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EXISTENCE OF SOLUTIONS OF PLANE TRACTION PROBLEMS FOR IDEAL COMPOSITES*

A. C. PIPKIN AND V. M. SANCHEZ†

Abstract. The theory of plane deformations of ideal fiber-reinforced composites involves hyperbolic equations, but boundary data are specified as in elliptic problems. When the surface tractions are given at every boundary point of a plane region, and thus given at two points on each characteristic, it is not obvious that the problem is well-set. We show that under the usual global equilibrium conditions on prescribed tractions, a solution does exist. This is done by reducing the problem to an integral equation whose kernel depends on the shape of the region, locating the spectrum of eigenvalues, and then invoking standard results of the Hilbert–Schmidt theory.

1. Introduction. In the theory of plane deformations of ideal fiber-reinforced composites [1], the equilibrium equations are hyperbolic but, as is usual in solid mechanics, the boundary conditions are of the kind normally associated with elliptic equations. This would tend to put the existence of solutions in doubt, and thus cast doubt on the validity of the mathematical model. However, solving problems within the idealized theory is so simple that the existence of a solution can usually be demonstrated by exhibiting it [2]–[12].

Pure traction boundary value problems are exceptional. Few traction problems have been solved, and it appears that in most cases explicit, exact solutions cannot be expected. When we must be content with an approximate solution, it becomes important to know that an exact solution actually exists. This cannot be taken for granted on physical grounds; rather, it is the physical validity of the theory that is at issue.

In the present note we demonstrate the existence of solutions of a class of traction boundary value problems. We consider infinitesimal, plane deformations of bodies with elastic shearing response, reinforced by inextensible fibers that are initially straight and parallel (see § 3). The composite is incompressible in bulk. The equations of the theory are hyperbolic, with the fibers as one family of characteristics, and normal lines, perpendicular to the fibers, as a second family. We restrict our attention to cases in which each characteristic intersects the boundary of the body at most twice.

As England [11] has shown, application of traction boundary conditions at two points on each characteristic leads to a pair of differential-difference equations of an unusual type (§ 5). England [11] has shown that for bodies symmetrical about a fiber or a normal line, the solution of these equations can be reduced to quadratures. England and Rogers [12] have applied this result to the solution of certain fracture problems. However, for unsymmetrical bodies it appears that the governing differential-difference equations cannot be solved explicitly.

In §§ 6 and 7 we reduce these equations to a single Fredholm integral equation, for which the existence theory is well known (see Courant and Hilbert [13], for example). Consequently, we merely need to determine all solutions of the

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† Division of Applied Mathematics, Brown University, Providence, Rhode Island 02912. This work was supported by the National Science Foundation under Grant GP-28881.

associated homogeneous equation and to verify that the inhomogeneous term of the given equation is orthogonal to these solutions.

In §§ 8 and 9 we show that the associated homogeneous equation has only one independent solution, corresponding to a rigid-body rotation. Thus, the extent of nonuniqueness is exactly what is to be expected in a pure traction problem. The condition that the inhomogeneous term must satisfy in order for a solution to exist is found to be equivalent to the usual rotational equilibrium condition on the applied tractions.

Approximate solutions of the Fredholm equation can be obtained by iteration. To verify that iteration will converge, we show that in problems of the present type, all eigenvalues of the integral operator lie between zero and unity, except for one eigenvalue equal to unity (§ 9), and we show that the latter does not affect the convergence of the iteration procedure (§ 10).

2. Notation. We consider plane deformations of a cylindrical body whose cross section is bounded by a smooth closed curve C. We restrict our attention to cases in which each line parallel to a coordinate axis intersects C at most twice, so that the equation of the boundary can be written as

$$y = y^+(x) \quad \text{and} \quad y = y^-(x), \quad \text{with } y^+ \geqq y^-, \tag{2.1}$$

and also as

$$x = x^+(y) \quad \text{and} \quad x = x^-(y), \quad \text{with } x^+ \geqq x^-, \tag{2.2}$$

for an appropriate range of x or y, respectively.

