarks from the  $j^\mu$  current density directly enter the state's wave function, in effect neglecting the Moeller operators (rescattering). This is known from nuclear physics to be a poor approximation, and many works in hadron physics have also lifted it [20] since it fails, for example, to reproduce simple instances of vector meson dominance.

In a second step, one takes the non-relativistic limit by sending the mass denominators  $m_\pi \rightarrow \infty$ , and substitutes the Lorentz boost by its Galilean equivalent. Since a Galilean change of reference frame is generated by

$$U[\mathbf{v}] = \exp(-im_\pi \hat{\mathbf{r}} \cdot \mathbf{v}), \quad (15)$$

the relevant boost operator in quantum mechanics becomes  $\hat{\mathbf{K}} \rightarrow \hat{\mathbf{K}}_{\text{Galilean}} = -m_\pi \hat{\mathbf{r}}$ . Upgrading it to non-relativistic field theory for quarks, and summing over spin, color and flavor, the boost operator is then

$$\hat{\mathbf{K}}_{\text{Galilean}} = \sum_\ell \int d^3x \hat{q}^{\ell\dagger}(x) (-m_{q_\ell} \mathbf{x}) \hat{q}^\ell(x). \quad (16)$$

(In terms of quadrispinors including antiparticles, there would be an additional Dirac  $\beta$ -matrix  $(-im_{q_j})\beta\partial/\partial k^i$  in the operator).

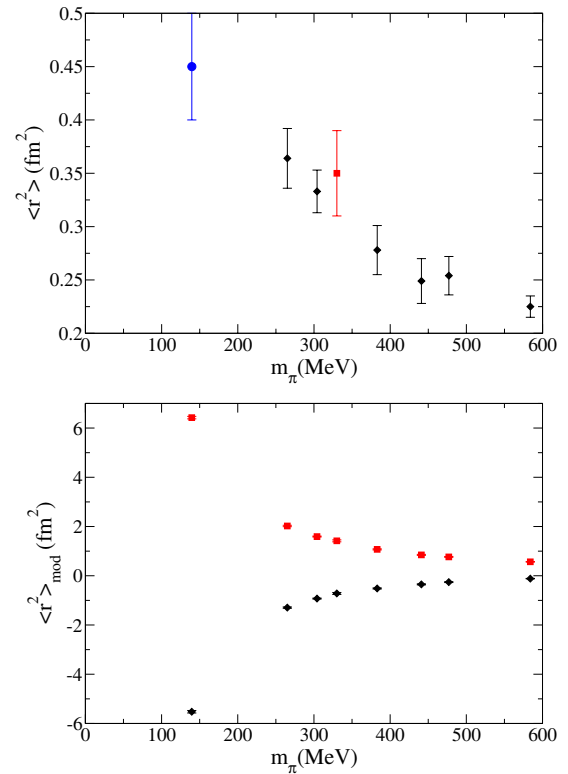
In this case, one obtains for the form factor a non-relativistic expression in terms of the wave function of the target. If the target  $|H\rangle$  is taken as a two-body scalar with non-relativistic wave function  $\mathcal{F}(r)$ , and two particles of mass  $m_1, m_2$  in the center-of-mass frame, with relative position  $\mathbf{r}$ , and opposite charge (as corresponds for example to the non-relativistic Hydrogen atom or the neutral  $\pi_0$  in the quark model)

$$\langle r^2 \rangle = e \int d^3x r^2 |\mathcal{F}(r)|^2 \frac{m_1 - m_2}{m_1 + m_2}. \quad (17)$$

However, returning to eq. (14) for the case of the pion,  $1/2m_\pi^2 \simeq 26 \text{ GeV}^{-2}$  is not negligible in any sense against  $-F'(0) \simeq 2 \text{ GeV}^{-2}$  taken from the experimental data quoted above. There need to be large cancellations among the two terms in the second of eq. (14) (see fig. 2). Its second non-constant term, that in the non-relativistic limit would correspond to the square radius, is therefore of order  $20 \text{ GeV}^{-2}$ . One needs to conclude that, in the case of the pion, there is no connection whatsoever between the derivative of the spacelike form factor and the square radius of the rest frame charge distribution.

## 2.2 Breit frame

In the study of the proton form factor, the Breit frame became very popular [21,22]. One reason is that, for a spin-1/2 target, a separation of the charge and magnetization densities is possible by employing the Sachs form factors. The point that concerns us here, the pion being a



**Fig. 2.** Top panel: accepted [23] value of  $-6F'(0)$ , the would-be pion square charge radius, and lattice calculations [24–26] as a function of the pion mass. Bottom panel: the two terms of eq. (14), that are way larger, and a cancelation must occur to give the physical form factor derivative. It is not clear that one can extract the square charge radius, except for large quark masses where one could think of the lattice data in non-relativistic terms.

scalar target, is that the  $1/m_\pi$  terms in eq. (14) for  $F'(0)$  are absent.

The Breit (or brick-wall) frame is defined as that in which the target bounces off the virtual photon with opposite incoming and outgoing momentum. As a consequence,  $E_{\text{Breit}} = E'_{\text{Breit}}$  and the factor  $(p+p')^0/2(EE')^{1/2}$  in eq. (7) and following becomes unity.

Therefore in this frame,

$$F(q^2) = \left\langle +\frac{p}{2} \left| \Omega_+^\dagger j^0(0) \Omega_- \right| -\frac{p}{2} \right\rangle \quad (18)$$

and therefore

$$\begin{aligned} F(0) &= \left\langle -\frac{p}{2} \left| \Omega_+^\dagger j^0 \Omega_- \right| -\frac{p}{2} \right\rangle, \\ F'(0) &= \frac{1}{2m_\pi^2} \left\langle -\frac{p}{2} \left| \hat{K}_z^2 \Omega_+^\dagger j^0 \Omega_- \right| -\frac{p}{2} \right\rangle, \\ F''(0) &= \frac{1}{12m_\pi^4} \left\langle -\frac{p}{2} \left| \hat{K}_z^4 \Omega_+^\dagger j^0 \Omega_- \right| -\frac{p}{2} \right\rangle + \frac{1}{6m_\pi^2} F'(0). \end{aligned} \quad (19)$$

In principle, the large term in  $F'(0)$  proportional to  $F(0)$  in eq. (14) is absent, although the interpretation of the remaining in terms of a square charge radius is still not possible due to the interactions (Moeller operators) and

the necessary change of momentum (and hence frame) of the incoming and outgoing target. Moreover, the curvature in  $F''(0)$  is still affected by  $1/m_\pi^2$  corrections. Therefore there is no clear advantage of interpretation of the lattice or experimental data in employing the Breit frame, at least in the case of the pion.

### 3 Poincaré generators of QCD

Having motivated their phenomenological importance, in this section we derive the generators of the Poincaré transformations in Coulomb gauge QCD. The method we follow is a modern path integral approach in the first-order formalism (with the path integration extending over phase space  $(q, p)$  [27], not only the field coordinates  $q$ ), based on the work of Zwanziger [28] and Watson and Reinhardt [29]. First, we obtain the result for Yang-Mills theory in the functional approach. Then, we derive the result for the boost generators appropriate for canonical quantization, in agreement with [30]. Finally, we add the quarks to complete the Boost operators in canonically quantized QCD.

#### 3.1 The Hamiltonian and momentum generators of a pure Yang-Mills theory

Yang-Mills theory is based on the renormalizable Lagrangian density of a spin-1, color-octet field  $A_\mu^a(x)$ , and written down in a gauge-invariant manner in terms of the Maxwell tensor  $\mathfrak{F}_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c$  as

$$\mathcal{S}_{\text{YM}} = \int d^4x \mathcal{L}_{\text{YM}} \quad \text{with} \quad \mathcal{L}_{\text{YM}} = -\frac{1}{4} \mathfrak{F}_{\mu\nu}^a \mathfrak{F}^{a\mu\nu}. \quad (20)$$

The canonical stress-energy tensor conserved by translational invariance of  $\mathcal{L}_{\text{YM}}$  reads

$$\begin{aligned} \mathcal{T}_{\text{YM}}^{\mu\nu} &= \frac{\partial \mathcal{L}_{\text{YM}}}{\partial(\partial_\mu A_\lambda^a)} \partial^\nu A_\lambda^a - g^{\mu\nu} \mathcal{L}_{\text{YM}} \\ &= -\mathfrak{F}_{\lambda}^{\mu a} \partial^\nu A^{\lambda a} + \frac{1}{4} g^{\mu\nu} \mathfrak{F}_\lambda^{\sigma a} \mathfrak{F}_\lambda^{\sigma a}, \end{aligned} \quad (21)$$

$g^{\mu\nu}$  being the Minkowski metric with signature  $+- - -$ . Associated to space-time translational invariance are the conserved Hamiltonian and momentum

$$H_{\text{YM}} = \int d^3x \mathcal{T}_{\text{YM}}^{00} \quad \text{and} \quad \mathcal{P}_{\text{YM}} = \int d^3x \mathcal{T}_{\text{YM}}^{0i}. \quad (22)$$

In the path integral framework the Green's functions are generated from the functional

$$\mathcal{Z} = \int \mathcal{D}\Phi \exp \{i\mathcal{S}_{\text{YM}}\} \quad (23)$$

( $\Phi$  representing generically an integral over all fields). Since we are interested in exposing the explicit transverse gluons, the degrees of freedom of the intuitive Coulomb

gauge, we divide the Faraday-Maxwell tensor in terms of chromoelectric and chromomagnetic fields

$$\begin{aligned} \mathbf{E}^a &= -\partial^0 \mathbf{A}^a - \nabla A^{0a} + gf^{abc} \mathbf{A}^b A^{0c}, \\ B_i^a &= \epsilon_{ijk} \left[ \nabla_j A_k^a - \frac{1}{2} gf^{abc} A_j^b A_k^c \right]. \end{aligned} \quad (24)$$

The Coulomb gauge is fixed with the help of the Faddeev-Popov mechanism by first introducing a gauge-fixing term in the action

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}\Phi \exp \{i\mathcal{S}_{\text{YM}} + i\mathcal{S}_{\text{FP}}\}, \\ \mathcal{S}_{\text{FP}} &= \int d^3x \left[ -\xi^a \nabla \cdot \mathbf{A}^a - \bar{c}^a \nabla \cdot \mathbf{D}^{ab} c^b \right]. \end{aligned} \quad (25)$$

Here  $c^a$  are Grassmann (ghost) fields that keep track of the Faddeev-Popov determinant for now, involving the covariant-derivative operator  $\mathbf{D}^{ab} = \delta^{ab} \nabla + gf^{abc} \mathbf{A}^c$  and  $\xi^a$  is a Lagrange multiplier to assist with the Coulomb gauge constraint.

As is known, Coulomb gauge fixing is not unique in non-Abelian gauge theories. Gribov [31] was the first to point out that there remain physically equivalent gauge configurations that are related by finite (as opposed to infinitesimal) gauge transformations. A further restriction must be imposed to the configuration space of gauge fields to a region where

$$\Omega \equiv \{ \mathbf{A} : \nabla \cdot \mathbf{A} = 0 \mid -\nabla \cdot \mathbf{D} \geq 0 \}. \quad (26)$$

This simple domain is still not totally free of Gribov copies. Instead one should consider the fundamental modular region  $\Lambda$  [32]. It turns out though that functional integrals are dominated by configurations on the common boundary of  $\Omega$  and  $\Lambda$  [29] so that, in practice, it is enough to consider the domain defined by eq. (26).

