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ON THE LIMIT OF SOLUTIONS OF $u_t = \Delta u^m$ AS $m \to \infty$

Abstract. We consider the Cauchy problem

(1) $$u_t = \Delta u^m,\ u(0) = f\ .$$

wehere $f \in L^1(\mathbb{R}^N)$, $N \geq 1$ and $f \geq 0$ a.e., and prove that as $m \to \infty$, the corresponding solutions $u_m(t)$ converge in L^1, uniformly for t in a compact set in $]0,\infty[$, to the solution of a suitable limit problem.
We also show similar results for the Cauchy-Dirichlet and Cauchy-Neumann boundary value problems for (1) in bounded domains.

Key words and phrases. Porous medium equation, singular limit, mesa problem, asymptotic behavior.

Let $f \in L^1(\mathbb{R}^N)$, $f \geq 0$ be given and consider the problem

(1) $$u_t = \Delta u^m \text{ on }]0,\infty[\times\mathbb{R}^N, \quad u(0,.) = f \text{ on } \mathbb{R}^N\ ..$$

It is well known (see for instance [1]) that for any $m > 1$, there exists a unique "strong solution" of (1), that is a function $u(t)(x) = u(t,x)$ satisfying

$$u \in C([0,\infty[, L^1(\mathbb{R}^N)) \cap C(]0,\infty[\times\mathbb{R}^N),\ u \geq 0 \text{ on }]0,\infty[\times\mathbb{R}^N,\ u(0,.) = f \text{ on } \mathbb{R}^N,$$

for any $\tau > 0$, $u \in L^\infty(]\tau,\infty[\times\mathbb{R}^N)$, $u_t, \Delta u^m \in L^\infty(]\tau,\infty[, L^1(\mathbb{R}^N))$ and

$$u_t = \Delta u^m \text{ a.e. on }]0,\infty[\times\mathbb{R}^N\ .$$

We note u_m the solution of (1) and prove the following

THEOREM 1. *As* $m \to \infty$,

$$u_m(t) \to \underline{u} = f + \Delta \underline{w} \text{ in } L^1(\mathbb{R}^N)$$

uniformly for t *in a compact set in* $]0, \infty[$, *where* $\underline{w}$ *is the solution of the variational inequality*

(2) $\underline{w} \in L^1(\mathbb{R}^N)$, $\Delta \underline{w} \in L^1(\mathbb{R}^n)$, $0 \leq f + \Delta \underline{w} \leq 1$, $\underline{w} \geq 0$, $\underline{w}(f + \Delta \underline{w} - 1) = 0$ a.e. .

Existence and uniqueness of a solution $\underline{w}$ of (2) follows by the results in [3]: indeed the problem may be rewritten under the form

$$\underline{u},\, \underline{w} \in L^1(\mathbb{R}^N)^+,\ \underline{u} - \Delta \underline{w} = f \text{ in } \mathcal{D}(\mathbb{R}^N),\ \underline{u} \in \beta(\underline{w}) \text{ a.e. .} \tag{3}$$

where β is the sign graph.

If $\underline{w} \in W^{2,1}_{\text{loc}}(\mathbb{R}^N)$, which is the case if $N = 1$ or $f \in L^{1+\varepsilon}_{\text{loc}}(\mathbb{R}^N)$ with some $\varepsilon > 0$, then $\Delta \underline{w} = 0$ a.e. on $\{\underline{w} = 0\}$ so that

$$\underline{u} = \chi_\Sigma + f\,\chi_{\mathbb{R}^N \setminus \Sigma} \qquad \text{with } \Sigma = \mathbb{R}^N \setminus \{\underline{w} > 0\} . \tag{4}$$

The fact that for m large the solution of the porous medium equation develops "mesas" on the set of noncoincidence of the solution of the variational inequality (2), and tends to f on the complementary set, has been noticed in [8]. In [7], it has been proved that for f bounded and satisfying strong geometric assumptions, $u_m(t) \to \underline{u}$ given by (4) in the weak-* topology of $L^\infty(\mathbb{R}^N)$ as $m \to \infty$, uniformly for t in a compact set in $]0, \infty[$. In [9], Theorem 1 has been proved in the cases $N = 1$ and $N \geq 2$ with f radially symmetric.