For the boundary values of any function $f(x, y)$ we use the notation

$$\begin{aligned} f^+(x) &= f[x, y^+(x)], \qquad f^-(x) = f[x, y^-(x)], \\ f^+(y) &= f[x^+(y), y], \qquad f^-(y) = f[x^-(y), y]. \end{aligned} \tag{2.3}$$

The problem to be considered is hyperbolic, with lines parallel to the coordinate axes as characteristics. The differences between two boundary values at points on the same characteristic are denoted by

$$\Delta f(x) = f^+(x) - f^-(x) \quad \text{and} \quad \Delta f(y) = f^+(y) - f^-(y). \tag{2.4}$$

In particular, $\Delta x(y)$ is the width of the body along a line $y = \text{const.}$, and $\Delta y(x)$ is the width along $x = \text{const.}$

Let s be the arc length measured counterclockwise along C from an arbitrary point $s = 0$. If $x = x(s)$ and $y = y(s)$ are the parametric equations of C, then the components of the unit outward normal vector are

$$n_x = dy/ds \quad \text{and} \quad n_y = -dx/ds. \tag{2.5}$$

3. Basic equations. We consider infinitesimal plane deformations of ideal fiber-reinforced composites [1]. The fibers are continuously distributed, lying parallel to the x-axis, and they are inextensible. Consequently, the strain component ε_{xx} is zero, and thus the displacement u parallel to the x-axis is constant along each fiber. The composite is incompressible in bulk; since ε_{zz} is zero in plane deformations, and ε_{xx} is also zero, the constraint of incompressibility then implies that ε_{yy} is zero as well. Hence, the displacement v parallel to the y-axis depends only on x:

$$u = u(y), \quad v = v(x). \tag{3.1}$$

We suppose that the shearing response of the material is elastic, so that the shearing stress σ_{xy} is proportional to the shearing strain:

$$\sigma_{xy} = 2G\varepsilon_{xy} = G[u'(y) + v'(x)]. \tag{3.2}$$

The normal stress components σ_{xx} and σ_{yy} are reactions to the constraint conditions, to be determined from the equilibrium equations, which are

$$\sigma_{xx,x} = -\sigma_{xy,y} = -Gu''(y) \tag{3.3}$$

and

$$\sigma_{yy,y} = -\sigma_{xy,x} = -Gv''(x). \tag{3.4}$$

From these equations we obtain

$$\sigma_{xx} = \sigma_{xx}^-(y) - Gu''(y)[x - x^-(y)] \tag{3.5}$$

and

$$\sigma_{yy} = \sigma_{yy}^-(x) - Gv''(x)[y - y^-(x)]. \tag{3.6}$$

Then, remembering the notational conventions explained in § 2, we obtain

$$\Delta\sigma_{xx}(y) = -Gu''(y)\Delta x(y) \quad \text{and} \quad \Delta\sigma_{yy}(x) = -Gv''(x)\Delta y(x). \tag{3.7}$$

4. Boundary conditions. We consider traction boundary value problems, with boundary conditions of the form $\sigma_{ij}n_j = T_i$ on C. The equilibrium equations by themselves imply that a solution exists only if the prescribed tractions T_i satisfy the global equilibrium conditions

$$\oint T_i\, ds = 0 \quad \text{and} \quad \oint (xT_y - yT_x)\, ds = 0. \tag{4.1}$$

We assume that these conditions are satisfied. In addition, the displacements and rotation at some point can be specified arbitrarily.

With (3.2), the boundary conditions can be written as

$$n_x\sigma_{xx} = T_x - G[u'(y) + v'(x)]n_y \quad \text{on } C \tag{4.2}$$

and

$$n_y\sigma_{yy} = T_y - G[u'(y) + v'(x)]n_x \quad \text{on } C. \tag{4.3}$$

Let $F_x(y)$ be the x-component of the resultant traction on that part of C above the line $y =$ const., and let $F_y(x)$ be the y-component of the total traction on the part of C to the right of the line $x =$ const. Then if x and y have the ranges $L \leqq x \leqq R$ and $B \leqq y \leqq T$ on C, with (4.1a) we have

$$F_x(B) = F_x(T) = F_y(L) = F_y(R) = 0. \tag{4.4}$$

The functions F_i are related to the tractions by

$$F'_x(y) = -\Delta(T_x/n_x) \quad \text{and} \quad F'_y(x) = -\Delta(T_y/n_y). \tag{4.5}$$

By using (2.5) and (4.5), we find that the moment condition (4.1b) can be written as

$$\int_L^R F_y(x)\,dx = \int_B^T F_x(y)\,dy. \tag{4.6}$$

5. First integration. By using the boundary conditions (4.2) and (4.3) to evaluate $\Delta\sigma_{xx}$ and $\Delta\sigma_{yy}$, and then using these values in (3.7), we obtain two differential-difference equations for u and v. The forms of these equations are fairly complicated when written out in full.