The canonical momentum associated to the transverse  $\mathbf{A}^a$  fields is introduced as an auxiliary field thanks to the following identity:

$$\begin{aligned} \exp \left\{ i \int d^4x \frac{1}{2} \mathbf{E}^a \cdot \mathbf{E}^a \right\} &= \\ \int \mathcal{D}\boldsymbol{\pi} \exp \left\{ i \int d^4x \left[ -\frac{1}{2} \boldsymbol{\pi}^a \cdot \boldsymbol{\pi}^a - \boldsymbol{\pi}^a \cdot \mathbf{E}^a \right] \right\}. \end{aligned} \quad (27)$$

Two more auxiliary variables  $\Omega, \tau$  will allow dividing the chromoelectric part of the action into the dynamical transverse potential of Coulomb gauge and a constrained longitudinal part. One needs the further identity

$$\begin{aligned} \text{const} &= \int \mathcal{D}\Omega \delta(\nabla \cdot \boldsymbol{\pi} + \nabla^2 \Omega) \\ &= \int \mathcal{D}\{\Omega, \tau\} \exp \left\{ -i \int d^4x \tau^a (\nabla \cdot \boldsymbol{\pi}^a + \nabla^2 \Omega^a) \right\} \end{aligned}$$

and a change of variables  $\boldsymbol{\pi} \rightarrow \boldsymbol{\pi} - \nabla\Omega$  to arrive at the equivalent form of the Yang-Mills action

$$\mathcal{S}_{\text{YM}} = \int d^4x \left[ -\frac{1}{2} \mathbf{B}^a \cdot \mathbf{B}^a - \tau^a \nabla \cdot \boldsymbol{\pi}^a - \frac{1}{2} (\boldsymbol{\pi}^a - \nabla\Omega^a) \cdot (\boldsymbol{\pi}^a - \nabla\Omega^a) + (\boldsymbol{\pi}^a - \nabla\Omega^a) \cdot (\partial^0 \mathbf{A}^a + \mathbf{D}^{ab} A^{0b}) \right]. \quad (28)$$

We now perform the integral over the ghost fields in eq. (25) to recover the Jacobian of the change of variables. Likewise we integrate over the Lagrange multipliers  $\xi^a$  and  $\tau^a$  to recover the  $\delta$ -functions associated to the constraints, that enforce transversality of the physical fields (the Coulomb gauge condition  $\nabla \cdot \mathbf{A} = \nabla \cdot \boldsymbol{\pi} = 0$ ). Thereafter one can set to zero any terms involving these divergences in the action.

Taking these details into account  $\mathcal{Z}$  can be expressed as

$$\mathcal{Z} = \int \mathcal{D}\Phi \text{Det} [-\nabla \cdot \mathbf{D} \delta^4(x-y)] \delta(\nabla \cdot \mathbf{A}) \times \delta(\nabla \cdot \boldsymbol{\pi}) \exp\{i\mathcal{S}\} \quad (29)$$

with

$$\mathcal{S} = \int d^4x \left[ -\frac{1}{2} \mathbf{B}^a \cdot \mathbf{B}^a - \frac{1}{2} \boldsymbol{\pi}^a \cdot \boldsymbol{\pi}^a + \frac{1}{2} \Omega^a \nabla^2 \Omega^a + \boldsymbol{\pi}^a \cdot \partial^0 \mathbf{A}^a + A^{0a} (\nabla \cdot \mathbf{D}^{ab} \Omega^b + g \varrho_{\mathfrak{g}}^a) \right] \quad (30)$$

and  $\varrho_{\mathfrak{g}}^a = f^{ade} \mathbf{A}^d \cdot \boldsymbol{\pi}^e$  the color-charge density carried by the gluons.

The functional integral over  $A^0$  can also be performed to yield the constraint equivalent to Poisson's equation in quantum electrodynamics in the form of a new  $\delta$ -function on the path integral,  $\delta(-\nabla \cdot \mathbf{D}^{ab} \Omega^b - g \varrho_{\mathfrak{g}}^a)$ . The Coulomb instantaneous potential of QED,  $1/|\mathbf{x} - \mathbf{y}|$ , the inverse of the Laplacian, is generalized in QCD to the inverse of the Faddeev-Popov operator

$$[-\nabla \cdot \mathbf{D}^{ab}] M^{bc} = \delta^{ac}, \quad (31)$$

and the constraint can be formally solved for  $\Omega$ , now taking the place of the scalar potential  $A^0$ ,

$$\Omega^a = g M^{ab} \varrho_{\mathfrak{g}}^b \quad (32)$$

The associated  $\delta$ -function factorizes in a useful way [33]:

$$\delta(-\nabla \cdot \mathbf{D}^{ab} \Omega^b - g \varrho_{\mathfrak{g}}^a) = \text{Det} [-\nabla \cdot \mathbf{D} \delta^4(x-y)]^{-1} \delta(\Omega^a - g M^{ab} \varrho_{\mathfrak{g}}^b) \quad (33)$$

to cancel the determinant in eq. (29) yielding

$$\mathcal{Z} = \int \mathcal{D}\mathbf{A} \mathcal{D}\boldsymbol{\pi} \delta(\nabla \cdot \mathbf{A}) \delta(\nabla \cdot \boldsymbol{\pi}) \exp\{i\mathcal{S}_0\}, \quad (34)$$

where

$$\mathcal{S}_0 = \int d^4x \left[ -\frac{1}{2} \mathbf{B}^a \cdot \mathbf{B}^a - \frac{1}{2} \boldsymbol{\pi}^a \cdot \boldsymbol{\pi}^a - \frac{1}{2} g^2 \varrho_{\mathfrak{g}}^b M^{ba} (-\nabla^2) M^{ac} \varrho_{\mathfrak{g}}^c + \boldsymbol{\pi}^a \cdot \dot{\mathbf{A}}^a \right]. \quad (35)$$

Clearly, the term  $\boldsymbol{\pi}^a \cdot \dot{\mathbf{A}}^a$  is equivalent to  $p\dot{q}$  in classical mechanics, and the remaining part can be identified as the classical Hamiltonian of the pure Yang-Mills theory

$$H_{\text{YM}} = \int \mathcal{H}_{\text{YM}}(\mathbf{x}, t) d^3x, \quad \mathcal{H}_{\text{YM}} = \frac{1}{2} \boldsymbol{\pi}^a(\mathbf{x}, t) \cdot \boldsymbol{\pi}^a(\mathbf{x}, t) + \frac{1}{2} \mathbf{B}^a(\mathbf{x}, t) \cdot \mathbf{B}^a(\mathbf{x}, t) + \frac{1}{2} g^2 \varrho_{\mathfrak{g}}^b M^{ba} (-\nabla^2) M^{ac} \varrho_{\mathfrak{g}}^c. \quad (36)$$

We now take our expressions to the canonical functional quantization formalism. At this point  $\boldsymbol{\pi}$  ceases to be an integration variable and must be considered as the standard classical canonical momentum  $\boldsymbol{\pi}^a = \partial \mathcal{L}_{\text{YM}} / \partial(\dot{\mathbf{A}}^a)$ . After canonical quantization, the transverse field  $\boldsymbol{\pi}^a$  has to be substituted by the momentum conjugate  $\hat{\boldsymbol{\Pi}}^a$  to the transverse field  $\hat{\mathbf{A}}^a$ . This is, however, not trivial, but the problem has already been solved in the past.

Indeed, comparing this Hamiltonian with that of Christ and Lee [34], or in recent papers [35–38], one notices the absence of the Faddeev-Popov determinant in our expression. In order to introduce  $\mathcal{J}[\mathbf{A}] = \text{Det}(-\nabla \cdot \hat{\mathbf{D}})$  we first note that the path integral representation in eq. (34) involves a Cartesian integration measure over gauge field. Let us consider, then, the corresponding quantized version of eq. (36)

$$\hat{\mathcal{H}}_{\text{YM}} = \frac{1}{2} \hat{\boldsymbol{\Pi}}^{a\dagger}(\mathbf{x}) \cdot \hat{\boldsymbol{\Pi}}^a(\mathbf{x}) + \frac{1}{2} \hat{\mathbf{B}}^a(\mathbf{x}) \cdot \hat{\mathbf{B}}^a(\mathbf{x}) + \frac{1}{2} g^2 f^{bde} \hat{\mathbf{A}}^d(\mathbf{x}) \cdot \hat{\boldsymbol{\Pi}}^{e\dagger}(\mathbf{x}) \hat{M}^{ba} (-\nabla^2) \hat{M}^{ac} \hat{\varrho}_{\mathfrak{g}}^c. \quad (37)$$

One is not entitled to use these operators (that we have covered with a tilde) as the canonical momentum since  $\hat{\boldsymbol{\Pi}}^a \neq -i\delta/\delta\hat{\mathbf{A}}^a(\mathbf{x})$  (we repair this below in eq. (43)). In a Hilbert space, where the scalar product of the complex-valued wave functionals  $\tilde{\Psi}[\mathbf{A}]$  is taken as ‘‘Cartesian’’,

$$\langle \tilde{\Psi}_1 | \tilde{\Psi}_2 \rangle = \int \mathcal{D}\mathbf{A} \tilde{\Psi}_1^*[\mathbf{A}] \tilde{\Psi}_2[\mathbf{A}]. \quad (38)$$

(Note that in eq. (37) we have fixed  $t = 0$ .)

We now introduce

$$\Psi_i[\mathbf{A}] \equiv \mathcal{J}^{-1/2}[\mathbf{A}] \tilde{\Psi}_i[\mathbf{A}], \quad (39)$$

which is the wave functional one would find if, as Christ and Lee did, one had first quantized in  $A^0 = 0$  gauge and then transformed the gauge fields to Coulomb gauge, since in this variable change the scalar product picks up the determinant eq. (38)

$$\langle \Psi_1 | \Psi_2 \rangle = \int \mathcal{D}\mathbf{A} \mathcal{J}[\mathbf{A}] \Psi_1^*[\mathbf{A}] \Psi_2[\mathbf{A}] \quad (40)$$

that we would have instead absorbed in the wave functional. However, by doing the change of wave functional eq. (39) one should, at the same time, preserve the expectation value of observables  $\langle \tilde{\Psi}_1 | \hat{\mathcal{O}} | \tilde{\Psi}_2 \rangle = \langle \Psi_1 | \hat{\mathcal{O}} | \Psi_2 \rangle$  with

$$\hat{\mathcal{O}} = \mathcal{J}^{-1/2} \hat{\mathcal{O}} \mathcal{J}^{1/2}. \quad (41)$$

This has to be true, in particular, for the Hamiltonian density. As a consequence

$$\begin{aligned} \hat{\mathcal{H}} &= \mathcal{J}^{-1/2} \hat{\mathcal{H}} \mathcal{J}^{1/2}, \\ \hat{\mathcal{H}} &= \frac{1}{2} \mathcal{J}^{-1/2} \hat{\Pi}^{a\dagger}(\mathbf{x}) \hat{\Pi}^a(\mathbf{x}) \mathcal{J}^{1/2} + \frac{1}{2} \hat{\mathbf{B}}^a(\mathbf{x}) \cdot \hat{\mathbf{B}}^a(\mathbf{x}) \\ &\quad + \frac{1}{2} g^2 \mathcal{J}^{-1/2} \hat{\varrho}_g^b \hat{M}^{ba} (-\nabla^2) \hat{M}^{ac} \hat{\varrho}_g^c \mathcal{J}^{1/2}. \end{aligned} \quad (42)$$

