Documento de Trabajo

8 8 2 0

GENERALIZATION OF THE KALMAN FILTER FOR A KIND OF RATIONAL EXPECTATIONS MODELS

Emilio Cerdá Tena

FACULTAD DE CIENCIAS ECONOMICAS Y EMPRESARIALES.- UNIVERSIDAD COMPLUTENSE
Campus de Somosaguas. 28023 - MADRID

Nº C → X-53-226310-2

Nº E → 53074030 1X

GENERALIZATION OF THE KALMAN FILTER FOR A KIND OF RATIONAL EXPECTATIONS MODELS

Emilio CERDA TENA
Universidad Complutense de Madrid

ABSTRACT:

In this paper we obtain a generalization of the Kalman Filter for a kind of models in which the value of the vector variable in period t is explained linearly by the value it had in the previous period, by the rational expectations about the value that the variable y would take in period t, that the economic agents had in previous periods and by additive Gaussian noise. Then we try to get rid of the Gaussian hypothesis and we find a kind of systems in which we don't need that hypothesis, although these systems will not be, in general, rational expectations models.

1.- Introduction

In the papers by Aoki-Canzoneri (1979), Visco (1981, 1984), Broze-Szafarz (1984) and Schonfeld (1984) appear models, containing rational expectations of the following type:

$$y_t = Ay_{t-1} + B_1\, y^*_{t/t-1} + \dots + B_p\, y^*_{t/t-p} + u_t$$

,where $y^*_{t/t-i} = E(y_t | I_{t-i})$

In these models, then, the value of the variable y in period t is explained by the value it had in the previous period, by the (rational) expectations about the value that the variable y would take in period t, that the economic agents had in previous periods, and by additive noise.

These papers consider complete information. We are going to consider incomplete information and we want to solve the problem of estimation.

2.- Problem 1

Consider the model

(2.1) $$y_t = A_t y_{t-1} + \sum_{t=1}^{p} B_{it} y^*_{t/t-i} + u_t \qquad , \; t=p,p+1,\dots,T$$

Consider also the measurement equation:

(2.2) $$z_t = M_t y_t + v_t \qquad , \; t=0,1,2,\dots,T$$

We assume that $y_0, y_1, \dots, y_{p-1}, u_p, u_{p+1}, \dots, u_T$; $v_0, v_1, \dots, v_T$ are mutually independent, Gaussian random vectors, with:

$Eu_t=0$; $Eu_t u_t' = R_t$ $(t=p,p+1,\dots,T)$

$Ev_t=0$; $Ev_t v_t' = V_t$ $(t=0,1,2,\dots,T)$

$Ey_t=m_t$; $E(y_t-m_t)(y_t-m_t)'= S_t$ $(t=0,1,2,\dots,p-1)$

$A_t, B_{1t},\dots, B_{pt}, M_t$ are known matrices.

In this case, $y^*_{t/k} = E(y_t|I_k)$, where -- $I_k= \{z_k, z_{k-1},\dots, z_o\}$. but $y_i \notin I_k$ $i=0,1,2,\dots,k$, because they are unknown.

We want to find the linear least-squares estimator of y_t, given the information I_t $(\hat{y}_{t/t})$

3.- Solution of problem 1

We will proceed as in the Kalman Filter for systems without expectations and we will arrive to an expression of the Kalman Filter that will be function of the vectors $y^*_{t/t-i}$ which are not observables. But, in this particular case, the noises are Gaussian and then $y^*_{t/t-i}=E(y_t|I_{t-i})= \hat{y}_{t/t-i}$, and we can calculate these expressions using the predictor of Kalman

We will proceed as follows:

- We calculate $\hat{y}_{o/o}$. Then we calculate $\hat{y}_{1/o}, \hat{y}_{2/o},\dots, \hat{y}_{p/o}$
- We calculate $\hat{y}_{1/1}$. Then we calculate $\hat{y}_{2/1}, \hat{y}_{3/1},\dots, \hat{y}_{p+1,1}$

⋮

- We calculate $\hat{y}_{t-1/t-1}$. Then we calculate $\hat{y}_{t/t-1}, \hat{y}_{t+1/t-1},\dots$ $\dots, \hat{y}_{t+p-1/t-1}$

- We calculate $\hat{y}_{t/t}$ (that will depend on $\hat{y}_{t-1/t-1}$ and also on $y^*_{t/t-1}, \ldots, y^*_{t/t-p}$ that in this case are $\hat{y}_{t/t-1}, \ldots, \hat{y}_{t/t-p}$ which we'll have calculated previously). Then we calculate $\hat{y}_{t+1/t}, \hat{y}_{t+2/t}, \ldots, \hat{y}_{t+p/t}$.