Each of the resulting equations can be integrated once explicitly. However, the integrated equations have such simple interpretations that it is easier to derive them directly, without reference to (3.7).

Consider the part of the body below a line $y = \text{const}$. The x-component of the resultant force on that part must vanish for equilibrium. The contribution of the traction is $-F_x(y)$, and the shearing force along the line $y = \text{const.}$ is found from (3.2):

$$u'(y)\Delta x(y) + \Delta v(y) = F_x(y)/G = f_x(y) \quad \text{(say)}. \tag{5.1}$$

(Recall that $\Delta v(y)$ means the difference between the two values of $v(x)$ at the ends of the line $y = \text{const.}$) In a similar way we obtain

$$v'(x)\Delta y(x) + \Delta u(x) = F_y(x)/G = f_y(x) \quad \text{(say)}. \tag{5.2}$$

By differentiating (5.1) and (5.2) we recover (3.7), with the stress differences as specified by the boundary data (4.2) and (4.3). To verify this, use (2.5), (4.5), and

$$(d/dy)\Delta v(y) = \Delta[v'(x(y))x'(y)], \tag{5.3}$$

with a similar expression for the derivative of $\Delta u(x)$.

The relations (5.1) and (5.2) are essentially the same as equations previously derived by England [11].

6. Second integration. By integrating (5.1) and (5.2), we obtain

$$u(y) = u(0) + \int_0^y [(f_x - \Delta v)/\Delta x]\,dy' \tag{6.1}$$

and

$$v(x) = v(0) + \int_0^x [(f_y - \Delta u)/\Delta y]\,dx', \tag{6.2}$$

where the constants of integration $u(0)$ and $v(0)$ may be specified arbitrarily.

From (6.1) we obtain

$$\Delta u(x) = \int_{y^-(x)}^{y^+(x)} [(f_x - \Delta v)/\Delta x]\,dy', \tag{6.3}$$

and a similar expression for $\Delta v(y)$ can be obtained from (6.2).

To avoid complicated limits of integration, we introduce the characteristic function $U(x, y)$, equal to unity at points in the body and equal to zero outside it. Then (6.3) is

$$\Delta u(x) = \int U(x, y)[(f_x - \Delta v)/\Delta x]\, dy, \tag{6.4}$$

and the corresponding equation obtained from (6.2) is

$$\Delta v(y) = \int U(x, y)[(f_y - \Delta u)/\Delta y]\, dx. \tag{6.5}$$

The limits of integration are now $-\infty$ and $+\infty$, or any other fixed limits including the whole body.

If the displacement differences $\Delta u(x)$ and $\Delta v(y)$ can be determined from (6.4) and (6.5), then the displacements are given by (6.1) and (6.2), and the stresses by (3.2), (3.5), (3.6), (4.2) and (4.3). Thus, the problem is reduced to that of solving (6.4) and (6.5) for the displacement differences.

7. The fundamental integral equation. We wish to show that solutions of (6.4) and (6.5) exist, and that approximate solutions can be obtained by iteration. To do this, we combine (6.4) and (6.5) into a single Fredholm integral equation of the second kind and then rely largely on well-known facts about such equations (see Courant and Hilbert [13] for relevant theorems). The equation is

$$\Delta y(x)\phi(x) = f(x) + \int K(x, x')\phi(x')\, dx', \tag{7.1}$$

where the unknown $\phi(x)$ is the difference quotient

$$\phi(x) = \Delta u(x)/\Delta y(x). \tag{7.2}$$

The symmetric kernel $K(x, x')$ is defined by the integral

$$K(x, x') = \int U(x, y)U(x', y)[\Delta x(y)]^{-1}\, dy. \tag{7.3}$$

The inhomogeneous term is

$$f(x) = \int U(x, y)[f_x(y)/\Delta x(y)]\, dy - \int K(x, x')[f_y(x')/\Delta y(x')]\, dx'. \tag{7.4}$$