Replacing the operator  $\hat{\Pi}^a(\mathbf{x})$  by the transformed one

$$\begin{aligned} \hat{\Pi}^a(\mathbf{x}) &= \mathcal{J}^{1/2} \hat{\Pi}^a(\mathbf{x}) \mathcal{J}^{-1/2} \\ &= \hat{\Pi}^a(\mathbf{x}) + \frac{i}{2} \frac{\delta \ln \mathcal{J}}{\delta \hat{\mathbf{A}}^a(\mathbf{x})} \end{aligned} \quad (43)$$

we find that the quantum Hamiltonian density features the Faddeev-Popov determinant

$$\begin{aligned} \hat{\mathcal{H}}_{\text{YM}} &= \frac{1}{2} \mathcal{J}^{-1} \hat{\Pi}^a(\mathbf{x}) \mathcal{J} \hat{\Pi}^a(\mathbf{x}) + \frac{1}{2} \hat{\mathbf{B}}^a(\mathbf{x}) \cdot \hat{\mathbf{B}}^a(\mathbf{x}) \\ &\quad + \frac{1}{2} g^2 \mathcal{J}^{-1} \hat{\varrho}_g^b \mathcal{J} \hat{M}^{ba} (-\nabla^2) \hat{M}^{ac} \hat{\varrho}_g^c. \end{aligned} \quad (44)$$

Now, the dynamical operators in eq. (44) are  $\hat{\mathbf{A}}^a(\mathbf{x})$  and  $\hat{\Pi}^a(\mathbf{x}) = -i\delta/\delta\hat{\mathbf{A}}^a(\mathbf{x})$ , that do satisfy the equal-time commutation relation

$$[\hat{A}_i^a(\mathbf{x}), \hat{\Pi}_j^b(\mathbf{y})] = i\delta^{ab} \left( \delta_{ij} - \frac{\nabla_i \nabla_j}{\nabla^2} \right) \delta^3(\mathbf{x} - \mathbf{y}). \quad (45)$$

A prize to pay is that  $\hat{\Pi}^a(\mathbf{x})$  is not a Hermitian operator for the curvilinear scalar product. Indeed,  $\hat{\Pi}^{a\dagger}(\mathbf{x}) = \mathcal{J}^{-1} \hat{\Pi}^a(\mathbf{x}) \mathcal{J} \neq \hat{\Pi}^a(\mathbf{x})$  (see details in appendix A).

We turn our attention to  $\mathcal{P}_{\text{YM}}$  defined in eq. (22). Taking into account eq. (21) we can express

$$\mathcal{P}_{\text{YM}} = \int d^3x E^{\ell a}(\mathbf{x}, t) \nabla A^{\ell a}(\mathbf{x}, t). \quad (46)$$

The structure of the above expression does not differ from those obtained for the case of a pure  $U(1)$  gauge theory. New here is the sum running over the color indices.

We then split the chromoelectric field into the transverse and longitudinal part  $\mathbf{E}^a \equiv \mathbf{E}_{\text{tr}}^a + \nabla \Omega^a$ . From a classical point of view  $\mathbf{E}_{\text{tr}}^a = -\boldsymbol{\pi}^a$ . Under these conditions

$$\begin{aligned} \mathcal{P}_{\text{YM}} &= - \int d^3x \{ \boldsymbol{\pi}^{\ell a}(\mathbf{x}, t) \nabla A^{\ell a}(\mathbf{x}, t) \\ &\quad + \partial^\ell \Omega^a(\mathbf{x}, t) A^{\ell a}(\mathbf{x}, t) \}. \end{aligned} \quad (47)$$

Next, we integrate by part the second term of eq. (47) and use the identity  $\nabla \cdot \mathbf{A}^a = 0$ . As a consequence, it vanishes

identically and only the first term of eq. (47) remains. The quantized version in the flat gauge field configuration takes the form

$$\hat{\mathcal{P}}_{\text{YM}} = - \int d^3x \hat{\Pi}^{\ell a}(\mathbf{x}, t) \nabla \hat{A}^{\ell a}(\mathbf{x}, t). \quad (48)$$

Finally, after applying the transformation

$$\hat{\mathcal{P}}_{\text{YM}} = \mathcal{J}^{-1/2} \hat{\mathcal{P}}_{\text{YM}} \mathcal{J}^{1/2} \quad (49)$$

and by considering eq. (43) we obtain

$$\hat{\mathcal{P}}_{\text{YM}} = - \int d^3x \hat{\Pi}^{\ell a}(\mathbf{x}, t) \nabla \hat{A}^{\ell a}(\mathbf{x}, t) \quad (50)$$

(in agreement with [30] and generalizing the momentum vector of electrodynamics).

### 3.2 Rotation and boost generators of pure Yang-Mills theory

In this subsection we obtain the generators associated to the homogeneous Lorentz group. We again invoke Noether's current for the field  $A^\lambda(x)$ , (with special transformation  $\delta A^\lambda = 1/2 \omega^{\alpha\beta} (\mathcal{J}_{\alpha\beta})^\lambda_\sigma A^\sigma$ ):

$$\begin{aligned} \mathcal{J}_{\text{YM}}^{\mu\alpha\beta}(x) &= \frac{\partial \mathcal{L}_{\text{YM}}}{\partial (\partial_\mu A^{\nu a})} \{ \partial_\lambda A^{\nu a} (\mathcal{J}^{\alpha\beta})^\lambda_\sigma x^\sigma \\ &\quad - (\mathcal{J}^{\alpha\beta})^\nu_\zeta A^{\zeta a} \} - (\mathcal{J}^{\alpha\beta})^\mu_{\nu'} x^{\nu'} \mathcal{L}_{\text{YM}}, \end{aligned} \quad (51)$$

where  $(\mathcal{J}_{\alpha\beta})^\ell_{\bar{\ell}} = \delta_\alpha^\ell g_{\beta\bar{\ell}} - \delta_{\beta\bar{\ell}}^\ell g_{\alpha\bar{\ell}}$  is the vectorial representation of the Lie algebra generator of  $SO(3, 1)$ . Recalling eq. (21) we can write

$$\begin{aligned} \mathcal{J}_{\text{YM}}^{\mu\alpha\beta} &= \mathfrak{L}^{\mu\alpha\beta} + \mathfrak{S}^{\mu\alpha\beta}, \\ \mathfrak{L}_{\text{YM}}^{\mu\alpha\beta} &= x^\alpha \mathcal{F}_{\text{YM}}^{\mu\beta} - x^\beta \mathcal{F}_{\text{YM}}^{\mu\alpha}, \\ \mathfrak{S}^{\mu\alpha\beta} &= \mathfrak{F}^{\mu\beta a} A^{\alpha a} - \mathfrak{F}^{\mu\alpha a} A^{\beta a}. \end{aligned} \quad (52)$$

Here  $\mathfrak{L}^{\lambda\mu\nu}$  is the orbital part, whereas  $\mathfrak{S}^{\lambda\mu\nu}$  corresponds to the intrinsic spin. Note that  $\mathcal{J}_{\text{YM}}^{\mu\alpha\beta}$  include both rotations and boosts belonging to the Lorentz group. We will start by extracting the first. This is obtained from eq. (52) by fixing  $\mu = 0$  and taking the spatial part of the remaining tensor. As a consequence the structure of the angular momentum does not differ from an Abelian group except by a sum running over the color indices

$$\begin{aligned} \mathfrak{J}_{\text{YM}} &= \mathcal{L}_{\text{YM}} + \mathcal{S}_{\text{YM}}, \\ \mathcal{L}_{\text{YM}} &= \int d^3x \{ \mathbf{x} \times [E^{\ell a}(\mathbf{x}, t) \nabla A^{\ell a}(\mathbf{x}, t)] \}, \end{aligned} \quad (53)$$

$$\mathcal{S}_{\text{YM}} = - \int d^3x \{ \mathbf{E}^a(\mathbf{x}, t) \times \mathbf{A}^a(\mathbf{x}, t) \},$$

where we have used  $\mathfrak{F}^{i0a} = E^{ai}$ . We again decompose  $\mathbf{E}^a \rightarrow -\boldsymbol{\pi}^a + \nabla \Omega^a$ . Terms arising from the longitudinal part,  $\nabla \Omega^a$ , cancel each other. In the end,

$$\begin{aligned} \hat{\mathfrak{J}}_{\text{YM}} &= - \int d^3x \left\{ \mathbf{x} \times \left[ \hat{\Pi}^{\ell a}(\mathbf{x}, t) \nabla \hat{A}^{\ell a}(\mathbf{x}, t) \right] \right. \\ &\quad \left. - \hat{\Pi}^a(\mathbf{x}, t) \times \hat{\mathbf{A}}^a(\mathbf{x}, t) \right\}. \end{aligned} \quad (54)$$

Next we address the Boost generator in pure Yang-Mills theory, fixing  $\mu = \alpha = 0$  in eq. (52). This procedure leads to

$$\mathcal{J}_{\text{YM}}^{00i} = x^0 \mathcal{F}_{\text{YM}}^{0i} - x^i \mathcal{F}_{\text{YM}}^{00} + \mathfrak{F}^{i0a} A_0^a, \quad (55)$$

and  $\mathcal{F}_{\text{YM}}^{00}$  must be understood as the Hamiltonian density in eq. (36). Quantizing again at  $t = 0$ , the corresponding charge is

$$\begin{aligned} \mathbf{K}_{\text{YM}} &= \mathcal{K}_{\text{YM}} + \mathfrak{K}_{\text{YM}}, \\ \mathcal{K}_{\text{YM}} &= - \int d^3x \left\{ \mathbf{x} \left( \frac{1}{2} \mathbf{B}^a \cdot \mathbf{B}^a + \frac{1}{2} \boldsymbol{\pi}^a \cdot \boldsymbol{\pi}^a \right. \right. \\ &\quad \left. \left. + \frac{1}{2} g^2 \varrho_{\mathfrak{g}}^b M^{ba} (-\nabla^2) M^{ac} \varrho_{\mathfrak{g}}^c \right) \right\}, \\ \mathfrak{K}_{\text{YM}} &= \int d^3x \mathbf{E}^a(\mathbf{x}) A_0^a(\mathbf{x}). \end{aligned} \quad (56)$$

Returning to eq. (20), the classical Euler-Lagrange equation of motion associated to  $A_0$ , Gauss's law, is a constraint imposed on the quantization (above in eq. (33) we used it in the form of the classical Poisson equation)

$$\nabla \cdot \mathbf{E}^a - g f^{abc} \mathbf{A}^b \cdot \mathbf{E}^c = \mathbf{D}^{ab} \cdot \mathbf{E}^b = 0. \quad (57)$$

Substitution of  $\mathbf{E}^a = -\dot{\mathbf{A}} - \mathbf{D}^{ab} A_0^b$  in the first term of eq. (57) returns

$$-\nabla \cdot \mathbf{D}^{ab} A_0^b = g \varrho_{\mathfrak{g}}^a, \quad (58)$$

where we have used the transversality condition ( $\nabla \cdot \mathbf{A}^a = 0$ ) and identified  $\varrho_{\mathfrak{g}}^a = f^{abc} \mathbf{A}^b \cdot \boldsymbol{\pi}^c$ . Once more we decompose the electric field into transverse and longitudinal parts

$$\mathbf{E}^a = -\boldsymbol{\pi}^a + \nabla \Omega^a \quad (59)$$

in the middle term of eq. (57) to derive the equation

$$\mathbf{D}^{ab} \cdot \nabla \Omega^b = g \varrho_{\mathfrak{g}}^a \quad (60)$$

(at this point  $A_0$  and  $\Omega$  are interchangeable, but we keep the distinction for another instant). Taking into account eq. (59) the spin piece of the boost, eq. (56), becomes

$$\begin{aligned} \mathfrak{K}_{\text{YM}} &= \mathfrak{K}_{\perp} + \mathfrak{K}_{\parallel}, \\ \mathfrak{K}_{\perp} &= -g \int d^3x \boldsymbol{\pi}^a M^{ab} \varrho_{\mathfrak{g}}^a, \\ \mathfrak{K}_{\parallel} &= g \int d^3x (\nabla \Omega^a) A_0^a. \end{aligned} \quad (61)$$

We express  $\mathfrak{K}_{\perp}$  in a more symmetric form

$$\mathfrak{K}_{\perp} = -\frac{g}{2} \int d^3x \boldsymbol{\pi}^a M^{ab} \varrho_{\mathfrak{g}}^b - \frac{g}{2} \int d^3x \varrho_{\mathfrak{g}}^b M^{ba} \boldsymbol{\pi}^a. \quad (62)$$

As for the longitudinal part, eq. (62), substitution of solutions of eq. (60) and eq. (58) give

$$\mathfrak{K}_{\parallel} = -g^2 \int d^3x \nabla (\varrho_{\mathfrak{g}}^b M^{ba}) M^{ac} \varrho_{\mathfrak{g}}^c. \quad (63)$$

After integration by parts  $\mathfrak{K}_{\parallel} = g^2 \int d^3x \varrho_{\mathfrak{g}}^b M^{ba} \nabla (M^{ac} \varrho_{\mathfrak{g}}^c)$ . Therefore  $\mathfrak{K}_{\parallel}$  is a total derivative and vanishes identically.