We also consider equation (1) on a bounded open set Ω in $\mathbb{R}^N$ with Dirichlet or Neumann boundary conditions, and prove the results corresponding to Theorem 1; for the Cauchy-Dirichlet boundary value problem, such result has been shown in [9] in the case $N = 1$.

The paper is organized as follows:

1. Proof of Theorem 1.
2. The Cauchy-Dirichlet boundary value problem.
3. The Cauchy-Neumann boundary value problem.

SECTION 1. Proof of Theorem 1

We first recall that the map $f \to u_m(t)$ is a contraction in $L^1(\mathbb{R}^N)$ for any $m > 1$ and $t \geq 0$; a similar result holds for the map $f \to \underline{u}$. Therefore, as it was noticed in [9], it is enough to prove the Theorem assuming that f is bounded and compactly supported. Namely, we will assume throughout this Section that

$$(5) \qquad 0 \leq f \leq M \text{ a.e. on } \{|x| < R_0\}, \quad f = 0 \text{ a.e. on } \{|x| > R_0\} .$$

By the maximum principle we have

$$(6) \qquad 0 \leq u_m(t) \leq M \text{ a.e. for any } t \geq 0 \text{ and } m > 1 .$$

Fix now $T > 0$ and $m_0 > 1$. It follows from Lemma 2.1 in [9] that there exists R, depending on N, M, R_o, T and m_0, such that

$$(7) \qquad u_m(t) = 0 \text{ on } \{|x| > R\} \text{ for any } t \in [0, T] \text{ and } m \geq m_0 .$$

By the translation invariance and the L^1-contractivity of the maps $f \to u_m(t)$, we have that for any $t \geq 0$ and $m > 1$

$$(8) \quad \int |u_m(t, x + y) - u_m(t, x)|\, dx \leq \int |f(x + y) - f(x)|\, dx \qquad \text{for any } y \in \mathbb{R}^N .$$

Therefore, as in [9], it follows from (6)-(8) that

$$(9) \qquad \{u_m(t);\ t \in [0, T],\ m \geq m_0\} \text{ is precompact in } L^1(\mathbb{R}^N) .$$

We now recall the following oneside estimate (see [1]) for the solution $u = u_m$ of (1)

$$(10) \qquad u_t = \Delta u^m \geq -u/(m - 1 + 2/N)t \text{ a.e. } .$$

Since $\Delta u(t)^m \in L^1(\mathbb{R}^N)$ and $\int \Delta u(t)^m = 0$ a.e. $t > 0$, one then has

(11)
$$\|u_t(t)\|_{L^1} = \|\Delta u(t)^m\|_{L^1} = 2\|(\Delta u(t)^m)^-\|_{L^1}$$
$$\leq 2\|u(t)\|_{L^1} \Big/ \left(m - 1 + \frac{2}{N}\right) t \leq 2\|f\|_{L^1} \Big/ \left(m - 1 + \frac{2}{N}\right) t \quad \text{a.e. } t > 0.$$

From (6), (7) and (10), it follows that

$$(12) \quad (u_m(t, x))^m \leq ME(x) \Big/ \left(m - 1 + \frac{2}{N}\right) t \qquad \text{on }]0, T[\, \times \mathbb{R}^N \text{ for } m \geq m_0 .$$

where $E \in \mathcal{C}(\mathrm{I\!R}^N)$ is the solution of

$$E = 0 \text{ on } \{|x| \geq R\},\ -\Delta E = 1 \text{ in } \mathcal{D}'(\{|x| < R\}) .$$

In particular for $0 < \tau < T$, we have

(13) $$(u_m)^m \to 0 \text{ uniformly on } [\tau, T] \times \mathrm{I\!R}^N \text{ as } m \to \infty .$$

Thanks to (6) we have $(u_m)^m \in \mathcal{C}([0, \infty[, L^1(\mathrm{I\!R}^N))$ and we may define, for $t > 0$ and $m > 1$