We follow the notation and previous results of Bertsekas (1976).

Theorem 1 (Generalization of the Kalman Filter for the kind of models we are considering)

For problem 1, we obtain the following recurrent equations:

$$\hat{y}_{t/t}=\hat{y}_t(I_t)= (I-D_tM_t)\ (A_t\hat{y}_{t-1/t-1}+\ \sum_{i=1}^{p} B_{it}\ \hat{y}_{t/t-i})+D_tz_t$$

(for $t=p, p+1, \ldots, T$)

with: $\hat{y}_{o/-1}= m_o \Rightarrow \hat{y}_{o/o} =(I-D_oM_o)m_o+D_oz_o$

$\hat{y}_{1/o}= m_1 \Rightarrow \hat{y}_{1/1} =(I-D_1M_1)m_1+D_1z_1$

$\hat{y}_{2/o}= \hat{y}_{2/1}= m_2 \Rightarrow \hat{y}_{2/2}=(I-D_2M_2)m_2+D_2z_2$

$\vdots$

$\hat{y}_{p-1/o}= \hat{y}_{p-1/1}=\cdots\cdots= \hat{y}_{p-1/p-2}=m_{p-1} \Rightarrow \hat{y}_{p-1/p-1} =$

$= (I-D_{p-1}M_{p-1})m_{p-1}+D_{p-1}z_{p-1}$

where:

$$D_t = \Sigma_{t/t-1} M_t' (M_t \Sigma_{t/t-1} M_t' + V_t)^{-1}$$

$$\Sigma_{t/t-1} = A_t \Sigma_{t-1/t-1} A_t' + R_t$$

$$\Sigma_{t/t} = \Sigma_{t/t-1} - \Sigma_{t/t-1} M_t' (M_t \Sigma_{t/t-1} M_t' + V_t)^{-1} M_t \Sigma_{t/t-1}$$

with: $\Sigma_{o/-1} = S_o$; $\Sigma_{1/o} = S_1$; $\Sigma_{2/1} = S_2, \ldots, \Sigma_{p-1/p-2} = S_{p-1}$

Also:

$$\hat{y}_{t+1/t} = (I - B_{1,t+1})^{-1} (A_{t+1} \hat{y}_{t/t} + \sum_{i=2}^{p} B_{i,t+i} \hat{y}_{t+1/t+1-i})$$

$$\hat{y}_{t+2/t} = (I - B_{1,t+2} - B_{2,t+2})^{-1} (A_{t+2} \hat{y}_{t+1/t} + \sum_{i=3}^{p} B_{i,t+2} \hat{y}_{t+2/t+2-i})$$

$$\vdots$$

$$\hat{y}_{t+p/t} = (I - \sum_{i=1}^{p} B_{i,t+p})^{-1} A_{t+p} \hat{y}_{t+p-1/t}$$

Proof:

Suppose that we have computed the estimates $\hat{y}_{t/t-1}, \hat{y}_{t/t-2}, \ldots, \hat{y}_{t/t-p}$ together with the covariance matrix

$$\Sigma_{t/t-1} = E(y_t - \hat{y}_{t/t-1})(y_t - \hat{y}_{t/t-1})'$$

At time t we receive the additional measurement

$$z_t = M_t y_t + v_t$$

We have:

$$\hat{y}_{t/t} = \hat{y}_{t/t-1} + \hat{y}_t \left[z_t - \hat{z}_t (I_{t-1}) \right] - E(y_t)$$

$$\hat{z}_t(I_{t-1}) = M_t \hat{y}_{t/t-1}$$

$$E \left\{ z_t - \hat{z}_t(I_{t-1}) \right\} = E \left\{ M_t(y_t - \hat{y}_{t/t-1}) + v_t \right\} = 0$$