In order to be able to apply the Hilbert–Schmidt theory in its simplest form, we suppose that the boundary and the applied tractions are smooth, so that the equation is at most weakly singular. The main requirement is that at the extreme values of x and y, the widths $\Delta y(x)$ and $\Delta x(y)$ approach zero slowly enough that their reciprocals are integrable. To illustrate the kinds of singularities that can occur, let us briefly consider boundaries of the form

$$|y| = 1 - |x|^p, \qquad p > 0. \tag{7.5}$$

For these boundaries, the kernel K is given for $p \neq 1$ by

$$K(x, x') = (1 - M^{p-1})/(1 - p^{-1}), \tag{7.6}$$

and for $p = 1$ by

$$K(x, x') = \log(1/M), \tag{7.7}$$

where

$$M = \max(|x|, |x'|). \tag{7.8}$$

The kernel is bounded in the cases $p > 1$ for which the boundary is smooth at the extreme points $y = \pm 1$. However, the equation remains singular because for any p, the boundary (7.5) has corners at the extreme points $x = \pm 1$; since $\Delta y(x)$ approaches zero linearly at these corners, the presence of this factor in the left-hand member of (7.1) makes the equation singular. We restrict our attention to smooth boundaries in order to avoid detailed discussion of matters of this kind.

8. Existence and uniqueness. Some simple relations involving the characteristic function U and the kernel K are useful. First, notice that

$$\int U(x, y)\,dx = \Delta x(y) \quad \text{and} \quad \int U(x, y)\,dy = \Delta y(x). \tag{8.1}$$

From these relations, with (7.3), it follows that

$$\int K(x, x')\,dx' = \Delta y(x). \tag{8.2}$$

We first consider the eigenvalue problem associated with (7.1):

$$\int K(x, x')\psi(x')\,dx' = \lambda \Delta y(x)\psi(x). \tag{8.3}$$

From (8.2) it is evident that $\psi =$ const. is an eigenfunction with eigenvalue $\lambda = 1$. In § 9 we show that there is no other eigenfunction with eigenvalue unity. Consequently, (7.1) has a solution, unique up to an additive constant, if and only if the inhomogeneous term f satisfies the condition

$$\int f(x)\,dx = 0. \tag{8.4}$$

These results have a natural physical interpretation. In the present problem it is known from the outset that the solution is nonunique at least to the extent of an arbitrary rigid displacement field. Rigid translations do not concern us here, because they do not affect displacement differences. However, the difference quotient $\Delta u/\Delta y$ changes by an additive constant when the body is rotated. Consequently, if ϕ satisfies (7.1), so does $\phi + c$, as we immediately verify by using (8.2).

As for the condition (8.4), by using (7.4), (8.1), and (8.2) we find that the existence condition is merely the global condition of rotational equilibrium, (4.6).

9. Eigenvalues. We now verify that there are no eigenfunctions with eigenvalue unity except $\psi =$ const. In the course of doing this, we also show that all other eigenvalues lie between zero and unity.

By multiplying (8.3) by $\psi(x)$ and integrating, with (7.3) we obtain

$$\lambda \int \Delta y(x)\psi^2(x)\,dx = \int [I^2(y)/\Delta x(y)]\,dy, \tag{9.1}$$

where

$$I(y) = \int \psi(x)U(x, y)\,dx. \tag{9.2}$$

Since both integrals in (9.1) are positive, λ is positive.

By writing $U = U^2$ in (9.2) and using the Schwarz inequality, we obtain

$$I^2 \leqq \int \psi^2(x)U^2(x, y)\,dx \int U^2(x', y)\,dx' = \Delta x(y) \int \psi^2(x)U(x, y)\,dx. \tag{9.3}$$

By using this result in (9.1) and performing the y-integration by using (8.1) again, we obtain

$$\lambda \int \Delta y(x)\psi^2(x)\,dx \leqq \int \Delta y(x)\psi^2(x)\,dx. \tag{9.4}$$

Thus, $\lambda \leqq 1$.