Wrapping up, the combination of eq. (56) and eq. (62) in  $\mathbf{K}_{\text{YM}}$  allows us to express it as

$$\begin{aligned} \mathbf{K}_{\text{YM}} &= - \int d^3x \left\{ \mathbf{x} \left( \frac{1}{2} \mathbf{B}^a \cdot \mathbf{B}^a + \frac{1}{2} \boldsymbol{\pi}^a \cdot \boldsymbol{\pi}^a \right. \right. \\ &\quad \left. \left. + \frac{g^2}{2} \varrho_{\mathfrak{g}}^b M^{ba} (-\nabla^2) M^{ac} \varrho_{\mathfrak{g}}^c \right) + \frac{g}{2} \boldsymbol{\pi}^a M^{ab} \varrho_{\mathfrak{g}}^b \right. \\ &\quad \left. + \frac{g}{2} \varrho_{\mathfrak{g}}^b M^{ba} \boldsymbol{\pi}^a \right\}. \end{aligned} \quad (64)$$

To match the expression of Besting and Schütte it suffices to add to eq. (64) a vanishing term proportional to  $\mathfrak{K}_{\parallel}$ :  $0 = \mathcal{N} g^2 \int d^3x \nabla (\varrho_{\mathfrak{g}}^b M^{ba}) M^{ac} \varrho_{\mathfrak{g}}^c$ , and choosing  $\mathcal{N} = 1/2$ , so that the corresponding boost operator for a flat scalar product can be written as

$$\begin{aligned} \hat{\mathbf{K}}_{\text{YM}} &= - \int d^3x \left\{ \mathbf{x} \left( \frac{1}{2} \hat{\mathbf{B}}^a \cdot \hat{\mathbf{B}}^a + \frac{1}{2} \hat{\boldsymbol{\Pi}}^{a\dagger} \cdot \hat{\boldsymbol{\Pi}}^a \right) \right. \\ &\quad \left. - \frac{g^2}{2} \hat{\varrho}_{\mathfrak{g}}^{b\dagger} \hat{M}^{ba} (\partial_{\ell} \mathbf{x} \partial_{\ell}) \hat{M}^{ac} \hat{\varrho}_{\mathfrak{g}}^c + \frac{g}{2} \hat{\boldsymbol{\Pi}}^{a\dagger} \hat{M}^{ab} \hat{\varrho}_{\mathfrak{g}}^b \right. \\ &\quad \left. + \frac{g}{2} \hat{\varrho}_{\mathfrak{g}}^{b\dagger} \hat{M}^{ba} \hat{\boldsymbol{\Pi}}^a \right\}. \end{aligned} \quad (65)$$

Making the change  $\hat{\mathbf{K}}_{\text{YM}} = \mathcal{J}^{-1/2} \hat{\mathbf{K}}_{\text{YM}} \mathcal{J}^{1/2}$  and minding the correct ordering of the various, non-commuting operators we obtain

$$\begin{aligned} \hat{\mathbf{K}}_{\text{YM}} &= - \int d^3x \left\{ \frac{1}{2} \mathcal{J}^{-1} \hat{\boldsymbol{\Pi}}^a \mathcal{J} \mathbf{x} \hat{\boldsymbol{\Pi}}^a + \frac{1}{2} \hat{\mathbf{B}}^a \mathbf{x} \hat{\mathbf{B}}^a \right. \\ &\quad \left. - \frac{1}{2} g^2 \mathcal{J}^{-1} \hat{\varrho}_{\mathfrak{g}}^b \mathcal{J} \hat{M}^{ba} (\partial_k \mathbf{x} \partial_k) \hat{M}^{ac} \hat{\varrho}_{\mathfrak{g}}^c \right. \\ &\quad \left. + \frac{1}{2} g \mathcal{J}^{-1} \hat{\boldsymbol{\Pi}}^a \mathcal{J} \hat{M}^{ab} \hat{\varrho}_{\mathfrak{g}}^b + \frac{1}{2} g \mathcal{J}^{-1} \hat{\varrho}_{\mathfrak{g}}^b \mathcal{J} \hat{M}^{ba} \hat{\boldsymbol{\Pi}}^a \right\}. \end{aligned} \quad (66)$$

Equation (66) coincides with the expression derived in [30]. To derive the above expression we have replaced the operator  $\hat{\boldsymbol{\Pi}}^a(\mathbf{x})$  by the transformed one (see eq. (43)).

### 3.3 Inclusion of the quark sector

To complete the discussion on quantum chromodynamics we include the matter sector interacting with the gauge field. This is achieved by the matter Lagrangian

$$\mathcal{L}_{\text{Matter}} = \hat{\mathbf{q}}^{\ell} (i \gamma^{\mu} \partial_{\mu} - m) \hat{\mathbf{q}}^{\ell} + g \hat{\mathbf{q}}^{\ell} \gamma^{\mu} \frac{\lambda^a}{2} \hat{A}_{\mu}^a \hat{\mathbf{q}}^{\ell} \quad (67)$$

with  $\hat{\mathbf{q}} = \hat{\mathbf{q}}^{\dagger} \gamma^0$  and  $\lambda^a/2$  representing the  $SU(3)$  Gell-Mann matrices.

Its contribution to the stress-energy tensor is

$$\begin{aligned} \mathcal{T}_{\text{Matter}}^{\mu\nu} &= \frac{\partial \mathcal{L}_{\text{Matter}}}{\partial (\partial_{\mu} \hat{\mathbf{q}}^{\ell})} \partial^{\nu} \hat{\mathbf{q}}^{\ell} - g^{\mu\nu} \mathcal{L}_{\text{Matter}} \\ &= i \hat{\mathbf{q}}^{\ell} \gamma^{\mu} \partial^{\nu} \hat{\mathbf{q}}^{\ell} - g^{\mu\nu} \hat{\mathbf{q}}^{\ell} (i \gamma^{\lambda} \partial_{\lambda} - m) \hat{\mathbf{q}}^{\ell} \\ &\quad - g g^{\mu\nu} \hat{\mathbf{q}}^{\ell} \gamma^{\mu} \frac{\lambda^a}{2} \hat{A}_{\mu}^a \hat{\mathbf{q}}^{\ell}. \end{aligned} \quad (68)$$



The conserved charges associated to space-time translational invariance are then

$$\begin{aligned}\hat{H}_{\text{Matter}} &= \int d^3x \mathcal{T}_{\text{Matter}}^{00} = \int d^3x \left\{ \hat{\mathcal{H}}_{\text{Dirac}} + \hat{\mathcal{H}}_{\text{Int}} \right\}, \\ \hat{\mathcal{P}}_{\text{Matter}} &= \int d^3x \mathcal{T}_{\text{Matter}}^{0i} = \int d^3x \hat{q}^{\ell\dagger}(\mathbf{x})(-i\nabla)\hat{q}^\ell(\mathbf{x}),\end{aligned}\quad (69)$$

where  $\hat{\mathcal{H}}_{\text{Dirac}}$  is the free Dirac Hamiltonian density:

$$\hat{\mathcal{H}}_{\text{Dirac}}(\mathbf{x}) = \hat{q}^{\ell\dagger}(\mathbf{x})(-i\boldsymbol{\alpha} \cdot \nabla + \beta m)\hat{q}^\ell(\mathbf{x}), \quad (70)$$

whereas

$$\begin{aligned}\hat{\mathcal{H}}_{\text{Int}}(\mathbf{x}) &= g\hat{q}^{\ell\dagger}(\mathbf{x})\frac{\lambda^a}{2}\boldsymbol{\alpha} \cdot \hat{\mathbf{A}}^a(\mathbf{x})\hat{q}^\ell(\mathbf{x}) \\ &\quad - g\hat{q}^{\ell\dagger}(\mathbf{x})\frac{\lambda^a}{2}\hat{q}^\ell(\mathbf{x})\hat{\sigma}^a(\mathbf{x})\end{aligned}\quad (71)$$

arises due to the interaction between the gauge field and quarks. The last term of the above equation introduces the quark color charge density  $\hat{\varrho}_q^a = \hat{q}^{\ell\dagger}\frac{\lambda^a}{2}\hat{q}^\ell$  that must be summed to the right-hand side (source) of Poisson's equation in eq. (32) and following. As a consequence  $\Omega^a \rightarrow \hat{\Omega}^a = \hat{M}^{ab}\hat{\varrho}^b$  with  $\hat{\varrho}^a = \hat{\varrho}_g^a + \hat{\varrho}_q^a$ . The second term of eq. (71) adds up to the second term of eq. (44) and both together have the same functional form as the latter but with  $\hat{\varrho}_g^a$  replaced  $\hat{\varrho}^a = \hat{\varrho}_g^a + \hat{\varrho}_q^a$ .

The complete QCD Hamiltonian and momentum densities are the sum of both

$$\hat{H}_{\text{QCD}} = \hat{H}_{\text{YM}} + \hat{H}_{\text{Matter}}, \quad (72)$$

$$\hat{\mathcal{P}}_{\text{QCD}} = \hat{\mathcal{P}}_{\text{YM}} + \hat{\mathcal{P}}_{\text{Matter}}. \quad (73)$$

where  $\hat{H}_{\text{YM}}$  is the corresponding operational version of eq. (36) with  $\hat{\mathcal{H}}_{\text{YM}}$  given by eq. (44). In turn, eq. (73) combines eq. (50) and eq. (69).

