A NOTE ON THE $\gamma\gamma \to \pi^0\pi^0$ REACTION IN THE $1/N$ EXPANSION OF χPT

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Abstract

In this work we present the results of a complete calculation of the $\gamma\gamma \to \pi^0\pi^0$ amplitude to leading order in the large N approximation (N being the number of Goldstone bosons) up to order m_π^2/F^2. The amplitude turns to be proportional to that of $\pi^+\pi^- \to \pi^0\pi^0$. In spite of the fact that this factorization property cannot hold in general (as it was recently pointed out by Morgan and Penington), it appears here since in the large N limit only the $I = J = 0$ channel contributes to the $\gamma\gamma \to \pi^0\pi^0$ reaction. Moreover it seems to be a reasonable approximation in this case since it is possible to reproduce, as a prediction, the experimental data starting from a one-parameter fit of the $\pi\pi$ scattering data.

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1 Introduction

In the last years a great deal of work has been devoted to the $\gamma\gamma \to \pi^0\pi^0$ reaction. The reasons for that are the experimental data available from the the Crystal Ball Collaboration [1] and that this process, being a pure finite one-loop effect, is a very good test for the modern Chiral Perturbation Theory (χPT) [2] which successfully describes many low-energy hadronic reactions. However, for $\gamma\gamma \to \pi^0\pi^0$, the one-loop prediction [3] disagrees with the experimental data even near threshold. This is due to the pion rescattering which is very strong in the $I = J = 0$ channel. Nevertheless it has been shown that the use of dispersion relation techniques [4] can provide a good description of this reaction and reconcile data with χPT [5]. More recently it has been shown that a two loop-computation produces a cross-section that agrees rather well with the available data and with dispersion theoretical calculations even substantially above threshold [6].

Another completely different approach that has been proposed for this reaction by Im [7] is the so called large N expansion (N being the number of Nambu-Goldstone bosons). This method was first used for elastic pion scattering in [8] in the context of the Linear Sigma Model (LSM) and in [9] in the Non Linear Sigma Model (NLSM). Pion mass effects have also been included in [10] for the LSM and in [11] for the NLSM. In this last work it was shown how it is possible to fit the $J = 0$ pion scattering phase-shifts up to very high energies.

Nevertheless Im's work has been strongly critizised by Pennington and Morgan in [12] because some of his assumptions are only true for point interactions which is not the case of pions. Since Im's computation is not in fact a complete large N computation, these assumptions cannot be supported and thus we consider Pennington and Morgan criticisms appropriate. In this note we clarify this issue by making a complete large N computation of the $\gamma\gamma \to \pi^0\pi^0$ reaction. Our main conclusion is that at the end the original Im's result is correct at leading order of the large N expansion and to first order in m_π^2/F^2 so both Pennington and Morgan on one side as Im on the other, are simultaneously right. In addition we will show how the large N approximation provides a very good description of the Crystal Ball experimental data.

2 The Chiral Lagrangian in Standard S^3 Coordinates

As it is well known, the low-energy interactions of pions and photons can be described by a $U(1)_{em}$ gauged NLSM in the context of modern χPT. This NLSM is based in the coset space $SU(2)_L \times SU(2)_R/SU(2)_{L+R} = O(4)/O(3) = S^3 = SU(2)$. In order to define the large N limit we will extend the coset space to $O(N+1)/O(N) = S^N$ and we will define the corresponding lagrangian as follows. First we start from the LSM

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^T \partial^\mu \phi + \sqrt{N} F m^2 \sigma \tag{1}$$

where $\phi^T = (\pi_1, \pi_1^{'}, \pi_2, \pi_2^{'}, .., \pi_k, \pi_k^{'}, \pi_N, \sigma)$ which is $O(N+1)$ invariant if it was not for the last piece that explicitly breaks this symmetry to $O(N)$ (with $N = 2k+1$). This contribution will provide the mass term for the pions and other interactions forced by the chiral symmetry. With $\pi_0 = \pi_N$ the above lagrangian can be written as:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \pi_a \partial^\mu \pi^a + \frac{1}{2}\partial_\mu \pi_a^{'} \partial^\mu \pi^{'a} + \frac{1}{2}\partial_\mu \pi^0 \partial^\mu \pi^0 + \frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma + \sqrt{N} F m^2 \sigma \tag{2}$$

with $a = 1, 2, .., k$, or, in terms of the complex charged fields $\varphi^a = (\pi^a + i\pi^{'a})/\sqrt{2}$, as

$$\mathcal{L} = \partial_\mu \varphi_a^* \partial^\mu \varphi^a + \frac{1}{2}\partial_\mu \pi^0 \partial^\mu \pi^0 + \frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma + \sqrt{N} F m^2 \sigma \tag{3}$$