(14) $$w_m(t) = \int_0^t (u_m(s))^m ds .$$

which satisfies

(15) $$u_m(t) - \Delta w_m(t) = f \text{ in } \mathcal{D}'(\mathrm{I\!R}^N) .$$

If for a subsequence $m_k \to \infty$ we have $u_{m_k}(1) \to \underline{u}$ in $L^1(\mathrm{I\!R}^N)$, then by (11)

(16) $$u_{m_k}(t) \to \underline{u} \text{ in } L^1(\mathrm{I\!R}^N) \text{ uniformly for } t \in [\tau, T] .$$

whereas, by (7) and (15)

(17) $$w_{m_k}(1) \to \underline{w} \text{ in } L^1(\mathrm{I\!R}^N) .$$

with

(18) $$\underline{u} - \Delta \underline{w} = f \text{ in } \mathcal{D}'(\mathrm{I\!R}^N) , \quad \underline{w} \geq 0 \text{ a.e. on } \mathrm{I\!R}^N .$$

and using (13)

(19) $$0 \leq \underline{u} \leq 1 \text{ a.e. on } \mathrm{I\!R}^N .$$

We claim that

(20) $$\underline{w} = 0 \text{ a.e. on } \{\underline{u} < 1\} .$$

This will end the proof the Theorem 1.

In order to prove (20), we first remark that according to (10), the map $t \to t^{(n+1+\frac{2}{N})^{-1}} u(t)$ is nondecreasing so that

(21) $$u_m(t) \leq t^{-1/(m-1+(\frac{2}{N}))} u_m(1) \quad \text{for any } 0 < t \leq 1 .$$

By (6)

$$u_m(t)^m \leq M u_m(t)^{m-1} . \tag{22}$$

so that by the definition (14) of $w_m(t)$ and (21)

$$w_m(1) \leq M u_m(1)^{m-1}(1 + N(m-1)/2) . \tag{23}$$

Property (20) is now clear: we may assume $u_{m_k}(1) \to \underline{u}$ a.e., such that, a.e. $x \in \{\underline{u} < 1\}$ we willl have for k large, $u_{m_k}(1)(x) \leq \delta < 1$ and then, using (23), $w_{m_k}(1)(x) \to 0$ as $k \to \infty$.

REMARK 1. Under assumption (5), we have justified the definition (14) of $w_m(t)$, and actually proved that

$$w_m(t) \to \underline{w} \text{ in } W^{2,p}(\mathbb{R}^N) \text{ for any } 1 \leq p < \infty, \text{ as } m \to \infty . \tag{24}$$

uniformly for t in a compact set in $]0, \infty[$.

Actually, according to (15), one has that for any $f \in L^1(\mathbb{R}^N)$, $w_m(t)$ is well defined and converges to $\underline{w}$ in $W^{1,p}_{\text{loc}}(\mathbb{R}^N)$ for any $1 \leq p < N/(N-1)$.

SECTION 2. The Cauchy-Dirichlet boundary value problem

In this section Ω will be an open set in $\mathbb{R}^N$ and $f \in L^1(\Omega)$, $f \geq 0$. We consider now the problem

$$u_t = \Delta u^m \text{ on }]0, \infty[\times\Omega, \quad u(0,.) = f \text{ on } \Omega, \quad u = 0 \text{ on }]0,\infty[\times\partial\Omega . \tag{25}$$

For simplicity we will assume Ω bounded with smooth boundary $\partial\Omega$, although the results which follow can be easily extended to a general open set Ω.

Using for instance the results of [2], it follows that for $m > 1$ there exists a unique "strong solution" of (25) satisfying

$u \in \mathcal{C}([0, \infty[, L^1(\Omega)) \cap \mathcal{C}(]0, \infty[\times\overline{\Omega})$, $u \geq 0$ on $]0, \infty[\times\Omega$, $u = 0$ on $]0, \infty[\times\partial\Omega$,

$u(0,.) = f$ on Ω; for any $\tau > 0$, $u_t, \Delta u^m \in L^\infty(]\tau, \infty[, L^1(\Omega))$ and

$$u_t = \Delta u^m \text{ a.e. on }]0, \infty[\times\Omega .$$

We shall denote by u_m the strong solution of (25).