Equality holds in (9.3), and thus $\lambda = 1$, only if $\psi(x)U(x, y)$ is a multiple $c(y)$ of $U(x, y)$. This is true only if $\psi(x)$ is a constant. Thus, there are no eigenfunctions with $\lambda = 1$ except constants.

10. Convergence of iteration. The solution of (7.1) is the limit as $n \to \infty$ of the sequence of functions ϕ_n defined recursively by

$$\Delta y(x)\phi_{n+1}(x) = f(x) + \int K(x, x')\phi_n(x')\,dx', \tag{10.1}$$

with any first approximation ϕ_0. This result would follow immediately if all eigenvalues were less than unity in magnitude, but that is not the case in the present problem.

The eigenfunctions are orthogonal with respect to the weighted inner product defined by

$$(f, g) = \int \Delta y(x) f(x) g(x)\,dx. \tag{10.2}$$

With this inner product, the normalized version of the rotational eigenfunction $\psi = \text{const.}$ is $\psi = A^{-1/2}$, where A is the area enclosed by C. Then the rotational component of the function ϕ_n is

$$r_n = (\phi_n, A^{-1/2}). \tag{10.3}$$

By integrating both sides of (10.1) and using (8.2) and (8.4), we find that $r_{n+1} = r_n$. Hence, all iterates have the same rotational component as the first approximation, which is arbitrary.

In the subspace orthogonal to $\psi = A^{-1/2}$, the integral operator is a contraction since all eigenfunctions in this subspace have eigenvalues less than unity in magnitude. Thus the projection of the sequence ϕ_n on this subspace converges, to a unique limit. Since it converges trivially in the complement, the sequence ϕ_n converges. The lack of uniqueness of the limit amounts to an additive constant representing the rotational component of the first approximation ϕ_0.

11. Concluding remarks. The result obtained in the present note adds support to the following conjecture: A boundary value problem for infinitesimal, plane deformations of an ideal fiber-reinforced composite is well-set if the boundary data yields a well-set problem in infinitesimal elasticity theory, provided that u is specified at no more than one point on each fiber and v is specified at no more than one point on each normal line.

The iterative method of solution proposed here is likely to be useful only with automatic numerical computation. More effective methods of solution remain to be devised.

REFERENCES

[1] A. C. PIPKIN AND T. G. ROGERS, *Plane deformations of incompressible fiber-reinforced materials*, J. Appl. Mech., 38 (1971), pp. 634–640.

[2] ———, *A mixed boundary value problem for fiber-reinforced materials*, Quart. Appl. Math., 29 (1971), pp. 151–155.

[3] T. G. ROGERS AND A. C. PIPKIN, *Small deflections of fiber-reinforced beams or slabs*, J. Appl. Mech., 38 (1971), pp. 1047–1048.

[4] G. C. EVERSTINE AND T. G. ROGERS, *A theory of machining of fiber-reinforced materials*, J. Comp. Mat., 5 (1971), pp. 94–106.

[5] T. G. ROGERS AND A. C. PIPKIN, *Finite lateral compression of a fiber-reinforced tube*, Quart. J. Mech. Appl. Math., 24 (1971), pp. 311–330.

[6] G. C. EVERSTINE AND A. C. PIPKIN, *Stress channelling in transversely isotropic elastic composites*, ZAMP, 22 (1971), pp. 825–834.

[7] B. C. KAO AND A. C. PIPKIN, *Finite buckling of fiber-reinforced columns*, Acta Mech., 13 (1972), pp. 265–280.

[8] A. J. M. SPENCER, *Plane strain bending of laminated fiber-reinforced plates*, Quart. J. Mech. Appl. Math., to appear.

[9] ———, *Deformations of Fibre-Reinforced Materials*, Oxford University Press, London, 1972.

[10] A. C. PIPKIN, *Finite deformations of ideal fiber-reinforced composites*, Micromechanics, G. P. Sendeckyi, ed., Academic Press, New York, 1973.

[11] A. H. ENGLAND, *The stress boundary value problem for an ideal fibre-reinforced material*, J. Inst. Math. Appl., 9 (1972), pp. 310–322.

[12] A. H. ENGLAND AND T. G. ROGERS, *Plane crack problems for ideal fibre-reinforced materials.*

[13] R. COURANT AND D. HILBERT, *Methods of Mathematical Physics*, vol. I, Interscience, New York, 1953.