The electromagnetic interactions are included by introducing the corresponding covariant derivative

$$\partial_\mu \varphi^a \rightarrow D_\mu \varphi^a \equiv \partial_\mu \varphi^a + ieA_\mu \varphi^a \tag{4}$$

and adding the photon kinetic term

$$\begin{aligned} \mathcal{L}_g &= \partial_\mu \varphi_a^* \partial^\mu \varphi^a + \frac{1}{2}\partial_\mu \pi^0 \partial^\mu \pi^0 + \frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma + \sqrt{N} F m^2 \sigma \\ &+ e^2 A^2 \varphi_a^* \varphi^a + ieA_\mu \partial_\mu \varphi_a^* \varphi^a + h.c. - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \end{aligned} \tag{5}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu$. Now we obtain the NLSM describing the low-energy pion dynamics by setting $\phi^T \phi = NF^2 = f_\pi^2$ to force the system to live in the S^N sphere i.e.

$$\pi^2 + \sigma^2 = NF^2 \tag{6}$$

where $\pi^2 = \sum_{a=1}^{k}(\pi^{a2} + \pi'^{a2})$. Thus we have $\sigma = (NF^2 - \pi^2)^{1/2}$

and the lagrangian of the $U(1)_{em}$ gauged NLSM becomes

$$\begin{aligned}
\mathcal{L}_g &= \partial_\mu\varphi^+\partial^\mu\varphi + \frac{1}{2}\partial_\mu\pi^0\partial^\mu\pi^0 + \frac{1}{2}\frac{\left(\partial_\mu\varphi^+\varphi + \varphi^+\partial^\mu\varphi + \pi^0\partial_\mu\pi^0\right)^2}{NF^2 - 2\varphi^+\varphi} \\
&+ NF^2m^2\sqrt{1 - \frac{2\varphi^+\varphi + \pi_0^2}{NF^2}} + e^2A^2\varphi^+\varphi + ieA_\mu\partial^\mu\varphi^+\varphi + h.c. \\
&- \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad (7)
\end{aligned}$$

with $\varphi^T = (\varphi_1, ..., \varphi_k)$.

This way to write the lagrangian describing the low-energy pion dynamics corresponds to a particular coordinate choice on the coset space S^N but in principle any other parametrization connected with this one through an analytical change of coordinates on the coset manifold will yield the same S matrix elements (but not necessarily the same Green functions) [13]. In particular, for the $N = 3$ case a very popular coordinate choice consists in defining the following $SU(2) = S^3$ matrix:

$$\Sigma = e^{\frac{i\tau^a\omega^a(x)}{f_\pi}} \qquad (8)$$

The change of coordinates $\phi^i = \phi^i(\omega)$, with $i = 1, 2, 3$ relating both parametrizations of the coset space is given by $f_\pi\Sigma = \phi^4 + i\phi^i\tau^i$ with $\phi^4 = \sqrt{f_\pi^2 - \phi^2}$ where $\phi^2 = \sum_{i=1}^{3}\phi^{i2}$ with $\phi^T = [\phi_1, \phi_2, \phi_3, \phi_4] = [\pi_1, \pi_1', \pi_0, \sigma]$. The above lagrangian written in terms of these new coordinates reads

$$\mathcal{L} = \frac{f_\pi^2}{4}TrD_\mu\Sigma\left(D^\mu\Sigma\right)^+ + \frac{f_\pi^2}{4}Trm^2\left(\Sigma + \Sigma^+\right) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad (9)$$

where

$$\begin{aligned}
D_\mu\Sigma &= \partial_\mu\Sigma + ie\left[Q, \Sigma\right]A_\mu \\
Q &= diag\left(2/3, -1/3\right) \qquad (10)
\end{aligned}$$

