Analytic completion of an open set

José M. Ansemil, Jerónimo López-Salazar,

and Socorro Ponte

Departamento de Análisis Matemático
Facultad de Ciencias Matemáticas
Universidad Complutense de Madrid
E-28040 Madrid, Spain
ansemil@mat.ucm.es
jlopezsalazar@mat.ucm.es
ponte@mat.ucm.es

Dedicado al profesor Juan Tarrés con ocasión de su jubilación.

ABSTRACT

Let f be a holomorphic function on a complex normed space E. The possibility of extending f to the completion of E has been studied by Hirschowitz [6], Noverraz [10], Dineen [3] and Dineen-Noverraz [5]. Here, following the ideas in [4, Section 6.1], we study the extension problem for functions defined on open subsets of E. Moreover, through a complexification process, we use these results to obtain the analogous ones for analytic functions on real normed spaces. We note that in the real case we do not have, among other things, Cauchy inequalities and they are essential in the complex case.

Key words: Normed Space, Holomorphic Function, Bounding set.

2010 Mathematics Subject Classification: 46G20, 26E05.

1. Notations

The letter E will denote a normed space over the real or the complex field and $\widehat{E}$ will represent the completion of E. The open ball in E with center in x and radius r will be denoted by $B_E(x,r)$ and the corresponding ball in $\widehat{E}$ by $B_{\widehat{E}}(x,r)$. If n is a natural number n, $\mathcal{P}(^n E)$ will denote the space of all $n-$homogeneous continuous polynomials on E. If U is a non void open subset of E, then $\mathcal{A}(U)$ will denote the space of all analytic functions on U; when E is a complex space, we will write $\mathcal{H}(U)$ instead of $\mathcal{A}(U)$. Given $f \in \mathcal{A}(U)$ and $S \subset U$, let

$$|f|_S = \sup\{|f(z)| : z \in S\}.$$

The second author has been supported by University Complutense de Madrid, grant BE45/08.

We recall that a subset $A \subset U$ is a *bounding* subset in U if every $f \in \mathcal{A}(U)$ is bounded on A. If U is an open subset of E, $\widehat{U}$ will be the following open subset of $\widehat{E}$:

$$\widehat{U} = \bigcup \{B_{\widehat{E}}(x,r) : x \in U,\ r > 0 \text{ and } B_E(x,r) \subset U\}.$$

Note that $\widehat{U}$ is the biggest open subset of $\widehat{E}$ such that $\widehat{U} \cap E = U$ and $\widehat{U} \subset \overline{U}^{\widehat{E}}$.

2. Extension to a neighborhood of each point

Throughout sections 2 and 3, the letter E will always represent a complex normed space and U will be an open subset of E.

Let $z \in U$. If f is a holomorphic function on U, then the polynomial coefficients of the Taylor series of f at z, $\frac{d^n f(z)}{n!}$, are locally uniformly continuous [4, Proposition 1.11]. Therefore, each $\frac{\widehat{d^n f(z)}}{n!}$ admits an extension to $\widehat{E}$ which will also be denoted by $\frac{\widehat{d^n f(z)}}{n!}$. Thus, we can consider the set

$$A_f = \left\{ B_{\widehat{E}}(z,r) : B_E(z,r) \subset U \text{ and } \sum_{n=0}^{\infty} \left| \frac{\widehat{d^n f}(z)}{n!} \right|_{B_{\widehat{E}}(0,r)} < \infty \right\}.$$

The next proposition shows that A_f is non void.

Proposition 2.1. *For every $z \in U$ and every $f \in \mathcal{H}(U)$ there exists $r > 0$ such that $B_{\widehat{E}}(z,r) \in A_f$.*

Proof. Let $z \in U$ and $f \in \mathcal{H}(U)$. As U is open and f is continuous on U, there exists $r > 0$ such that $B_E(z,2r) \subset U$ and $|f|_{B_E(z,2r)} < \infty$. By the density of $B_E(0,r)$ in $B_{\widehat{E}}(0,r)$ and the Cauchy inequalities,

$$\sum_{n=0}^{\infty} \left| \frac{\widehat{d^n f}(z)}{n!} \right|_{B_{\widehat{E}}(0,r)} = \sum_{n=0}^{\infty} \left| \frac{\widehat{d^n f}(z)}{n!} \right|_{B_E(0,r)} \leq \sum_{n=0}^{\infty} \frac{1}{2^n} \cdot |f|_{B_E(z,2r)} < \infty.$$

Then $B_{\widehat{E}}(z,r) \in A_f$. □

Proposition 2.2. *If $f \in \mathcal{H}(U)$, then there exist an open subset $\Omega_f \subset \widehat{E}$ and $\widehat{f} \in \mathcal{H}(\Omega_f)$ such that $U \subset \Omega_f \subset \widehat{U}$ and $\widehat{f}|_U = f$.*

Proof. We define

$$\Omega_f = \bigcup \{B_{\widehat{E}}(z,r) : B_{\widehat{E}}(z,r) \in A_f\}.$$

Let $z_0 \in U$ and $r > 0$ be such that $B_{\widehat{E}}(z_0,r)$ belongs to A_f. Then

$$\sum_{n=0}^{\infty} \left| \frac{\widehat{d^n f}(z_0)}{n!} \right|_{B_{\widehat{E}}(0,r)} < \infty,$$

so the series

$$\sum_{n=0}^{\infty} \frac{\widehat{d^n f}(z_0)}{n!}(z - z_0)$$

is uniformly convergent on $B_{\widehat{E}}(z_0,r)$. This implies that the function

$$\widehat{f}(z) = \sum_{n=0}^{\infty} \frac{\widehat{d^n f}(z_0)}{n!}(z - z_0), \tag{1}$$

is holomorphic on $B_{\widehat{E}}(z_0,r)$. By definition, $\widehat{f} = f$ on $B_E(z_0,r)$.