$$\Rightarrow \hat{y}_t \left[z_t - \hat{z}_t(I_{t-1}) \right] = \hat{y}_t \left[\tilde{z}_t(I_{t-1}) \right] = E(y_t) + \Sigma_{y\tilde{z}} \Sigma^{-1}_{\tilde{z}\tilde{z}} \cdot$$

$$\cdot \left[z_t - \hat{z}_t(I_{t-1}) \right]$$

,where $\Sigma_{y\tilde{z}} = \Sigma_{t/t-1} M_t'$

$$\Sigma_{\tilde{z}\tilde{z}} = M_t \Sigma_{t/t-1} M_t' + V_t$$

Then:

$$\hat{y}_t \left[z_t - \hat{z}_t (I_{t-1}) \right] = E(y_t) + D_t \left[z_t - M_t \hat{y}_{t/t-1} \right]$$, where

$$D_t = \Sigma_{t/t-1} M_t' (M_t \Sigma_{t/t-1} M_t' + V_t)^{-1}$$

We have:

$$\hat{y}_{t/t} = \hat{y}_{t/t-1} + D_t (z_t - M_t \hat{y}_{t/t-1}) = (I - D_t M_t) \hat{y}_{t/t-1} + D_t z_t$$

Let us consider now the system (2.1). Taking conditional expectations of both sides, given I_{t-1}, and knowing that $y^*_{t/t-i} = \hat{y}_{t/t-i}$, we have:

$$y^*_{t/t-1} = \hat{y}_{t/t-1} = A_t\, y^*_{t-1/t-1} + \sum_{i=1}^{p} B_{it}\, y^*_{t/t-i} = A_t\, \hat{y}_{t-1/t-1} + $$

$$+ \sum_{i=1}^{p} B_{it}\, \hat{y}_{t/t-i}$$

$\Rightarrow y_t - \hat{y}_{t/t-1} = A_t\, (y_{t-1} - \hat{y}_{t-1/t-1}) + u_t$ (as in systems without expectations)

$$\Rightarrow \Sigma_{t/t-1} = A_t\, \Sigma_{t-1/t-1}\, A'_t + R_t$$

$$y_t - \hat{y}_{t/t-1} = y_t - \hat{y}_{t/t-1} - D_t\, (z_t - M_t\, \hat{y}_{t/t-1}) = y_t - \hat{y}_{t/t-1} -$$

$$- D_t\, M_t\, (y_t - \hat{y}_{t/t-1}) - D_t v_t$$

$$\Rightarrow \Sigma_{t/t} = \Sigma_{t/t-1} - \Sigma_{t/t-1}\, M'_t\, (M_t\, \Sigma_{t/t-1}\, M'_t + V_t)^{-1}\, M_t\, \Sigma_{t/t-1}$$

then:

$$\hat{y}_{t/t} = (I - D_t M_t)\, (A_t \hat{y}_{t-1/t-1} + \sum_{i=1}^{p} B_{it} \hat{y}_{t/t-i}) + D_t z_t \quad \text{(for } t=p,p+1,\ldots,T)$$

Let us calculate now $\hat{y}_{t+1/t}, \hat{y}_{t+2/t}, \ldots, \hat{y}_{t+p/t}$, (supposing that $t+p \leq T$) . If $t+p > T$, then we calculate $\hat{y}_{t+1/t}, \hat{y}_{t+2/t}, \ldots, \hat{y}_{T/t}$

From (2.1)

$$y_{t+1} = A_{t+1} y_t + \sum_{i=1}^{p} B_{i,t+1}\, y^*_{t+1/t+1-i} + u_{t+1}$$

Taking conditional expectations of both sides, given I_t:

$$y^*_{t+1/t} = A_{t+1}\, E(y_t | I_t) + \sum_{i=1}^{p} B_{i,t+1}\, y^*_{t+1/t+1-i} = A_{t+1}\, \hat{y}_{t/t} + $$
$$+ \sum_{i=1}^{p} B_{i,t+1}\, \hat{y}_{t+1/t+1-i}$$

$$\Rightarrow \hat{y}_{t+1/t} = y^*_{t+1/t} = (I-B_{1,t+1})^{-1} (A_{t+1}\, \hat{y}_{t/t} + \sum_{i=2}^{p} B_{i,t+1} \cdot$$
$$\cdot \hat{y}_{t+1/t+1-i})$$