As it was commented above, both lagrangians yield the same S matrix elements and the same observables like phase-shifts, cross sections, etc. However, the Green functions could be different. Indeed the computations can be much easier using one set of coordinates than another since the Feynman rules and the diagrams contributing to some given process will be different in general. In particular, the computation of the one-loop $\gamma\gamma \to \pi^0\pi^0$ amplitude is simpler using the ϕ standard S^3 coordinates than the Σ (chiral) ones, which were used in [3]

3 The one-loop scattering amplitude

The $\gamma\gamma \to \pi\pi$ transition amplitude is defined as

$$\mathcal{A}\left(\gamma\gamma \to \pi^0\pi^0, \pi^+\pi^-\right) = \epsilon_{1\mu}\left(k_1\right)\epsilon_{2\nu}\left(k_2\right)\mathcal{M}^{\mu\nu}_{\pi^0\pi^0,\pi^+\pi^-} \tag{11}$$

where $\epsilon_{1\mu}, \epsilon_{2\nu}$ and k_1, k_2 are the polarization vectors and the momenta of the two photons.

The two one-loop diagrams for the process $\gamma\gamma \to \pi^0\pi^0$ using standard S^3 coordinates are shown in Fig.1.a. Note that when this computation is done with chiral coordinates there are two more Feynman diagrams [3]. In our case, using dimensional regularization, the diagrams are given by:

$$\begin{aligned}
\mathcal{N}^a_{\mu\nu} &= \frac{2e^2}{\left(NF^2\right)^2}\left(s-m_\pi^2\right)\int d\tilde{\ell}\frac{g_{\mu\nu}}{\left[\left(\ell+k_1\right)^2-m_\pi^2\right]\left[\left(\ell-k_2\right)^2-m_\pi^2\right]} \\
\mathcal{N}^b_{\mu\nu} &= \frac{-e^2}{\left(NF^2\right)^2}\left(s-m_\pi^2\right)\int d\tilde{\ell}\frac{\left(2\ell+k_1\right)_\mu\left(2\ell-k_2\right)_\nu}{\left(\ell^2-m_\pi^2\right)\left[\left(\ell+k_1\right)^2-m_\pi^2\right]\left[\left(\ell-k_2\right)^2-m_\pi^2\right]} \\
&+ \left[(k_1,\mu) \longleftrightarrow (k_2,\nu)\right]
\end{aligned} \tag{12}$$

with

$$\int d\tilde{\ell} \equiv \mu^\epsilon \int \frac{d^{4-\epsilon}\ell}{\left(2\pi\right)^{4-\epsilon}} \tag{13}$$

It is not difficult to see that the above $\mathcal{N}^a_{\mu\nu}$ and $\mathcal{N}^b_{\mu\nu}$ correspond respectively to the terms $\mathcal{M}^a_{\mu\nu} + \mathcal{M}^c_{\mu\nu}$ and $\mathcal{M}^b_{\mu\nu} + \mathcal{M}^d_{\mu\nu}$ of the second reference in [3]. The complete matrix element is given by

$$\mathcal{M}_{\mu\nu} = \frac{2e^2}{\left(NF^2\right)^2}\left(s-m_\pi^2\right)\int d\tilde{\ell}\frac{g_{\mu\nu}\left(l^2-m_\pi^2\right)-\left(2\ell+k_1\right)_\mu\left(2\ell-k_2\right)_\nu}{\left(\ell^2-m_\pi^2\right)\left[\left(\ell+k_1\right)^2-m_\pi^2\right]\left[\left(\ell-k_2\right)^2-m_\pi^2\right]} \tag{14}$$

which is proportional to the χPT lowest order $\pi^+\pi^- \to \pi^0\pi^0$ amplitude given by $(s-m_\pi^2)/f_\pi^2$.