On the other hand there is existence and uniqueness of a solution of the variational inequality

(26) $\underline{w} \in W_0^{1,1}(\Omega),\ 0 \le f + \Delta \underline{w} \le 1$ in $\mathcal{D}'(\Omega)$, $\underline{w} \ge 0$, $\underline{w}(f + \Delta \underline{w} - 1) = 0$ a.e. on Ω .

This follows by the result of [6].

We have the following

THEOREM 2. *With the notations of this section, as* $m \to \infty$

$$u_m(t) \to \underline{u} = f + \Delta \underline{w} \ \textit{in} \ L^1(\Omega)$$

uniformly for t *in a compact set in* $]0, \infty[$.

To show this we will adapt the proof of Section 1. Using the contraction property in $L^1(\Omega)$ of the maps $f \to u_m(t)$ and $f \to \underline{u}$, we may assume f bounded, namely $0 \le f \le M$, such that (6) still holds.

The oneside estimate (10), with the constant $\left(m - 1 + \frac{2}{N}\right)$ there is no more true, but as it is proved in [4] for general homogeneous evolution equation, we have for $u = u_m$

$$u_t = \Delta u^m \ge -u/(m-1)t \ \text{ a.e. .} \tag{27}$$

and also

$$\|u_t(t)\|_{L^1} = \|\Delta u(t)^m\|_{L^1} \le 2\|f\|_{L^1}/(m-1)t \qquad \text{a.e. } t > 0 \ . \tag{28}$$

From (27) and (6), we deduce

$$(u_m(t,x))^m \le M E(x)/(m-1)t \qquad \text{on }]0,T] \times \Omega \text{ for } m > 1 \ . \tag{29}$$

where E is now the solution of the Dirichlet problem on Ω

$$-\Delta E = 1 \text{ on } \Omega,\ E = 0 \text{ on } \partial\Omega$$

and it follows that for $0 < \tau < T$

$$(u_m)^m \to 0 \ \text{ uniformly on } [\tau, T] \times \Omega \text{ as } m \to \infty \ . \tag{30}$$

We may define again $w_m(t)$ by (14), and we have

$$u_m(t) - \Delta w_m(t) = f \text{ on } \Omega, \ w_m(t) = 0 \text{ on } \partial\Omega \ . \tag{31}$$

If for a subsequence $m_k \to \infty$ we have $u_{m_k}(1) \to \underline{u}$ in $L^1(\Omega)$, then we will have by (30), (29) and (31)

$$\begin{aligned} &0 \leq \underline{u} \leq 1 \text{ a.e. on } \Omega \\ &u_{m_k}(t) \to \underline{u} \text{ in } L^1(\Omega) \text{ uniformly for } t \in [\tau, T] \\ &w_{m_k}(1) \to \underline{w} \text{ in } L^1(\Omega) \end{aligned}$$

where $\underline{w} \geq 0$ is the solution of

$$\underline{u} - \Delta \underline{w} = f \text{ on } \Omega, \quad \underline{w} = 0 \text{ on } \partial\Omega \ .$$

The proof of (20) can be done as in Section 1, with slight modifications: according to (27), the map $t \to t^{1/(m-1)}u(t)$ is nondecreasing, so that replacing (22) by $u_m(t)^m \leq M^2 u_m(t)^{m-2}$, we will have in place of (23)

$$w_m(1) \leq M^2 u_m(1)^{m-2}(m-1) \ . \tag{32}$$

which gives also (20) exactly in the same way.