Similarly:

$$y_{t+2} = A_{t+2}\, y_{t+1} + \sum_{i=1}^{p} B_{i,t+2}\, y^*_{t+2/t+2-i} + u_{t+2}$$

$$\Rightarrow y^*_{t+2/t} = A_{t+2}\, y^*_{t+1/t} + B_{1,t+2}\, y^*_{t+2/t} + B_{2,t+2}\, y^*_{t+2/t} +$$
$$+ \sum_{i=3}^{p} B_{i,t+2}\, y^*_{t+2/t+2-i}$$

$$\Rightarrow \hat{y}_{t+2/t} = y^*_{t+2/t} = (I-B_{1,t+2}-B_{2,t+2})^{-1} (A_{t+2}\, \hat{y}_{t+1/t} +$$
$$+ \sum_{i=3}^{p} B_{i,t+2}\, \hat{y}_{t+2/t+2-i})$$

In general, for $k \in \{1,2,\ldots,p\}$ we have:

$$y_{t+k} = A_{t+k}\, y_{t+k-1} + \sum_{i=1}^{p} B_{i,t+k}\, y^*_{t+k/t+k-i} + u_{t+k}$$

$$y^*_{t+k/t} = A_{t+k}\, y^*_{t+k-1/t} + \sum_{i=1}^{k} B_{i,t+k}\, y^*_{t+k/t} + \sum_{i=k+1}^{p} B_{i,t+k}\, y^*_{t+k/t+k-i}$$

$$\Rightarrow \hat{y}_{t+k/t} = (I- \sum_{i=1}^{k} B_{i,t+k})^{-1} (A_{t+k}\, \hat{y}_{t+k-1/t} +$$
$$+ \sum_{i=k+1}^{p} B_{i,t+k}\, \hat{y}_{t+k/t+k-i})$$

4.- Problem 2

In theorem 1 we needed the hypothesis of Gaussian noises. We wonder if it's possible to get rid of that hypothesis. We'll see in this and the next section that it's possible if we have $\hat{y}_{i/t-1}$ substituting $y^*_{i/t-1}$ in the system. In that case the system will not be, in general, a system with rational expectations but, anyway, we are going now to study this case

Consider the model:

$$(4.1) \qquad y_t = A_t\, y_{t-1} + \sum_{i=1}^{p} B_{it}\, \hat{y}_{t/t-i} + u_t \quad , \qquad t=p,p+1,\dots,T$$

Consider also the measurement equation:

$$(4.2) \qquad z_t = M_t\, y_t + v_t \qquad\qquad t=0,1,2,\dots,T$$

We assume that $y_o, y_1, \dots, y_{p-1}$, $u_p, u_{p+1}, \dots$ $\dots, u_T$, $v_o, v_1, \dots, v_T$ are random vectors mutually uncorrelated, with:

$E\, u_t=0 \quad ; \; E\, u_t\, u_t' = R_t \qquad t= p,\ p+1, \dots,\ T$

$E\, v_t=0 \quad ; \; E\, v_t\, v_t' = V_t \qquad t= 0,1,2,\dots,\ T$

$E\, y_t=m_t \quad ; \; E(y_t-m_t)(y_t-m_t)'=S_t \qquad t=0,1,2,\dots,P$

A_t, $B_{1t}, \dots, B_{pt}$, M_t are known matrices.

In this case $y^*_{t/k} = E(y_t | I_k)$, where $I_k = \{z_k,\ z_{k-1}, \dots,\ z_o\}$

The problem is to find the linear least-squares estimator of y_t, given the information I_t $(\hat{y}_{t/t})$.

5.- Solution of problem 2

Lemma:

$$\text{Consider } z = Cx^1 + D\hat{x}^3(y_1) + u$$

where z, x^1, x^3, u are random vectors; C, D are matrices of constants.

$$\Rightarrow \hat{z}(y_1,y_2) = C\,\hat{x}^1(y_1y_2) + D\,\hat{x}^3(y_1)$$

We assume that the vector u has $Eu=0$ and it's uncorrelated with y_1, y_2.