As it is well known, the integral above is finite and therefore no renormalization is needed. This is because there is no four derivative term in the gauged NLSM contributing to $\gamma\gamma \to \pi^0\pi^0$. Therefore there is no any possible counterterm to absorb the divergences appearing in the one-loop computation

and thus the consistency of χPT renders this one-loop amplitude completely finite. The result is given by

$$\begin{aligned}\mathcal{M}_{\mu\nu} &= \frac{2e^2}{\left(NF^2\right)^2}\left(s-m_\pi^2\right)\left(\frac{-i}{16\pi^2}\right)\left(g_{\mu\nu}-\frac{k_{2\mu}k_{1\nu}}{k_1k_2}\right)\left[1+\frac{m_\pi^2}{s}\left\{\ln Q_\pi - i\pi\right\}^2\right] \\ Q_\pi &= \frac{1+\sqrt{1-\frac{4m_\pi^2}{s}}}{1-\sqrt{1-\frac{4m_\pi^2}{s}}} \end{aligned} \tag{15}$$

From this amplitude one can calculate the total cross-section which is found to be

$$\sigma\left(\gamma\gamma\to\pi^0\pi^0\right)=\frac{\alpha^2}{8\pi^2}\sqrt{1-\frac{4m_\pi^2}{s}}\left[1+\frac{m_\pi^2}{s}f\left(s\right)\right]\sigma_0\left(\pi^+\pi^-\to\pi^0\pi^0\right) \tag{16}$$

where

$$\sigma_0\left(\pi^+\pi^-\to\pi^0\pi^0\right)=\frac{1}{32\pi\left(NF^2\right)^2}\frac{\left(s-m_\pi^2\right)^2}{s} \tag{17}$$

which is the lowest order $\pi^+\pi^-\to\pi^0\pi^0$ scattering cross-section and

$$f\left(s\right)=2\left[\ln^2\left(\frac{1+\sqrt{1-\frac{4m_\pi^2}{s}}}{1-\sqrt{1-\frac{4m_\pi^2}{s}}}\right)-\pi^2\right]+\frac{m_\pi^2}{s}\left[\ln^2\left(\frac{1+\sqrt{1-\frac{4m_\pi^2}{s}}}{1-\sqrt{1-\frac{4m_\pi^2}{s}}}\right)+\pi^2\right]^2 \tag{18}$$

Thus, from the above equations one can realize that, as it was remarked in the second reference in [3], the lowest order $\pi^+\pi^-\to\pi^0\pi^0$ amplitude and cross-section factorize the one-loop $\gamma\gamma\to\pi^0\pi^0$ ones.

From the diagrams in Fig.1. it is also possible to compute the reaction $\gamma\gamma\to\pi^+\pi^-$. However, in this case we have a tree level contribution (Fig.1.b) and, in addition, we have to consider at the one-loop level (Fig.1.c) the contribution coming from the pion wave-function renormalization (Fig.1.d). In the reference frame where the photon polarization vectors ϵ_1,ϵ_2 satisfy

$$k_1\cdot\epsilon_1=k_1\cdot\epsilon_2=k_2\cdot\epsilon_1=k_2\cdot\epsilon_2=0 \tag{19}$$

the amplitude at tree level is given by

$$\mathcal{A}^{tree}_{(\gamma\gamma\to\pi^+\pi^-)}=2ie^2\left\{\epsilon_1\cdot\epsilon_2-\frac{p^+\cdot\epsilon_1 p^-\cdot\epsilon_2}{k_1\cdot p^+}-\frac{p^-\cdot\epsilon_1 p^+\cdot\epsilon_2}{k_2\cdot p^+}\right\} \tag{20}$$

where k_1, k_2 are the photon momenta and p^+, p^- are the π^+, π^- outgoing momenta. Now including the one-loop corrections.

$$\begin{aligned}\mathcal{A}^{one-loop}_{(\gamma\gamma\to\pi^+\pi^-)} &= \frac{-ie^2}{16\pi^2 NF^2}\left\{\epsilon_1\cdot\epsilon_2\left(s+m_\pi^2\ln^2 Q_\pi\right)\right. \\ &+ \left. 2A_0\left(m_\pi^2\right)\left[\epsilon_1\cdot\epsilon_2 - \frac{p^+\cdot\epsilon_1 p^-\cdot\epsilon_2}{k_1\cdot p^+} - \frac{p^-\cdot\epsilon_1 p^+\cdot\epsilon_2}{k_1\cdot p^-}\right]\right\} \quad (21)\end{aligned}$$