We now turn to the angular momentum and boost operators. The conserved current due to quarks is obtained from the tensor

$$\begin{aligned}\mathcal{J}_{\text{Matter}}^{\mu\alpha\beta} &= x^\alpha \mathcal{T}_{\text{Matter}}^{\mu\beta} - x^\beta \mathcal{T}_{\text{Matter}}^{\mu\alpha} + \frac{i}{4} \frac{\partial \mathcal{L}_{\text{Matter}}}{\partial(\partial_\mu \hat{q}^\ell)} \sigma^{\alpha\beta} \hat{q}^\ell \\ &\quad - \frac{i}{4} \hat{q}^\ell \sigma^{\alpha\beta} \frac{\partial \mathcal{L}_{\text{Matter}}}{\partial(\partial_\mu \hat{q}^\ell)}.\end{aligned}\quad (74)$$

Here  $\partial \mathcal{L}_{\text{Matter}} / \partial(\partial_\mu \hat{q}^\ell) = i\hat{q}^\mu$ , whereas  $\partial \mathcal{L}_{\text{Matter}} / \partial(\partial_\mu \hat{q}^\ell) = -i\gamma^\mu \hat{q}$  therefore

$$\begin{aligned}\mathcal{J}_{\text{Matter}}^{\mu\alpha\beta}(x) &= \mathfrak{L}_{\text{Matter}}^{\mu\alpha\beta} + \mathfrak{S}_{\text{Matter}}^{\mu\alpha\beta}, \\ \mathfrak{L}_{\text{Matter}}^{\mu\alpha\beta} &= x^\alpha \mathcal{T}_{\text{Matter}}^{\mu\beta} - x^\beta \mathcal{T}_{\text{Matter}}^{\mu\alpha}, \\ \mathfrak{S}_{\text{Matter}}^{\mu\alpha\beta} &= -\frac{1}{4} \hat{q}^\ell \{ \gamma^\mu, \sigma^{\alpha\beta} \} \hat{q}^\ell.\end{aligned}\quad (75)$$

By fixing  $\mu = 0$  and taking the remaining indices as spatial, we obtain the quark angular momentum,  $\mathfrak{J}_{\text{Matter}}^i = 1/2 \epsilon^{ijk} \int d^3x (x^j \mathcal{T}_{\text{Matter}}^{0k} - \frac{1}{2} \hat{q}^{\ell\dagger} \sigma^{jk} \hat{q}^\ell)$ , where

$\mathfrak{T}_{\text{Matter}}^{0k}$  is the momentum density in eq. (69). Therefore

$$\begin{aligned}\hat{\mathfrak{J}}_{\text{Matter}} &= \hat{\mathcal{L}}_{\text{Matter}} + \hat{\mathcal{S}}_{\text{Matter}}, \\ \hat{\mathcal{L}}_{\text{Matter}} &= \int d^3x \{ \hat{q}^{\ell\dagger}(\mathbf{x}) [\mathbf{x} \times (-i\nabla)] \hat{q}^\ell(\mathbf{x}) \}, \\ \hat{\mathcal{S}}_{\text{Matter}} &= \int d^3x \hat{q}^{\ell\dagger}(\mathbf{x}) \left( \frac{1}{2} \boldsymbol{\Sigma} \right) \hat{q}^\ell(\mathbf{x}),\end{aligned}\quad (76)$$

with  $\Sigma_i = \frac{1}{2} \epsilon_{ijk} \sigma_{jk}$ .

Following a procedure similar to that used in deriving eq. (55) one obtains the Boost density easily. The generator is obtained integrating over the ( $t = 0$ ) surface  $\hat{K}_{\text{Matter}}^i = \int d^3x \{ -x^i \mathcal{T}_{\text{Matter}}^{00} - \frac{1}{2} \hat{q}^{\ell\dagger} \sigma^{0i} \hat{q}^\ell \}$ . Therefore

$$\begin{aligned}\hat{\mathcal{K}}_{\text{Matter}} &= \hat{\mathcal{K}}_{\text{Matter}} + \hat{\mathfrak{K}}_{\text{Matter}}, \\ \hat{\mathcal{K}}_{\text{Matter}} &= - \int d^3x \mathbf{x} \left\{ \hat{q}^{\ell\dagger}(\mathbf{x}) (-i\boldsymbol{\alpha} \cdot \nabla + \beta m) \hat{q}^\ell(\mathbf{x}) \right. \\ &\quad \left. + \frac{g}{2} \hat{q}^{\ell\dagger}(\mathbf{x}) \lambda^a \boldsymbol{\alpha} \cdot \hat{\mathbf{A}}^a(\mathbf{x}) \hat{q}^\ell(\mathbf{x}) \right\}, \\ \hat{\mathfrak{K}}_{\text{Matter}} &= \int d^3x \hat{q}^{\ell\dagger}(\mathbf{x}) \left( \frac{i}{2} \boldsymbol{\alpha} \right) \hat{q}^\ell(\mathbf{x})\end{aligned}\quad (77)$$

with  $\alpha_i = i\sigma_{0i}$ . For complete QCD, one needs to substitute again  $\hat{\varrho}_g^a$  by  $\hat{\varrho}^a = \hat{\varrho}_g^a + \hat{\varrho}_q^a$  in eq. (54), the boost generator for pure Yang-Mills theory. Finally, one finds

$$\hat{\mathfrak{J}}_{\text{QCD}} = \hat{\mathfrak{J}}_{\text{YM}} + \hat{\mathfrak{J}}_{\text{Matter}}, \quad (78)$$

$$\hat{\mathcal{K}}_{\text{QCD}} = \hat{\mathcal{K}}_{\text{YM}} + \hat{\mathcal{K}}_{\text{Matter}}. \quad (79)$$

(keeping in mind that  $\hat{\varrho}_g^a \rightarrow \hat{\varrho}^a = \hat{\varrho}_g^a + \hat{\varrho}_q^a$ ).

We also record the (possibly Bogoliubov rotated) normal mode expansions of the dynamical fields

$$\begin{aligned}\hat{\mathbf{A}}^a(\mathbf{x}) &= \int \frac{d\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}} [\mathbf{a}^a(\mathbf{k}) + \mathbf{a}^{a\dagger}(-\mathbf{k})] e^{i\mathbf{k}\cdot\mathbf{x}}, \\ \hat{\Pi}^a(\mathbf{x}) &= -i \int d\mathbf{k} \sqrt{\frac{\omega_{\mathbf{k}}}{2}} [\mathbf{a}^a(\mathbf{k}) - \mathbf{a}^{a\dagger}(-\mathbf{k})] e^{i\mathbf{k}\cdot\mathbf{x}}, \\ \hat{q}^\ell(\mathbf{x}) &= \sum_\lambda \int d\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} \left( \mathcal{B}_{\mathbf{k}\lambda}^\ell U_{\mathbf{k}\lambda} + \mathcal{D}_{-\mathbf{k}\lambda}^{\ell\dagger} V_{-\mathbf{k}\lambda} \right), \\ \hat{q}^{\ell\dagger}(\mathbf{x}) &= \sum_\lambda \int d\mathbf{k} e^{-i\mathbf{k}\cdot\mathbf{x}} \left( \mathcal{B}_{\mathbf{k}\lambda}^{\ell\dagger} U_{\mathbf{k}\lambda}^\dagger + \mathcal{D}_{-\mathbf{k}\lambda}^\ell V_{-\mathbf{k}\lambda}^\dagger \right)\end{aligned}\quad (80)$$

with  $d\mathbf{k} \equiv d^3k/(2\pi)^3$  and  $\omega_{\mathbf{k}} = |\mathbf{k}|$ . In momentum space eq. (45) reduces to

$$\left[ a_i^a(\mathbf{k}), a_j^{b\dagger}(\mathbf{k}') \right] = (2\pi)^3 \delta^{ab} \left( \delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2} \right) \delta^3(\mathbf{k} - \mathbf{k}'). \quad (81)$$

Transversality translates into  $\mathbf{k} \cdot \mathbf{a}^a(\mathbf{k}) = \mathbf{k} \cdot \mathbf{a}^{a\dagger}(\mathbf{k})$ . The operators  $\mathbf{a}^a(\mathbf{k})$  and  $\mathbf{a}^{a\dagger}(\mathbf{k})$  include the gluon polarization  $\mathbf{a}^a \equiv \sum_{r=1,2} \epsilon_r(\mathbf{k}) a_r^a(\mathbf{k})$  and the usual creation  $a_r^{a\dagger}(\mathbf{k})$  and annihilation  $a_r^a(\mathbf{k})$  operators (the normalization is as usual  $\epsilon_r^i \epsilon_s^i = \delta_{rs}$ ,  $\epsilon_r^i \epsilon_r^j = \delta^{ij} - k^i k^j / \mathbf{k}^2$ ).

Similarly, quark creation and annihilation operators satisfy the anticommutation relations

$$\left\{ \mathcal{B}_{\mathbf{k}\lambda}, \mathcal{B}_{\mathbf{k}'\lambda'}^\dagger \right\} = \left\{ \mathcal{D}_{\mathbf{k}\lambda}, \mathcal{D}_{\mathbf{k}'\lambda'}^\dagger \right\} = (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}') \delta_{\lambda\lambda'} \quad (82)$$

(others zero).

The renormalization of the boost generators also deserves a comment. It is clear that radiative corrections to the classical, conformal theory are infinite and the quantum theory is only well defined after a scale is chosen. Regularization and renormalization in fixed-gauge Hamiltonian dynamics is arduous and is being pursued independently with attention to the Slavnov-Taylor identities [39]. One should like to find a Lorentz-invariant regulator that would leave the Poincaré algebra intact (which could be established by looking into Schwinger's condition). Whatever strategy one adopts to regularize the Hamiltonian, for example a lattice regularization, one can similarly apply it to the boost operator. A large part of the same, coming from  $x^i H$ , is automatically regulated and requires no additional counterterms other than those in  $H$ , as the additional power of  $x^i$  lowers the mass dimension. This leaves the spin terms like  $\hat{\mathbf{K}}$ , and in principle, one should expect mass counterterms of equal or lower mass dimension than those spin terms to appear. The issue is postponed to future work.

## 4 $\phi$ radiative decays

### 4.1 The boosted wave functions of decay products

We consider again the radiative decay  $\phi \rightarrow \gamma\eta$  as an example, but it is obvious that the discussion is general. The boosted meson with velocity  $v$  is given (in terms of the rapidity  $\zeta$  defined in eq. (9)) by

$$\begin{aligned} |\eta_v\rangle &= e^{i\hat{\mathbf{K}}\cdot\zeta} |\eta_0\rangle \\ &\simeq |\eta_0\rangle + i\hat{\mathbf{K}}\cdot\zeta |\eta_0\rangle - \frac{(\hat{\mathbf{K}}\cdot\zeta)^2}{2} |\eta_0\rangle + \dots \end{aligned} \quad (83)$$

We want to establish that a gluonium term, not taken into account in past analysis, arises just because of the change of reference frame, even if  $|\eta_0\rangle$  contained only purely  $q\bar{q}$  configurations. Therefore we need to see what pieces of the QCD boost operator connect the  $q\bar{q}$  and  $g\bar{g}$  Fock subspaces.

At linear order in  $v$ , the two spaces are indeed disconnected, since the interacting part of  $\hat{\mathbf{K}}$  that could effect the change is proportional to the color charge densities in quarks and gluons, respectively,  $\hat{q}_q \hat{q}_g \propto 1/2 \hat{q}^\dagger \lambda^a \hat{q} f^{abc} \hat{\mathbf{A}}^b \cdot \hat{\mathbf{T}}^c$ , and upon closing the quark line, color arithmetic sets the contribution to zero since  $\text{Tr}(\lambda) = 0$ . One needs then to resource to the second-order term,  $\hat{\mathbf{K}}^2$ , where the square of the instantaneous Coulomb potential contributes, as does the square of the gauge coupling to transverse gluons.

We focus on the latter

$$\hat{K}_z = -\frac{g}{2} \int d^3x z \hat{q}^{\ell\dagger}(\mathbf{x}) \lambda^a \boldsymbol{\alpha} \cdot \hat{\mathbf{A}}^a(\mathbf{x}) \hat{q}^\ell(\mathbf{x}) \quad (84)$$

to show the phenomenon, not because the contribution of the  $\boldsymbol{\alpha} \cdot \hat{\mathbf{A}}$  Hamiltonian is larger than the  $\hat{q}\hat{q}$  piece, but just because of simplicity: this contribution is the one with the smallest number of loops that turns out not to vanish, which we here demonstrate. An added bonus is that, although formally in Coulomb gauge, the piece we calculate will be present in other gauges since it involves only the transverse gluons. The calculational methods are analogous to those of the estimate of the glueball width in Coulomb gauge models of QCD [40].