Proof:

$$\hat{z}(y_1,y_2) = \hat{z}(y_1) + \hat{z}(y_2 - \hat{y}_2(y_1)) - \bar{z}$$

Consider: $\tilde{y}_2(y_1) = y_2 - \hat{y}_2(y_1)$. We know that $\tilde{y}_2(y_1)$ is uncorrelated with y_1 and also with $\hat{y}_2(y_1)$

Now: $\hat{z}(y_1) = C\hat{x}^1(y_1) + D\hat{x}^3(y_1)$

Also: $\hat{z}(\tilde{y}_2) = C\hat{x}^1(\tilde{y}_2) + D\left[\widehat{\hat{x}^3(y_1)}\right](\tilde{y}_2)$

$$\left[\widehat{\hat{x}^3(y_1)}\right](\tilde{y}_2) = \bar{x}^3 + \Sigma_{\hat{x}^3 y_2}\,\Sigma^{-1}_{\tilde{y}_2 y_2}\,\tilde{y}_2(y_1) = \bar{x}^3,$$

because $\Sigma_{\hat{x}^3(y_1)\,\tilde{y}(y_1)} = 0$

Therefore:

$$\hat{z}(y_1,y_2) = C\hat{x}^1(y_1) + D\hat{x}^3(y_1) + C\hat{x}^1(\tilde{y}_2(y_1)) + \bar{x}^3 - \bar{z} \quad =$$

wh

$$= C\left[\hat{x}^1(y_1)+\hat{x}^1\ (y_2-\hat{y}_2(y_1))-\bar{x}^1\right]+D\hat{x}^3(y_1)=C\hat{x}^1(y_1,y_2)+D\ \hat{x}^3(y_1)$$

Theorem 2 (generalization of the Kalman Filter for the kind of models we are considering in section 3).

For problem 2, we obtain the following recurrent equations:

$$\hat{y}_{t/t}=\hat{y}_t(I_t)=(I-D_tM_t)(A_t\hat{y}_{t-1/t-1}+\sum_{i=1}^{p}B_{it}\hat{y}_{t/t-i})+D_tz_t$$

(for $t=p,p+1,\ldots,T$)

with:

$$\hat{y}_{o/-1}=m_o \Rightarrow \hat{y}_{o/o}=(I-D_oM_o)m_o+D_oz_o$$

$$\hat{y}_{1/o}=m_1 \Rightarrow \hat{y}_{1/1}=(I-D_1M_1)m_1+D_1z_1$$

$$\hat{y}_{2/o}=\hat{y}_{2/1}=m_2 \Rightarrow \hat{y}_{2/2}=(I-D_2M_2)m_2+D_2z_2$$

$$\vdots$$

$$\hat{y}_{p-1/o}=\hat{y}_{p-1/1}=\ldots=\hat{y}_{p-1/p-2}=m_{p-1} \Rightarrow \hat{y}_{p-1/p-1}=$$

$$=(I-D_{p-1}M_{p-1})m_{p-1}+D_{p-1}z_{p-1}$$

where:

$$D_t=\Sigma_{t/t-1}M_t'(M_t\Sigma_{t/t-1}M_t'+V_t)^{-1}$$

$$\Sigma_{t/t-1}=A_t\Sigma_{t-1/t-1}A_t'+R_t$$

$$\Sigma_{t/t}=\Sigma_{t/t-1}-\Sigma_{t/t-1}M_t'(M_t\Sigma_{t/t-1}M_t'+V_t)^{-1}M_t\Sigma_{t/t-1}$$

with $\Sigma_{o/-1}=S_o$; $\Sigma_{1/o}=S_1$; $\Sigma_{2/1}=S_2,\ldots,\Sigma_{p-1/p-2}=S_{p-1}$

Also:

$$\hat{y}_{t+1/t} = (I-B_{1,t+1})^{-1}(A_{t+1}\hat{y}_{t/t} + \sum_{i=2}^{p} B_{i,t+1}\ \hat{y}_{t+1/t+1-i})$$

$$\vdots$$

$$\hat{y}_{t+p/t} = (I - \sum_{i=1}^{p} B_{i,t+p})^{-1}\ A_{t+p}\ \hat{y}_{t+p-1/t}$$