with

$$\begin{aligned}A_0\left(m_\pi^2\right) &= m_\pi^2\left(\triangle - \ln m_\pi^2\right) \\ \triangle &= \frac{2}{\epsilon} + \log 4\pi - \gamma + 1 + \ln\mu^2 \quad (22)\end{aligned}$$

so that the full amplitude is

$$\mathcal{A}\left(\gamma\gamma\to\pi^+\pi^-\right) = \left[\mathcal{A}^{tree}_{(\gamma\gamma\to\pi^+\pi^-)} + \mathcal{A}^{one-loop}_{(\gamma\gamma\to\pi^+\pi^-)}\right]\mathcal{Z}_\pi \quad (23)$$

where the wave function renormalization constant is given by

$$\mathcal{Z}_\pi = 1 + \frac{1}{16\pi^2 NF^2}A_0\left(m_\pi^2\right) \quad (24)$$

so that we can write

$$\mathcal{A}\left(\gamma\gamma\to\pi^+\pi^-\right) = 2ie^2\left\{a\epsilon_1\cdot\epsilon_2 - \frac{p^+\cdot\epsilon_1 p^-\cdot\epsilon_2}{k_1\cdot p^+} - \frac{p^-\cdot\epsilon_1 p^+\cdot\epsilon_2}{k_2\cdot p^+}\right\} \quad (25)$$

where a is given by

$$a(s) = 1 - \frac{1}{16\pi^2 NF^2}\left(\frac{s}{2} + \frac{m_\pi^2}{2}\ln^2 Q_\pi\right) \quad (26)$$

4 The large N limit approximation

In order to compute the $\gamma\gamma \to \pi^0\pi^0$ cross-section at leading order in the $1/N$ expansion we go back to the lagrangian in eq.7. The pion scattering amplitudes to order $1/N$ and the lowest order in m_π^2/f_π^2 were computed in [11]. The contributing diagrams in this approximation are shown in Fig.2.a where the pion vertex also includes the infinite set of counter terms needed to renormalize the bare result. The renormalized amplitude can be written as

$$T_{abcd} = \delta_{ab}\delta_{cd}A\left(s\right) + \delta_{ac}\delta_{bd}A\left(t\right) + \delta_{ad}\delta_{bc}A\left(u\right) \tag{27}$$

where

$$A\left(s\right) = \frac{1}{NF^2}\frac{G^R(s;\mu)}{1-\frac{sG^R(s;\mu)}{2(4\pi)^2F^2}T(s;\mu)}\left\{s - \frac{m_\pi^2 G^R(s;\mu)}{1-\frac{sG^R(s;\mu)}{2(4\pi)^2F^2}T(s;\mu)}\left[J^R(s;\mu) - \frac{s}{\left(4\pi\right)^2F^2}\log\frac{m_\pi^2}{\mu^2}\right]\right\} \tag{28}$$

with

$$T\left(s;\mu\right) \equiv \sqrt{1-\frac{4m_\pi^2}{s}}\log\frac{\sqrt{1-\frac{4m_\pi^2}{s}}-1}{\sqrt{1-\frac{4m_\pi^2}{s}}+1} - \log\frac{m_\pi^2}{\mu^2} \tag{29}$$

The $G^R(s;\mu)$ and $J^R(s;\mu)$ are the generating functions of the counterterm couplings and they have a well defined dependence on the μ scale, in order to ensure the μ independence of the whole amplitude. The isospin channels for the model here considered are defined as [14]:

$$\begin{aligned} T_0 &= NA\left(s\right) + A\left(t\right) + A\left(u\right) \\ T_1 &= A\left(t\right) - A\left(u\right) \\ T_2 &= A\left(t\right) + A\left(u\right) \end{aligned} \tag{30}$$

and the partial waves are given by:

$$a_{IJ}(s) = \frac{1}{64\pi}\int_{-1}^{1} d\left(\cos\theta\right)T_I\left(s,\cos\theta\right)P_J\left(\cos\theta\right) \tag{31}$$

As it is well known, the requirement of unitarity strongly constraints the possible behaviour of these partial waves. In particular, they should have a cut along the positive real axis from the threshold to infinity. The physical amplitudes are the values just on the cut, and, in addition, the condition of elastic unitarity:

$$Ima_{IJ} = \sigma|a_{IJ}|^2 \tag{32}$$

(where $\sigma = \sqrt{1-4m_\pi^2/s}$) has to be satisfied in the physical region $s = E^2 + i\epsilon$ and $E^2 > 4m_\pi^2$, where E is the total center of mass energy. This equation is exact for energies below the next four pion threshold but even beyond that point it is approximately valid.