With  $v \simeq 0.55$  for our particular case,  $\frac{\zeta^2}{2} \simeq 0.15$ . For the rest-frame state is sufficient to take the color singlet, flavor singlet (through mixing) component of the  $\eta$ -meson

$$\begin{aligned} |\eta_0\rangle &= \sin(\phi_P) \int d^3k Y_0^0(\Omega_{\mathbf{k}}) \mathcal{F}_{\eta_0}(|\mathbf{k}\rangle) \\ &\times \sum_{\mu_1\mu_2} \left\langle \frac{1}{2}\mu_1 \frac{1}{2} - \mu_2 \middle| 00 \right\rangle (-1)^{1/2+\mu_2} \mathcal{B}_{\mathbf{k}\mu_1}^\dagger \mathcal{D}_{-\mathbf{k}\mu_2}^\dagger |0\rangle \end{aligned} \quad (85)$$

with color wave function  $\delta^{i_1 i_2} / (N_c)^{1/2}$ , and flavor wave function understood to be the singlet projection.

Substituting the normal-mode expansion eq. (80) for the fields  $\hat{q}$  and  $\hat{\mathbf{A}}^a$  in eq. (84), one obtains, in terms of BCS spinors  $U, V$ , a gluonium component of the  $\eta$  moving with velocity  $v$ ,

$$\begin{aligned} |\eta_v^g\rangle &= g^2 \frac{\zeta^2}{2} \sin(\phi_P) \int d\mathbf{q} \int d\mathbf{p} \int d^3k \int d^3k' \\ &\times \partial_{p_z} \left( \delta^{(3)}(\mathbf{k} + \mathbf{p} - \mathbf{q}) \right) \partial_{p_z} \left( \delta^{(3)}(\mathbf{k}' + \mathbf{q} - \mathbf{p}) \right) \\ &\times \sum_{\lambda\mu_1\mu_2} \frac{1}{\sqrt{4\pi}} \frac{1}{2\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}} \left\langle \frac{1}{2}\mu_1 \frac{1}{2} - \mu_2 \middle| 00 \right\rangle (-1)^{1/2+\mu_2} \\ &\times \mathcal{F}_{\eta_0}(|\mathbf{q}\rangle) V_{-\mathbf{q}\lambda}^\dagger \alpha^j \frac{\lambda^a}{2} U_{\mathbf{p}\mu_1} V_{-\mathbf{p}\mu_2}^\dagger \alpha^{j'} \frac{\lambda^b}{2} V_{-\mathbf{q}\lambda} a_{\mathbf{k}}^{\dagger aj} a_{\mathbf{k}'}^{\dagger bj'} |0\rangle. \end{aligned} \quad (86)$$

The color algebra is easy since  $\text{Tr}(\lambda^a \lambda^b) = 2\delta^{ab}$ . The spin work, however, is a little more tedious. Let us adopt a definition of the spinors in terms of a BCS angle, defined as  $s_{\mathbf{k}} \equiv \sin \varphi(|\mathbf{k}|) = \frac{m(|\mathbf{k}|)}{E(|\mathbf{k}|)}$ , with  $E(|\mathbf{k}|) = [\mathbf{k}^2 + m(|\mathbf{k}|)^2]^{1/2}$  and  $c_{\mathbf{k}} \equiv \cos \varphi(|\mathbf{k}|) = \frac{|\mathbf{k}|}{E(|\mathbf{k}|)}$ . The spinors read then

$$U_{\mathbf{k}\lambda} = \frac{1}{2^{1/2}} \begin{bmatrix} (1 + s_{\mathbf{k}})^{1/2} \chi_\lambda \\ (1 - s_{\mathbf{k}})^{1/2} \boldsymbol{\sigma} \cdot \mathbf{n}_{\mathbf{k}} \chi_\lambda \end{bmatrix}, \quad (87)$$

$$V_{-\mathbf{k}\lambda} = \frac{1}{2^{1/2}} \begin{bmatrix} -(1 - s_{\mathbf{k}})^{1/2} \boldsymbol{\sigma} \cdot \mathbf{n}_{\mathbf{k}} \chi_\lambda \\ (1 + s_{\mathbf{k}})^{1/2} \chi_\lambda \end{bmatrix}, \quad (88)$$

in terms of (bidimensional) Pauli matrices and Pauli spinors satisfying the closure relation  $\sum_\lambda \chi_\lambda \chi_\lambda^\dagger = 1_{2 \times 2}$ . Here  $\mathbf{n}_{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$  is a unit vector.

The spin combination needed yields

$$\begin{aligned}
& 4 \sum_{\lambda} V_{-\mathbf{q}\lambda}^{\dagger} \alpha^j U_{\mathbf{p}\mu_1} V_{-\mathbf{p}\mu_2}^{\dagger} \alpha^{j'} V_{-\mathbf{q}\lambda} = \\
& \sum_{\lambda} \chi_{\lambda}^{\dagger} \left\{ (1 - s_{\mathbf{p}}) c_{\mathbf{q}} \boldsymbol{\sigma} \cdot \mathbf{n}_{\mathbf{q}} \sigma^j \boldsymbol{\sigma} \cdot \mathbf{n}_{\mathbf{p}} \chi_{\mu_1} \chi_{\mu_2}^{\dagger} \boldsymbol{\sigma} \cdot \mathbf{n}_{\mathbf{p}} \sigma^{j'} \right. \\
& + (1 - s_{\mathbf{q}}) c_{\mathbf{p}} \boldsymbol{\sigma} \cdot \mathbf{n}_{\mathbf{q}} \sigma^j \boldsymbol{\sigma} \cdot \mathbf{n}_{\mathbf{p}} \chi_{\mu_1} \chi_{\mu_2}^{\dagger} \sigma^{j'} \boldsymbol{\sigma} \cdot \mathbf{n}_{\mathbf{q}} \\
& - (1 + s_{\mathbf{q}}) c_{\mathbf{p}} \sigma^j \chi_{\mu_1} \chi_{\mu_2}^{\dagger} \boldsymbol{\sigma} \cdot \mathbf{n}_{\mathbf{p}} \sigma^{j'} \\
& \left. - (1 + s_{\mathbf{p}}) c_{\mathbf{q}} \sigma^j \chi_{\mu_1} \chi_{\mu_2}^{\dagger} \sigma^{j'} \boldsymbol{\sigma} \cdot \mathbf{n}_{\mathbf{q}} \right\} \chi_{\lambda}. \quad (89)
\end{aligned}$$

Then, we take into account that the Clebsch-Gordan coefficient in eq. (86) is

$$\left\langle \frac{1}{2} \mu_1 \frac{1}{2} - \mu_2 \middle| 00 \right\rangle (-1)^{1/2 + \mu_2} = -\frac{1}{\sqrt{2}} \delta_{\mu_1 \mu_2} \quad (90)$$

which forces a trace over the Pauli matrices. Employing cyclicity of the trace and  $\text{Tr}(\sigma^i \sigma^j \sigma^k) = 2i \epsilon^{ijk}$ , and returning to the running mass  $m(|\mathbf{p}|)$  and energy  $E(|\mathbf{p}|)$ , we have

$$\begin{aligned}
|\eta_{\mathbf{v}}^{\mathbf{g}}\rangle &= \frac{v^2}{2} (4\pi\alpha_s) \sin(\phi_{\mathbf{P}}) \int \bar{d}\mathbf{q} \int \bar{d}\mathbf{p} \\
&\times \frac{1}{\sqrt{4\pi}} \frac{i\epsilon^{ijj'}}{\sqrt{2}} \frac{1}{2\sqrt{N_c}} \frac{q^i m(|\mathbf{p}|) - p^i m(|\mathbf{q}|)}{E(|\mathbf{q}|)E(|\mathbf{p}|)} \mathcal{F}_{\eta_0}(|\mathbf{q}|) \\
&\times \partial_{q_z} \left( \frac{1}{\sqrt{2\omega_{\mathbf{q}-\mathbf{p}}}} a_{\mathbf{q}-\mathbf{p}}^{\dagger aj} \right) (-\partial_{q_z}) \left( \frac{1}{\sqrt{2\omega_{\mathbf{p}-\mathbf{q}}}} a_{\mathbf{p}-\mathbf{q}}^{\dagger aj} \right) |0\rangle, \quad (91)
\end{aligned}$$

where  $\alpha_s = g^2/(4\pi)$  is the strong fine-structure constant.

## 4.2 The gluonium content

In calculating matrix elements of the boost operator one encounters unusual features. The presence of  $x^i$  turns, in the usual momentum representation, in a derivative respect to  $k^i$ , that affects in the end the  $\delta$ -functions from the field (anti)commutators. While more than one such momentum conservation functions appear, the following construction can be found often:  $\int dk F(k) \delta(k) \partial_k^2 \delta(k)$  that we at present ignore how to handle. If we ignore the term with second derivatives acting on the Dirac  $\delta$ -functions, we can complete the rest of the calculation by the usual per-parts integration method that systematically transfers derivatives from the  $\delta$ -functions to the other functions being integrated. To proceed we write the overlap as

$$\begin{aligned}
\langle \eta_{\mathbf{v}}^{\mathbf{g}} | \eta_{\mathbf{v}}^{\mathbf{g}} \rangle &= (4\pi\alpha_s)^2 \left( \frac{\zeta^2}{2} \right)^2 \sin^2(\phi_{\mathbf{P}}) \int \bar{d}\mathbf{q} \int \bar{d}\mathbf{p} \int \bar{d}\mathbf{q}' \int \bar{d}\mathbf{p}' \\
&\times \frac{1}{4\pi} \frac{1}{4N_c} \frac{\epsilon^{ijl} \epsilon^{i'j'l'}}{2E_{\mathbf{p}} E_{\mathbf{q}} E_{\mathbf{p}'} E_{\mathbf{q}'}} \mathcal{F}_{\eta_0}^*(|\mathbf{q}|) \mathcal{F}_{\eta_0}(|\mathbf{q}'|) \\
&\times (m(|\mathbf{p}|) q^i - m(|\mathbf{q}|) p^i) (m(|\mathbf{p}'|) q'^{i'} - m(|\mathbf{q}'|) p'^{i'})
\end{aligned}$$

$$\begin{aligned}
&\times \int \frac{d^3 k_1 d^3 k_2 d^3 k'_1 d^3 k'_2}{(2\omega_1 \omega_2 \omega'_1 \omega'_2)^{1/2}} \partial_{k_1^z} \delta^{(3)}(\mathbf{k}_1 + \mathbf{q} - \mathbf{p}) \\
&\times \partial_{k_2^z} \delta^{(3)}(\mathbf{k}_2 + \mathbf{p} - \mathbf{q}) \partial_{k'_1^z} \delta^{(3)}(\mathbf{k}'_1 + \mathbf{q}' - \mathbf{p}') \\
&\times \partial_{k'_2^z} \delta^{(3)}(\mathbf{k}'_2 + \mathbf{p}' - \mathbf{q}') (2\pi)^6 \left\{ \delta^{(3)}(\mathbf{k}_1 - \mathbf{k}'_1) \right. \\
&\times \mathbf{t}^{jj'}(\mathbf{k}_1) \delta^{(3)}(\mathbf{k}_2 - \mathbf{k}'_2) \mathbf{t}^{ll'}(\mathbf{k}_2) + \delta^{(3)}(\mathbf{k}_1 - \mathbf{k}'_2) \\
&\left. \times \mathbf{t}^{j'l'}(\mathbf{k}_1) \delta^{(3)}(\mathbf{k}_2 - \mathbf{k}'_1) \mathbf{t}^{l'j'}(\mathbf{k}_2) \right\}. \quad (92)
\end{aligned}$$

There are several things to remark concerning this expression. First, note that  $\mathbf{t}^{ij}(\mathbf{k}) \equiv (\delta^{ij} - k^i k^j / \mathbf{k}^2)$  is the transverse projector in momentum space which ensures transversality. Since the suite of Kronecker  $\delta$ 's will force  $\mathbf{k}_1 = -\mathbf{k}_2$ , and due to the spin contraction with anti-symmetric  $\epsilon^{ijl}$  symbols, the terms quartic in  $\hat{k}_n$  will not contribute. It is also convenient to define a shorthand notation regrouping the meson wave function and the quark propagator pieces,

$$\mathbf{G}(\mathbf{p}, \mathbf{q}) \equiv \mathcal{F}_{\eta_0}(|\mathbf{q}|) \frac{m(|\mathbf{p}|)\mathbf{q} - m(|\mathbf{q}|)\mathbf{p}}{E_{\mathbf{p}} E_{\mathbf{q}}} \quad (93)$$

(where the second wave function depending on  $|\mathbf{p}|$  arises due to the second contributing spin combination).