Proof:

Proceeding exactly as in theorem 1, we have:

$$\hat{y}_{t/t} = (I-D_tM_t)\ \hat{y}_{t/t-1} + D_t z_t$$

Let us consider now the system (4.1). Aplying the lemma, we have:

$$\hat{y}_{t/t-1} = A_t\ \hat{y}_{t-1/t-1} + \sum_{i=1}^{p} B_{it}\hat{y}_{t/t-i}$$, as in theorem 1

Then, we have the same expressions for $\Sigma_{t/t-1}$ and $\Sigma_{t/t}$

Therefore, we have:

$$\hat{y}_{t/t} = (I-D_tM_t)(A_t\hat{y}_{t-1/t-1} + \sum_{i=1}^{p} B_{it}\ \hat{y}_{t/t-i}) + D_t z_t$$

(for t=p,p+1,...,T)

Let us calculate now $\hat{y}_{t+1/t}, \hat{y}_{t+2/t}, \ldots, \hat{y}_{t+p/t}$ (if $t+p \leq T$). When $t+p > T$, let us calculate $\hat{y}_{t+1/t}, \hat{y}_{t+2/t}, \ldots\ldots, \hat{y}_{T/t}$)

Now, from (4.1)

$$y_{t+1}=A_{t+1}y_t+\sum_{i=1}^{p} B_{i,t+1}\hat{y}_{t+1/t+1-i}+u_{t+1}$$

Aplying the lemma:

$$\hat{y}_{t+1/t}= A_{t+1}\hat{y}_{t/t}+\sum_{i=1}^{p} B_{i,t+1}\hat{y}_{t+1/t+1-i}$$

$$\Rightarrow \hat{y}_{t+1/t}=(I-B_{1,t+1})^{-1}(A_{t+1}\hat{y}_{t/t}+\sum_{i=2}^{p}B_{i,t+1}\hat{y}_{t+1/t+1-i})$$

In general, for k=1,2,....,p

$$y_{t+k}=A_{t+k}y_{t+k-1}+\sum_{i=1}^{p} B_{i,t+k}\hat{y}_{t+k/t+k-i}+u_{t+k}$$

$$\Rightarrow \hat{y}_{t+k/t}=A_{t+k}\hat{y}_{t+k-1/t}+\sum_{i=1}^{k} B_{i,t+k}\hat{y}_{t+k/t}+$$

$$+\sum_{i=k+1}^{p} B_{i,t+k}\hat{y}_{t+k/t+k-i}$$

$$\Rightarrow \hat{y}_{t+k/t}=(I-\sum_{i=1}^{k} B_{i,t+k})^{-1}(A_{t+k}\hat{y}_{t+k-1/t}+\sum_{i=k+1}^{p} B_{i,t+k}\hat{y}_{t+k/t+k-i})$$

Corollary:

If the random vectors y_o, $y_1,\ldots,y_{p-1}$, u_p, u_{p+1},...., u_T, $v_o,v_1,\ldots,v_T$ are all Gaussian, we know that $y^*_{t/t-i}=\hat{y}_{t/t-i}$, then problem 1 is the same than problem 2, and the results given by theorem 1 and theorem 2 are also the same.

References:

Aoki, M. and Canzoneri, M. (1979): "Reduced forms of Rational Expectations models". The Quarterly Journal of Economics. February.

Bertsekas, D. (1976): "Dynamic Programming and Stochastic Control". Academic Press.

Broze, Szafarz (1984): "On linear models with rational expectations which admit a unique solution". European Economic Review - 24, pag. 103-111.

Schonfeld, P. (1984): "Dynamic linear models with rational expectations of current endogenous variables", in Operations Research and Economic theory. Edited by H. Hauptmann, W. Krelle and K.C. - Mosler. Springer-Verlag.

Visco, I. (1981): "On the derivation of reduced forms of rational expectations models". European Economic Review 16. Pag. 355-365.

Visco, I. (1984): "On linear models with rational expectations. An addendum". European Economic Review 24. Pag. 113-115.