In the level of approximation considered in this work (large N limit and first order in m_π^2/F^2), the dominant channel is the $I = J = 0$. As it can be

expected the elastic unitarity condition is not exact but it can be shown that one has

$$Ima_{00} = \sigma|a_{00}|^2 + O\left(\frac{1}{N}\right) + O\left[\left(\frac{m_\pi^2}{F^2}\right)^2\right] \tag{33}$$

From the partial waves it is possible to define in the standard way the phase shifts δ as

$$t_{IJ}(s) = e^{i\delta_{IJ}(s)} \sin\delta_{IJ}(s)/\sigma(s) \tag{34}$$

Now one can try to compare the results of our approximation with the experiment. Then one finds that the $I = J = 0$ pion scattering phase shifts can be fitted with the simple choice $G^R(s;\mu) = 1$ and $J^R(s;\mu) = 1$ for $\mu = 775MeV$, as it is shown in Fig.3. Concerning the $I = 0, J = 2$ channel (which is the other relevant for the $\gamma\gamma \to \pi^0\pi^0$ reaction), the prediction of the large N approximation is that it is suppressed at leading order in the $1/N$ expansion. In any case, one can use the above equations with the μ value fitted for the a_{00} channel. The result, which is in fact a prediction, is also shown in Fig.3 and, as it can be seen, the agreement with the experimental data is pretty good.

At the same level of approximation, the diagrams contributing to the $\gamma\gamma \to \pi^0\pi^0$ reaction can be found in Fig.2.b since it is not difficult to see that any other diagram is suppressed by extra $1/N$ or m_π^2/F^2 powers. Thus the amplitude corresponding to this reaction can be written as

$$\mathcal{M}_{\mu\nu} = e^2\left(2g_{\mu\nu}I(s) - I(k_1k_2)_{\mu\nu} - I'(k_1k_2)_{\mu\nu}\right)NA(s) + O(1/N) + O((\frac{m_\pi^2}{F^2})^2) \tag{35}$$

where

$$\begin{aligned} I(s) &= \int d\tilde{l}\frac{1}{(l-m_\pi^2)[(l-p_1-p_2)^2-m_\pi^2]} \\ I(k_1k_2)_{\mu\nu} &= \int d\tilde{l}\frac{(2l+k_1)_\mu(2l-k_2)_\nu}{(l-m_\pi^2)[(l+k_1)^2-m_\pi^2][(l-k_2)^2-m_\pi^2]} \\ I'(k_1k_2)_{\mu\nu} &= \int d\tilde{l}\frac{(2l-k_1)_\mu(2l+k_2)_\nu}{(l-m_\pi^2)[(l+k_2)^2-m_\pi^2][(l-k_1)^2-m_\pi^2]} \end{aligned} \tag{36}$$

The last two integrals have the same value, so that we can write

$$\mathcal{M}_{\mu\nu} = 2e^2NA(s)\int d\tilde{\ell}\frac{g_{\mu\nu}\left(l^2-m_\pi^2\right)-\left(2\ell+k_1\right)_\mu\left(2\ell-k_2\right)_\nu}{\left(\ell^2-m_\pi^2\right)\left[\left(\ell+k_1\right)^2-m_\pi^2\right]\left[\left(\ell-k_2\right)^2-m_\pi^2\right]} + O(1/N) + O((\frac{m_\pi^2}{F^2})^2) \tag{37}$$

Note that this is the same integral that appears in the one-loop computation in eq.14. Therefore we also find, in the approximation here considered, the factorization property that we found there. In particular, eq.16 can be applied again although now the lowest order $\pi^+\pi^- \to \pi^0\pi^0$ cross-section has to be replaced by the cross-section obtained from eq.28, which is given by:

$$\sigma\left(\pi^+\pi^- \to \pi^0\pi^0\right) = \frac{\mid A(s) \mid^2}{32\pi \left(NF^2\right)^2 s} \tag{38}$$