In other words, according to these remarks, the proof of Theorem 2 reduces to showing that

$$\{u_m(t);\ t \in [0, T], m \geq m_0\} \text{ is precompact in } L^1(\Omega) \ . \tag{33}$$

According to (6), it is actually enough to prove that

$$\{u_m(t);\ t \in [0, T],\ m \geq m_0\} \text{ is precompact in } L^1_{\text{loc}}(\Omega) \ .$$

To prove this, fix $\rho \in \mathcal{D}(\Omega)$, $\rho \geq 0$. Let $u = u_m$, and for $y \in \mathbb{R}^N$ with $\operatorname{supp}(\rho + y)$ contained in Ω, let $v(t, x) = \rho(x)|u(t, x+y) - u(t, x)|$. By Kato's inequality, we have

$$v_t \leq \rho \Delta w \quad \text{in } \mathcal{D}'(]0, \infty[\times\Omega) \quad \text{with } w(t, x) = |u(t, x+y)^m - u(t, x)^m|$$

and then integrating

$$\int \rho(x)|u(t, x+y) - u(t, x)|\, dx \leq \int \rho(x)|f(x+y) - f(x)|\, dx + R|y|$$

with

$$R = |y|^{-1} \int_0^t \int \Delta\rho(x) w(s, x)\, dx\, ds \le \|\Delta\rho\|_{L^\infty(\Omega)} \|\text{ grad } u^m\|_{L^1(]0,t[\times\Omega)} \cdot$$

Therefore (33) will follow from

LEMMA 1. *For* $T > 0$ *and* $m_0 > 1$, *there exists* C *such that*

$$\|\text{grad } (u_m)^m\|_{L^1(]0,T[\times\Omega)} \le C \quad \text{for } m \ge m_0 .$$

Proof of lemma 1.

Set $u = u_m$. For any $t \ge 0$, let $v(t)$ be the solution of

$$-\Delta v(t) = u(t) \text{ on } \Omega, \;\; v(t) = 0 \text{ on } \partial\Omega .$$

We have

(34) $$v \in C^1(]0, \infty[\times\Omega), \quad v_t = -u^m .$$

such that, taking integrating over $]0, T[\times\Omega$, we obtain

$$\iint 2u^{m+1} = \iint 2v_t\Delta v = -\iint (|\text{grad } v|^2)_t \le \int |\text{grad } v(0)|^2$$

and then

(35) $$\iint u^{m+1} \le C .$$

Using now (27) we have that for any $t > 0$

$$(m-1)t \int |\text{grad } u^m(t)|^2 \le \int u^{m+1}(t) \le \|u(t)\|_{L^{m+1}} \left(\int u^{m+1}(t)\right)^{m/m+1}$$

Then using Holder and the fact that $\|u(t)\|_{L^{m+1}} \le \|f\|_{L^{m+1}}$, we deduce that

$$(m-1)\left(\iint |\text{grad } u^m|\right)^2 \le$$

$$\le |\Omega|\, \|f\|_{L^{m+1}} \left(\iint u^{m+1}\right)^{m/m+1} \left(\int dt/t^{(m+1)/(m+2)}\right)^{(m+2)/(m+1)}$$

and

$$\left(\iint |\text{grad } u^m|\right)^2 \le M|\Omega|^{(m+2)/(m+1)} T^{1/m+1} C^{m/m+1} (m+2)^{(m+2)/(m+1)} (m-1)^{-1} ,$$

whence the Lemma follows.

SECTION 3. The Cauchy-Neumann boundary value problem

In this section we consider the problem

(36) $u_t = \Delta u^m$ on $]0,\infty[\times\Omega$, $u(0,.) = f$ on Ω, $\partial u^m/\partial n = 0$ on $]0,\infty[\times\partial\Omega$.

where Ω is a bounded connected open set with smooth boundary $\partial\Omega$, and $f \in L^1(\Omega)$, $f \geq 0$. Using again the results of [2], for any $m > 1$ there exists a unique "strong solution" of (36) satisfying

$$u \in C([0,\infty[, L^1(\Omega)) \cap C(]0,\infty[\times\overline{\Omega}),\ u \geq 0 \text{ on }]0,\infty[\times\Omega,\ u(0,.) = f \text{ on } \Omega,$$

$$u_t \in L^\infty(]\tau,\infty[, L^1(\Omega)) \text{ for any } \tau > 0,\ u^m \in L^\infty_{\text{loc}}(]0,\infty[, W^{2,1}(\Omega)) \text{ and}$$

$$u_t = \Delta u^m \text{ a.e. on }]0,\infty[\times\Omega,\ \partial u^m/\partial n = 0 \text{ a.e. on }]0,\infty[\times\partial\Omega .$$

We denote now by u_m this solution of (36).