Finally, as explained above,

$$\begin{aligned}
&\partial_{k_1^z} \delta^{(3)}(\mathbf{k}_1 + \mathbf{q} - \mathbf{p}) \partial_{k_2^z} \delta^{(3)}(\mathbf{k}_1 + \mathbf{q} - \mathbf{p}) \times [\dots] \\
&\rightarrow \delta^{(3)}(\mathbf{k}_1 + \mathbf{q} - \mathbf{p}) \delta^{(3)}(\mathbf{k}_1 + \mathbf{q} - \mathbf{p}) \partial_{p_z} \partial_{q_z} [\dots]
\end{aligned}$$

ignoring the effect of the derivatives over other  $\delta$ -functions, that would normally lead to the  $\delta \nabla^2 \delta$  term. Due to invariance under rotations, one can also substitute derivatives by the Laplacian  $\partial^2 / \partial_{k_z}^2 \rightarrow \nabla^2 / 3$ .

After isolating the  $\delta$ 's, one encounters two factors of the type

$$\frac{1}{9} \nabla_{\mathbf{k}_1}^2 \nabla_{\mathbf{k}_2}^2 \left\{ \frac{1}{2\omega_{\mathbf{k}_1} 2\omega_{\mathbf{k}_2}} \mathbf{t}^{j'l'}(\mathbf{k}_1) \mathbf{t}^{j'l}(\mathbf{k}_2) \right\}$$

that are handled by means of

$$\begin{aligned}
\nabla_{\mathbf{k}}^2 \left\{ \frac{1}{2\omega_{\mathbf{k}}} \mathbf{t}^{ij}(\mathbf{k}) \right\} &= \frac{\mathbf{t}^{ij}(\mathbf{k})}{\omega_{\mathbf{k}}^2} \left( \frac{(\omega'_{\mathbf{k}})^2}{4\omega_{\mathbf{k}}} - \frac{\omega''_{\mathbf{k}}}{2} - \frac{\omega'}{|\mathbf{k}|} \right) \\
&\quad - \frac{2}{\omega |\mathbf{k}|^2} \left( \delta^{ij} - 3 \frac{k^i k^j}{\mathbf{k}^2} \right). \quad (94)
\end{aligned}$$

Since there are two insertions of the boost operator, this is a three-loop calculation. The integration variables can be chosen as  $\mathbf{q}$ ,  $\mathbf{p}$  and  $\mathbf{q}'$ . The fourth,  $\mathbf{p}'$ , becomes a linear combination. There are two terms with a different value of  $\mathbf{p}'$  in each, that we will denote with a subindex

$$\begin{aligned}
\mathbf{p}'_1 &= \mathbf{p} - \mathbf{q} + \mathbf{q}', \\
\mathbf{p}'_2 &= -\mathbf{p} + \mathbf{q} + \mathbf{q}'.
\end{aligned}$$

We can then write down our partial calculation for the overlap as

$$\begin{aligned}
\frac{\langle \eta_{\mathbf{v}}^{\mathbf{g}} | \eta_{\mathbf{v}}^{\mathbf{g}} \rangle}{\langle 0 | 0 \rangle} &= (4\pi\alpha_s)^2 \left( \frac{\zeta^2}{2} \right)^2 \sin^2(\phi_{\mathbf{P}}) \frac{1}{4\pi} \frac{1}{4N_c} \\
&\times \int \bar{d}\mathbf{q} \int \bar{d}\mathbf{p} \int \bar{d}\mathbf{q}' \frac{1}{9} \left( \mathcal{I} - \tilde{\mathcal{I}} \right) \quad (95)
\end{aligned}$$

with spin work  $\mathcal{S}$  that can be obtained by ( $\mathbf{\Delta}_{\mathbf{q}-\mathbf{p}} \equiv (\mathbf{q} - \mathbf{p})/|\mathbf{q} - \mathbf{p}|$ )

$$\frac{\epsilon^{ijl}\epsilon^{i'j'l'}}{2}(\delta^{jj'} - 3\Delta^j\Delta^{j'}) (\delta^{ll'} - 3\Delta^l\Delta^{l'}) G^i G^{i'} = -2\mathbf{G} \cdot \mathbf{G}' + 3(\mathbf{\Delta} \cdot \mathbf{G})(\mathbf{\Delta} \cdot \mathbf{G}') \quad (96)$$

and similar relations, to yield ( $\omega \equiv \omega_{\mathbf{q}-\mathbf{p}}$ )

$$\begin{aligned} \mathcal{S} = & \left\{ \left[ \frac{1}{\omega^2} \left( \frac{\omega'^2}{4\omega} - \frac{\omega''}{2} - \frac{\omega'}{|\mathbf{q}-\mathbf{p}|} \right) \right]^2 [\mathbf{G}(\mathbf{p}, \mathbf{q}) \cdot \mathbf{\Delta}_{\mathbf{q}-\mathbf{p}}] \right. \\ & \times [\mathbf{G}(\mathbf{p}'_1, \mathbf{q}') \cdot \mathbf{\Delta}_{\mathbf{q}-\mathbf{p}}] - \frac{2}{\omega|\mathbf{q}-\mathbf{p}|^2} \left[ \frac{1}{\omega^2} \left( \frac{\omega'^2}{4\omega} - \frac{\omega''}{2} \right. \right. \\ & \left. \left. - \frac{\omega'}{|\mathbf{q}-\mathbf{p}|} \right) \right] (-\mathbf{G}(\mathbf{p}, \mathbf{q}) \cdot \mathbf{G}(\mathbf{p}'_1, \mathbf{q}') + 2[\mathbf{G}(\mathbf{p}, \mathbf{q}) \cdot \mathbf{\Delta}_{\mathbf{q}-\mathbf{p}}] \\ & \times [\mathbf{G}(\mathbf{p}'_1, \mathbf{q}') \cdot \mathbf{\Delta}_{\mathbf{q}-\mathbf{p}}]) + \frac{4}{\omega^2|\mathbf{q}-\mathbf{p}|^4} (-2\mathbf{G}(\mathbf{p}, \mathbf{q}) \\ & \left. \cdot \mathbf{G}(\mathbf{p}'_1, \mathbf{q}') + 3(\mathbf{G}(\mathbf{p}, \mathbf{q}) \cdot \mathbf{\Delta}_{\mathbf{q}-\mathbf{p}})(\mathbf{G}(\mathbf{p}'_1, \mathbf{q}') \cdot \mathbf{\Delta}_{\mathbf{q}-\mathbf{p}}) \right\} \quad (97) \end{aligned}$$

and with  $\tilde{\mathcal{S}}$  obtainable replacing the argument of  $\mathbf{G}(\mathbf{p}'_1, \mathbf{q}')$  by the second solution to the momentum  $\delta$ 's  $\mathbf{G}(\mathbf{p}'_2, \mathbf{q}')$ . With all these factors, one can proceed to a numerical evaluation of eq. (95). Given the now explicit invariance under rotations of the integrand, the nine-dimensional integral can be reduced by choosing the  $z$ -axis along  $\mathbf{p}$ , and the  $x$ -axis so that  $\mathbf{p}$ ,  $\mathbf{q}$  are in the  $xz$ -plane. The remaining integration variables are then the three moduli  $|\mathbf{p}|$ ,  $|\mathbf{q}|$ ,  $|\mathbf{q}'|$ , and the three integration angles  $\theta_{\mathbf{q}}$ ,  $\Omega_{\mathbf{q}'}$ . This six-dimensional integral can be carried out with the help of VEGAS, the standard adaptive Monte Carlo algorithm [41]. The running quark masses  $m(|\mathbf{q}|)$  and gluon (dispersive) energies  $\omega(|\mathbf{k}|)$  appearing satisfy their respective gap equations. We take them to be solutions of the truncated versions in [42] and [43], as later work, especially in the gluon sector, has improved the analysis but not changed the numerical results qualitatively.

Exploration of the six-dimensional numerical integral  $\mathfrak{S}$ , defined by

$$\langle \eta_v^g | \eta_v^g \rangle = (4\pi\alpha_s)^2 \left( \frac{\zeta^2}{2} \right)^2 \sin^2(\phi_P) \mathfrak{S} \quad (98)$$

and taking into account conceivable uncertainties in the quarkonium  $\eta$  wave function, quark mass gap function, and gluon dispersive function, allows us to make an order of magnitude estimate (note that  $\mathfrak{S}$ , after the extraction of the volume factor  $(2\pi)^3\delta^{(3)}(0)$ , is a pure, dimensionless number)  $\log \mathfrak{S} = 2.6_{-2.0}^{+0.5}$ . The prefactor shows the explicit dependence in the pseudoscalar mixing angle (through which the singlet component of the  $\eta$  arises), which we take at 39 degrees, the rapidity  $\zeta \simeq 0.62$  of the boosted  $\eta$ -meson, and  $\alpha_s$  at some low scale. This last value is not known with certainty, but since it is estimated to be 0.4 in  $\tau$  decays, we obtain a lower bound to the prefactor of  $\mathfrak{S}$  which is 0.4.

Putting all together, we obtain

$$\log \langle \eta_v^g | \eta_v^g \rangle = 1.8_{-2.0}^{+0.5} \quad (99)$$

which means that  $\langle \eta_v^g | \eta_v^g \rangle$  is larger than 1 by somewhat more than one sigma. This is of course not possible, as the sum of all coefficients of the wave function in eq. (4)  $\sum_i c_i^2 = 1$ , but one has to take into account that this is only a partial calculation including only one of the relevant terms of the boost operator, and that model uncertainties creep in the choice of wave and gap functions.

What one should conclude is that there is no reason why  $\langle \eta_v^g | \eta_v^g \rangle$  should be orders of magnitude smaller than one. By the size of the controlled factors one could guess it to be of order 0.1. That is, we believe we can argue with certain confidence that a gluonium component of the  $\eta$ -meson is induced by boosting it to a moving reference frame, as in  $\phi$  radiative decays, even if this component was not present in the rest-frame wave function.