The results obtained by using this formula, with the previously fitted value for μ in the $I = J = 0$ elastic pion scattering channel, are given in Fig.4. For completeness we also show the results obtained when the $J = 0, I = 2$ channel contribution is included by using the exact formula

$$\sigma\left(\pi^+\pi^- \to \pi^0\pi^0\right) = \frac{8\pi}{9(s - 4m_\pi^2)} \sin^2(\delta_{00} - \delta_{20}) \tag{39}$$

together with eq.16 and the phase shifts computed in the large N approximation described above which are shown in Fig.3. As it can be seen, the effect of the $I = 2$ channel is not very large, since it is not so strongly interacting as the $I = 0$. In any case our computations are compatible with the experimental data in both cases (with and without the introduction of the $I = 2$ channel) if one takes into account the large error bars. Thus, the large N approximation makes it possible to fit all the data shown in Fig.3 and Fig.4 with just one parameter, namely the scale $\mu = 775 MeV$.

5 Discussion

In some sense the aim of this work is to clarify to what extent the above mentioned factorization relation between the $\gamma\gamma \to \pi^0\pi^0$ and the $\pi^+\pi^- \to \pi^0\pi^0$ amplitudes occurs beyond the one-loop approximation. Note that, as it was stressed by Pennington and Morgan in [12], the computation of the exact amplitude will require the knowledge of the $\pi^+\pi^- \to \pi^0\pi^0$ amplitude not only on-shell but also off-shell in order to make loops. In addition, the partial waves of the first reaction behave at threshold as $s^{J/2}(s - 4m_\pi^2)^{J/2}$ whereas the second goes as $(s-4m_\pi^2)^J$ and therefore the factorization formula cannot be correct. Moreover, from elastic unitarity (at the lowest order in the electromagnetic interactions), we have the relation

$$Im\mathcal{F}_{IJ}^{\Lambda} = \sigma \mathcal{F}_{IJ}^{\Lambda *} t_{IJ} \tag{40}$$

involving the $\gamma\gamma \to \pi^0\pi^0$ and the $\pi\pi$ partial wave (we have used the standard definitions which can be found for instance in [5]). In this reference it was shown how it is possible to use dispersion relations to improve the unitary behaviour of one-loop χPT, both for elastic pion scattering and $\gamma\gamma \to \pi^0\pi^0$. As a result, an excellent agreement with the data can be found in both cases. These dispersion relations have to be applied to each channel independently. Therefore, as the total amplitude is a linear combination, with well defined coefficients, of the different channel partial waves, we arrive again to the conclusion that the factorization relation cannot be hold in general.

Therefore, for many different reasons, we known that the factorization formula which appears as a result of the large N approximation (as well as the original one-loop) cannot be true for the exact amplitudes. Then the natural question arises: Is there anything wrong with the large N approximation? and in that case: Why does it reproduce so well the experimental data?

Of course the answer to the first question is no. The large N approximation is a well defined expansion in the $1/N$ parameter and it is perfectly right. In particular the computations done by using this approximation satisfy the current constrains of any Quantum Field Theory like unitary, analyticity, crossing and so on, modulo higher order corrections in the expansion parameter, i.e. $1/N$ (see eq.33), exactly as it happens in any other perturbative expansion. The reason why it gives rise to the factorization formula is the following: according to eq.27 the A function, which in principle depends on s, t and u, is only a function of s at leading order in the $1/N$ expansion. In addition, from eq.30 we see that, strictly speaking, only the $I = 0$ channel contributes in this approximation. In other words we can say that to leading order in the large N expansion the only non-zero channel is $I = J = 0$. Then the $\pi^+\pi^- \to \pi^0\pi^0$ amplitude only depends on s and only has an isoscalar component. This explains why all the previous mentioned objections to the factorization formula do not apply to this case as it happens in the general case where more than one channel contributes to the reaction. Therefore there is no inconsistency in the factorization formula obtained in the large N limit. However, this formula should be corrected by higher order computations (both in $1/N$ and in m_π^2/F^2) which will also include other channels different from the $I = J = 0$. In that case the factorization formula is not justified any more.