On the other hand, consider the variational inequality

(37)
$$\underline{w} \in W^{1,1}(\Omega),\ 0 \leq f + \Delta\underline{w} \leq 1 \text{ in } \mathcal{D}'(\Omega),\ \underline{w} \geq 0,\ \underline{w}(f + \Delta\underline{w} - 1) = 0 \text{ a.e. on } \Omega$$

$$\text{and } \int \rho\Delta\underline{w} = -\int \text{grad}\,\rho\ \text{grad}\,\underline{w} \qquad \text{for any } \rho \in C^1(\overline{\Omega}) .$$

According to the results in [5], (37) has a solution if and only if

$$\fint f = |\Omega|^{-1}\int f \leq 1 . \tag{38}$$

Moreover,
if $\fint f < 1$, then the solution $\underline{w}$ of (37) is unique
if $\fint f = 1$, for any solution $\underline{w}$ of (37), $f + \Delta\underline{w} = 1$ a.e. on Ω.

We have

THEOREM 3. *With the notations of this section,*

i) *if* $\fint f \geq 1$, *then* $u_m(t) \to \fint f$ *in* $L^1(\Omega)$ *as* $m \to \infty$, *uniformly for* t *in a compact set in* $]0,\infty[$.

ii) *if* $\fint f < 1$, *then* $u_m(t) \to \underline{u} = f + \Delta\underline{w}$ *in* $L^1(\Omega)$ *as* $m \to \infty$, *uniformly for* t *in a compact set in* $]0,\infty[$.

To prove this Theorem we may assume again that f is bounded and then that (6) holds. According to the results in [4], (27) and (28), still are true in this case. In particular, it is enough to prove that the conclusion is satisfied at $t = 1$: it will then hold uniformly for t in a compact set in $]0, \infty[$.

We denote by G the Green operator in $L^1(\Omega)$ associated to the Neumann problem for the Laplacian: for $w \in L^1(\Omega)$, $v = Gw$ is the unique solution of the problem

(39)

$$v \in W^{1,1}(\Omega), \quad \int v = 0, \quad \int \rho \left(w - \fint w \right) = \int \operatorname{grad} \rho \operatorname{grad} v \text{ for any } \rho \in C^1(\overline{\Omega}).$$

It is clear that G is a bounded (actually compact) linear operator from $L^1(\Omega)$ into $W^{1,1}(\Omega)$ (see [6]).

Finaly, we set $I = \fint f$. We then have

(40) $$\fint u_m(t) = I \quad \text{for any } m > 1, \quad t \geq 0\,.$$

Proof of part i): case $I \geq 1$.

We note for simplicity $u_m = u_m(1)$; using (40), we have $\int |u_m - I| = 2\int (I - u_m)_+$, where $r_+ = \sup(r, 0)$, and then it is enough to prove that

(41) $$\int (I - u_m)_+ \to 0 \text{ as } m \to \infty\,.$$

Using (28), since $(u_m)^m - \fint (u_m)^m = G(-\Delta(U_m)^m)$, we see that

(42) $$\varepsilon_m = |(u_m)^m - \fint (u_m)^m| \to 0 \qquad \text{in } W^{1,1}(\Omega) \text{ as } m \to \infty\,.$$

Now, by convexity, we have

$$\fint (u_m)^m \geq \left(\fint u_m \right)^m = I^m \geq 1$$

so that

$$(u_m)^m \geq (1 - \varepsilon_m)_+$$

and

$$\varepsilon_m \geq I^m - (u_m)^m \geq m(u_m)^{m-1}(I - u_m) \ .$$

Then

$$I - u_m \leq \varepsilon_m (1 - \varepsilon_m)_+^{1-1/m} m^{-1}$$

and, thanks to (42), (41) holds by Lebesgue dominated convergence theorem.