## 5 Summary

In this work we have derived the boost generators of quantum chromodynamics quantized in Coulomb gauge in the path integral formalism. These confirm the result of Besting and Schütte for pure Yang-Mills theory, to which we add the fermion fields to complete the boost operator of canonically quantized QCD in Coulomb gauge

$$\begin{aligned} \hat{K}_{\text{QCD}} = & - \int d^3x \left\{ \frac{1}{2} \mathcal{J}^{-1} \hat{\Pi}^a \mathcal{J} \mathbf{x} \hat{\Pi}^a + \frac{1}{2} \hat{\mathbf{B}}^a \mathbf{x} \hat{\mathbf{B}}^a \right. \\ & \left. - \frac{1}{2} g^2 \mathcal{J}^{-1} \hat{\varrho}^b \mathcal{J} \hat{M}^{ba} (\partial_\ell \mathbf{x} \partial_\ell) \hat{M}^{ac} \hat{\varrho}^c \right. \\ & \left. + \frac{1}{2} g \mathcal{J}^{-1} \hat{\Pi}^a \mathcal{J} \hat{M}^{ab} \hat{\varrho}^b + \frac{1}{2} g \mathcal{J}^{-1} \hat{\varrho}^b \mathcal{J} \hat{M}^{ba} \hat{\Pi}^a \right\} \\ & - \int d^3x \mathbf{x} \left\{ \hat{q}^{\ell\dagger}(\mathbf{x}) (-i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta m_q) \hat{q}^\ell(\mathbf{x}) \right. \\ & \left. + g \hat{q}^{\ell\dagger}(\mathbf{x}) \frac{\lambda^a}{2} \boldsymbol{\alpha} \cdot \hat{\mathbf{A}}^a(\mathbf{x}) \hat{q}^\ell(\mathbf{x}) \right\} \\ & + \int d^3x \hat{q}^{\ell\dagger}(\mathbf{x}) \left( \frac{i}{2} \boldsymbol{\alpha} \right) \hat{q}^\ell(\mathbf{x}). \quad (100) \end{aligned}$$

We wished to call the attention of the community to the dynamical nature of these operators. We have focused on two aspects of current interest for the interpretation of hadron data in terms of their quark and gluon constituents. The first is the lack of a charge distribution interpretation of the pion form factor. The operator matrix elements that one obtains from such experiments are a much more complicated generalization. The second is the fact that Fock space expansions of hadron wave functions are tied to the rest frame of the hadron, and they change with its velocity. This is of current interest for the theoretical understanding of pseudoscalar mesons produced in  $\phi$  radiative decays at Frascati.

Many more applications come to mind where theoretical interpretation is complicated by the dynamical nature

of the boost operators. However, there is no known practical way of handling them to obtain even model results. We look forward to any progress in this direction by lattice methods or other means<sup>1</sup>.

Work supported by grants FPA 2008-00592, FIS2008-01323, FPA2007-29115-E, PR34-1856-BSCH, UCM-BSCH GR58/08 910309, PR34/07-15875 (Spain). S. Villalba-Chávez is supported by the Doktoratskolleg “Hadrons in Vacuum, Nuclei and Stars” of the Austrian science fund (FWF) under contract W1203-N08. This author and D. Schütte thank the hospitality of the group at Universidad Complutense where this work was completed. Part of it has been presented to the University in the Master’s thesis of M. Gómez Rocha.

## Appendix A. The Fadeev-Popov Determinant and the Hermiticity of $\hat{\Pi}^a(\mathbf{x})$

Let us first re-check for the reader not familiar with functional quantization how hermiticity of the canonical momentum comes about, in total analogy with elementary quantum mechanics,

$$\langle \Psi_1 | \hat{\Pi}^a(\mathbf{x}) | \Psi_2 \rangle = \int \mathcal{D}\mathbf{A} \mathcal{J} \Psi_1^*[\mathbf{A}] \frac{1}{i} \frac{\delta}{\delta \hat{\mathbf{A}}^a(\mathbf{x})} \Psi_2[\mathbf{A}]. \quad (\text{A.1})$$

Again integrating by parts

$$\begin{aligned} \langle \Psi_1 | \hat{\Pi}^a(\mathbf{x}) | \Psi_2 \rangle &= - \int \mathcal{D}\mathbf{A} \left[ \frac{1}{i} \frac{\delta}{\delta \hat{\mathbf{A}}^a(\mathbf{x})} \mathcal{J} \Psi_1^*[\mathbf{A}] \right] \Psi_2[\mathbf{A}] \\ &= \int \mathcal{D}\mathbf{A} \left[ \hat{\Pi}^a(\mathbf{x}) \mathcal{J} \Psi_1[\mathbf{A}] \right]^* \Psi_2[\mathbf{A}]. \quad (\text{A.2}) \end{aligned}$$

The introduction of a unit factor  $1 = \mathcal{J} \mathcal{J}^{-1}$  in front of the bracket of the above expression leads to

$$\langle \Psi_1 | \hat{\Pi}^a(\mathbf{x}) | \Psi_2 \rangle = \int \mathcal{D}\mathbf{A} \mathcal{J} \left[ \mathcal{J}^{-1} \hat{\Pi}^a(\mathbf{x}) \mathcal{J} \Psi_1[\mathbf{A}] \right]^* \Psi_2[\mathbf{A}]. \quad (\text{A.3})$$

It is to notice that

$$\begin{aligned} \langle \Psi_1 | \hat{\Pi}^{a\dagger}(\mathbf{x}) | \Psi_2 \rangle &= \langle \hat{\Pi}^a(\mathbf{x}) \Psi_1 | \Psi_2 \rangle \\ &= \int \mathcal{D}\mathbf{A} \mathcal{J} \left[ \hat{\Pi}^{a\dagger}(\mathbf{x}) \Psi_1[\mathbf{A}] \right]^* \Psi_2[\mathbf{A}] \quad (\text{A.4}) \end{aligned}$$

and we compare eq. (A.3) with eq. (A.4) to find that with the curvilinear scalar product, the canonical momentum is not Hermitian,  $\hat{\Pi}^{a\dagger}(\mathbf{x}) = \mathcal{J}^{-1} \hat{\Pi}^a(\mathbf{x}) \mathcal{J}$ .

<sup>1</sup> Of course, light front quantization or explicitly covariant approaches such as the Dyson-Schwinger equations are free of this problem by construction but they have other issues of their own that need to be overcome. Coulomb gauge QCD remains an attractive avenue of thought for non-perturbative problems since the physical Hilbert space makes transparent the application of the variational principle.

## References

1. M.E. Rose, Phys. Rev. **73**, 279 (1948).
2. J. Gasser, U.-G. Meißner, Nucl. Phys. B **357**, 90 (1991).
3. F.K. Guo, C. Hanhart, F.J. Llanes-Estrada, U.G. Meißner, Phys. Lett. B **678**, 90 (2009) [arXiv:0812.3270 [hep-ph]].
4. NA7 Collaboration (S.R. Amendolia *et al.*), Nucl. Phys. B **277**, 168 (1986).
5. A. Huber *et al.*, Phys. Rev. Lett. **80**, 468 (1998).
6. J.L. Friar, J. Martorell, D.W.L. Sprung, Phys. Rev. A **56**, 4579 (1997) [arXiv:nucl-th/9707016].
7. J.L. Friar, Ann. Phys. (N.Y.) **81**, 332 (1973).
8. S.J. Brodsky, J.R. Primack, Ann. Phys. (N.Y.) **52**, 315 (1969).
9. H.M. Choi, C.R. Ji, Phys. Rev. D **77**, 113004 (2008) [arXiv:0803.2604 [hep-ph]].
10. E.P. Biernat, W. Schweiger, K. Fuchsberger, W.H. Klink, Phys. Rev. C **79**, 055203 (2009) [arXiv:0902.2348 [nucl-th]].
11. W. Plessas, S. Boffi, L.Y. Glozman, W. Klink, M. Radici, R.F. Wagenbrunn, Nucl. Phys. A **699**, 312 (2002).
12. S.J. Brodsky, G.F. de Teramond, Phys. Rev. D **77**, 056007 (2008) [arXiv:0707.3859 [hep-ph]].
13. A. Szczepaniak, C.R. Ji, S.R. Cotanch, Phys. Rev. C **52**, 2738 (1995).
14. KLOE Collaboration (B. Di Micco), Acta Phys. Pol. Suppl. **2**, 63 (2009).
15. KLOE Collaboration (B. Di Micco), Eur. Phys. J. A **38**, 129 (2008).
16. A. Bramon, R. Escribano, M.D. Scadron, Phys. Lett. B **503**, 271 (2001) [arXiv:hep-ph/0012049].
17. R. Escribano, J. Nadal, JHEP **0705**, 006 (2007) [arXiv:hep-ph/0703187].
18. R. Escribano, Eur. Phys. J. C **65**, 467 (2010) [arXiv:0807.4201 [hep-ph]].
19. R. Escribano, Nucl. Phys. Proc. Suppl. **181-182**, 226 (2008) [arXiv:0807.4205 [hep-ph]].
20. P. Maris, P.C. Tandy, Phys. Rev. C **62**, 055204 (2000) [arXiv:nucl-th/0005015].
21. R.G. Sachs, Phys. Rev. **126**, 2256 (1962).
22. J.J. Kelly, Phys. Rev. C **66**, 065203 (2002) [arXiv:hep-ph/0204239].
23. Particle Data Group (C. Amsler *et al.*), Phys. Lett. B **667**, 1 (2008).
24. P.A. Boyle *et al.*, JHEP **0807**, 112 (2008) [arXiv:0804.3971 [hep-lat]].
25. JLQCD and TWQCD Collaborations (S. Aoki *et al.*), Phys. Rev. D **80**, 034508 (2009) [arXiv:0905.2465 [hep-lat]].
26. R. Frezzotti, V. Lubicz, S. Simula, Phys. Rev. D **79**, 074506 (2009) [arXiv:0812.4042 [hep-lat]].
27. S. Villalba-Chavez, R. Alkofer, K. Schwenzer, arXiv:0807.2146 [hep-th].
28. D. Zwanziger, Nucl. Phys. B **518**, 237 (1998).
29. P. Watson, H. Reinhardt, Phys. Rev. D **75**, 045021 (2007) [arXiv:hep-th/0612114].
30. P. Besting, D. Schutte, Phys. Rev. D **42**, 594 (1990).
31. V.N. Gribov, Nucl. Phys. B **139**, 1 (1978).
32. D. Zwanziger, Phys. Rev. D **69**, 016002 (2004) [arXiv:hep-ph/0303028].
33. H. Reinhardt, P. Watson, Phys. Rev. D **79**, 045013 (2009) [arXiv:0808.2436 [hep-th]].

34. N.H. Christ, T.D. Lee, Phys. Rev. D **22**, 939 (1980) [Phys. Scripta **23**, 970 (1981)].
35. C. Feuchter, H. Reinhardt, Phys. Rev. D **70**, 105021 (2004) [arXiv:hep-th/0408236].
36. A.P. Szczepaniak, E.S. Swanson, Phys. Rev. D **65**, 025012 (2002) [arXiv:hep-ph/0107078].
37. A.P. Szczepaniak, Phys. Rev. D **69**, 074031 (2004) [arXiv:hep-ph/0306030].
38. D.R. Campagnari, H. Reinhardt, A. Weber, Phys. Rev. D **80**, 025005 (2009) [arXiv:0904.3490 [hep-th]].
39. P. Watson, H. Reinhardt, Eur. Phys. J. C **65**, 567 (2010) [arXiv:0812.1989 [hep-th]].
40. P. Bicudo, S.R. Cotanch, F.J. Llanes-Estrada, D.G. Robertson, Eur. Phys. J. C **52**, 363 (2007) [arXiv:hep-ph/0602172].
41. G.P. Lepage, CLNS-80/447, 1980.
42. F.J. Llanes-Estrada, S.R. Cotanch, Nucl. Phys. A **697**, 303 (2002) [arXiv:hep-ph/0101078].
43. A. Szczepaniak, E.S. Swanson, C.R. Ji, S.R. Cotanch, Phys. Rev. Lett. **76**, 2011 (1996) [arXiv:hep-ph/9511422].