Of course one can ask how good the large N approximation to leading order in $1/N$ and in m_π^2/F^2 can be, since it only includes the $I = J = 0$ channel contribution. The answer can be found in Fig.4. There we see that the agreement with the experimental data is good, and in fact it considerably

improves the one-loop computation (the two-loop computation in [5] also reproduces well the data but depends on more parameters)

6 Conclusions

The main results of our work are the following:

In order to properly define the large N expansion for two flavor χPT we have parametrized the coset space $S^3 \to S^N$ by using standard coordinates instead of exponential (chiral) coordinates. We have reobtained the well known one-loop results for the $\gamma\gamma \to \pi\pi$ reactions with this parametrization in a much simpler way, thus illustrating, with a very non-trivial example, the S matrix invariance under reparametrizations of the NLSM.

We have computed the $\gamma\gamma \to \pi^0\pi^0$ scattering amplitude to leading order in the large N approximation up to order m_π^2/F^2. In this approximation this amplitude turns out to be proportional to that of $\pi^+\pi^- \to \pi^0\pi^0$ (the factorization formula) according to Im's conjecture (but in principle only for small m_π^2/F^2).

Using a one-parameter fit of the elastic pion scattering in the $I = J = 0$ channel obtained in the same approximation we reproduce, as a prediction, both the $I = 2, J = 0$ pion phase shift and the $\gamma\gamma \to \pi^0\pi^0$ cross-section experimental data (see Fig. 1 and 2).

From very general considerations it can be shown that the factorization formula cannot be exact and this was the basis for the Morgan and Pennington criticism to Im's results. However Im's result is correct in the framework of the large N approximation (for small m_π^2/F^2) since only the $I = J = 0$ channel contributes to the leading order in the $1/N$ expansion. As a matter of fact, as the $\gamma\gamma \to \pi^0\pi^0$ is dominated by this channel, the factorization formula yields a good approximation as it was suggested by Donoghue et al in [3]. This would not be the case if, for example, the ρ resonance played an important role in the reaction, since it cannot be reproduced by the leading order large N approximation [11].

In summary, we consider that the large N approach to χPT is a very interesting complementary alternative to the standard one-loop computations which can be useful in many cases in order to describe the relevant physics of the considered phenomena. This is particularly true in those cases where the $I = J = 0$ is dominant as it happens in the Higgs physics.

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Figure Captions

Fig. 1: Diagrams contributing to a) the $\gamma\gamma \to \pi^0\pi^0$ reaction at one-loop level, b) the tree level $\gamma\gamma \to \pi^+\pi^-$, c) the one-loop $\gamma\gamma \to \pi^+\pi^-$ and d) the one-loop pion wave-function renormalization.

Fig.2: a) Diagrams contributing to the $\pi - \pi$ elastic scattering at leading order in the $1/N$ expansion up to order m_π^2/F^2. The large black dot represents the pion coupling proportional to m_π^2. b) Diagrams contributing to the $\gamma\gamma \to \pi^0\pi^0$ reaction to the same level of approximation. The black square represents the addition of all the diagrams appearing in a).

Fig 3: Phase shifts for $J = 0$ elastic pion scattering. The dashed line represents the fit done with the leading order of the $1/N$ expansion up to the order m_π^2/F^2 for the $I = 0$ channel and the prediction for the $I = 2$ channel. The continuous lines are the phase shifts as obtained from the standard one-loop χPT (see [11] for the details). The experimental data corresponds to: $\triangle$ ref.[15], $\bigcirc$ ref.[16],$\Box$ ref.[17],$\diamondsuit$ ref.[18], $\bigtriangledown$ ref.[19],$\star$ ref.[20],$\times$ ref.[21],$\bullet$ ref.[22].

Fig 4: Cross section for the process $\gamma\gamma \to \pi^0\pi^0$ with $\mid cos\theta \mid \leq 0.8$. The continuous line represents the prediction of leading order of the $1/N$ expansion up to the order m_π^2/F^2. The dashed line represents the same but including the contribution of the $I = 2$ channel. The experimental data comes from Cristal Ball [1].

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