Proof of part ii): case $I < 1$.

We will prove

LEMMA 2. *With the notations of this Section 3, if $I < 1$ then for $T > 0$ there exists C such that*

$$\|(u_m)^{m+1}\|_{L^1(]0,T[\times\Omega)} \leq C \quad \text{for } m > 1 \ .$$

Using Lemma 2, and repeating the proof of Lemma 1, one sees that for $m_0 > 1$, there exists C such that

$$\|\text{grad}\,(u_m)^m\|_{L^1(]0,T[\times\Omega)} \leq C \quad \text{for } m \geq m_0$$

and then (33) holds also in this case.

Another consequence of Lemma 2 is that

$$\liminf_{m\to\infty} (u_m)^{m+1} < \infty \quad \text{a.e. on }]0, T[\times\Omega$$

which we will use instead of (30) to obtain, thanks to (28), that if for a subsequence $m_k \to \infty$ we have $u_{m_k}(1) \to \underline{u}$ in $L^1(\Omega)$, then we will have $0 \leq \underline{u} \leq 1$ a.e. on Ω.

The proof of Theorem 3 in this case will follow then exactly as that of Theorem 2. To end up, we give the

Proof of lemma 2.

Let $u = u_m$, $v(t) = Gu(t)$. We have that

$$v_t(t) = \int\!\!\!\!\!\!- \, u^m(t) - u^m(t), \quad u(t) = I - \Delta v(t)$$

and then

$$\int u^{m+1}(t) = \int u^m(t) \int u(t) - \int u(t) v_t(t) =$$
$$= I \int u^m(t) - I \int v_t(t) - \int \text{grad } v_t(t) \text{ grad } v(t) \ .$$

Using the convexity inequality $(m+1)u^m \leq mu^{m+1} + 1$, we obtain

$$(m + 1 - mI) \int\int u^{m+1} \leq IT + (m+1)\{I \int v(0) + 1/2 \int |\text{grad } v(0)|^2\}$$

whence the result.

REFERENCES

[1] D.G. Aronson, Ph. Bénilan. Régularité des solutions de l'équations des milieux poreux dans $\mathbb{R}^N$. C.R.Ac. Sc. Paris 288 (1979), pp. 103-105.

[2] Ph. Bénilan, A strong regularity L^p for solutions of the porous medium equation. Contribution nonlinear pde. Res. Notes in Math. 89. Pitman (1983), pp. 39-58.

[3] Ph. Bénilan, H. Brézis, M.G. Crandall. A semilinear elliptic equation in $L^1(\mathbb{R}^N)$. Ann. Sc. Norm. Sup. Pisa. V,2. (1975), pp. 523-555.

[4] Ph. Bénilan, M.G. Crandall. Regularizing effect of homogeneous evolutions equations. Contributions to Analysis and Geometry, D.N. Clarke & al eds, John Hopkins Un. Press, Baltimore (1981), pp. 23-30.

[5] Ph. Bénilan, M.G. Crandall, P.E. Sacks. Some L^1 existence and dependance results for semilinear elliptic equations under nonlinear boundary conditions. Appl. Math. Optim. 17 (1988), pp. 203-224.

[6] H. Brézis, W. Strauss. Semilinear elliptic equations in L^1. J. Math. Soc. Japan, 25 (1973), pp. 565-590.

[7] L.A. Caffarelli, A. Friedman. Asymptotic behavior of solutions of $u_t = \Delta u^m$ as $m \to \infty$. Indiana Un. Math. J. 36,4 (1987), pp. 711-718.

[8] C.M. Elliot, M.A. Herrero, J.R. King. J.R. Ockendon. The mesa problem: diffusion patterns for $u_t = \nabla(u^m \nabla u)$ as $m \to \infty$. IMA J. Appl. Maths 37 (1986), pp. 147-154.

[9] P.E. Sacks. A singular limit problem for the porous medium equation. J.Math. Anal. Appl. 140,2 (1989), pp. 456-466.

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