Let us assume now that $B_{\widehat{E}}(z_1,r_1)$ and $B_{\widehat{E}}(z_2,r_2)$ are two elements of A_f such that

$$B_{\widehat{E}}(z_1,r_1) \cap B_{\widehat{E}}(z_2,r_2) \neq \emptyset$$

and that $\widehat{f_1}$ and $\widehat{f_2}$ are extensions of f to $B_{\widehat{E}}(z_1,r_1)$ and $B_{\widehat{E}}(z_2,r_2)$ respectively. Let w be a point in $B_{\widehat{E}}(z_1,r_1) \cap B_{\widehat{E}}(z_2,r_2)$. Since $w \in \widehat{U} \subset \overline{U}^{\widehat{E}}$, there is a sequence $(w_n)_{n=1}^{\infty} \subset U$ which converges to w. Hence there is $n_0 \in \mathbb{N}$ such that

$$w_n \in B_{\widehat{E}}(z_1,r_1) \cap B_{\widehat{E}}(z_2,r_2)$$

for every $n \geq n_0$. This implies that

$$\widehat{f_1}(w_n) = f(w_n) = \widehat{f_2}(w_n)$$

for every $n \geq n_0$. As $\widehat{f_1}$ and $\widehat{f_2}$ are continuous at w, we have that $\widehat{f_1}(w) = \widehat{f_2}(w)$; that is, $\widehat{f_1} = \widehat{f_2}$ on $B_{\widehat{E}}(z_1,r_1) \cap B_{\widehat{E}}(z_2,r_2)$. Therefore, if we extend the function f to each $B_{\widehat{E}}(z,r) \in A_f$ by the formula (1), then we obtain a holomorphic function $\widehat{f}$ on Ω_f such that $\widehat{f}|_U = f$. □

3. Extension to a given open set in the completion

In this section we deal with the problem of the extension of holomorphic functions from an open subset of E to a subset of $\widehat{E}$.

Proposition 3.1. *If $f \in \mathcal{H}(U)$, $w \in E$ and $n \in \mathbb{N}$, then the function*

$$\begin{array}{rccl} g: & U & \to & \mathbb{C} \\ & z & \mapsto & g(z) = \frac{\widehat{d^n f(z)}}{n!}(w) \end{array}$$

is holomorphic on U.

Proof. Let us define

$$\begin{array}{rccl} F: & U & \to & \mathcal{P}(^nE) \\ & z & \mapsto & F(z) = \frac{\widehat{d^n f(z)}}{n!} \end{array}$$

and

$$\begin{array}{rccl} G: & \mathcal{P}(^nE) & \to & \mathbb{C} \\ & P & \mapsto & G(P) = P(w). \end{array}$$

The mapping F is holomorphic because f has derivatives of all orders. The function G is holomorphic because it is linear and continuous. Therefore, $g = G \circ F$ is holomorphic. □

Definition 3.2. The symbol τ_δ denotes the locally convex topology on $\mathcal{H}(U)$ defined by the seminorms p with this property: for every increasing countable open cover $(V_n)_{n=1}^\infty$ of U there are $C>0$ and $n_0 \in \mathbb{N}$ such that

$$p(f) \le C\,|f|_{V_{n_0}} \quad \text{for every } f \in \mathcal{H}(U).$$

This topology has been deeply studied in many papers, as it is possible to see in [4] and its list of references. It is the bornological topology associated with the compact open topology on $\mathcal{H}(U)$ [4, Proposition 3.18]. We recall that if A is a bounding subset A in U, the mapping

$$f \in \mathcal{H}(U) \mapsto |f|_A$$

is a τ_δ continuous seminorm on $\mathcal{H}(U)$ [4, Example 3.20(c)].

Theorem 3.3. *The following conditions are equivalent:*

a) Every $f \in \mathcal{H}(U)$ admits an extension $\widehat{f} \in \mathcal{H}(\widehat{U})$.

b) If $\widehat{z} \in \widehat{U}$, then every sequence in U which converges to $\widehat{z}$ is a bounding sequence in U.

c) Every point in $\widehat{U}$ is the limit of a bounding sequence in U.

Proof. $(a) \Rightarrow (b)$ Let $(z_n)_{n=1}^\infty$ be a sequence in U which converges to a point $\widehat{z} \in \widehat{U}$. If $f \in \mathcal{H}(U)$, by (a) there is a holomorphic extension $\widehat{f}$ of f to $\widehat{U}$. Then

$$\sup\{|f(z_n)| : n \in \mathbb{N}\} = \sup\left\{\left|\widehat{f}(z_n)\right|, \left|\widehat{f}(z)\right| : n \in \mathbb{N}\right\} < \infty.$$

This supreme is finite because $\{z_n : n \in \mathbb{N}\} \cup \{z\}$ is a compact subset in $\widehat{U}$ and $\widehat{f}$ is continuous on $\widehat{U}$.

$(b) \Rightarrow (c)$ If $\widehat{z} \in \widehat{U} \subset \overline{U}^{\widehat{E}}$, then there is a sequence $(z_n)_{n=1}^\infty$ in U which converges to $\widehat{z}$. By (b), the sequence $(z_n)_{n-1}^\infty$ is bounding in U.

$(c) \Rightarrow (a)$ Let us fix $f \in \mathcal{H}(U)$. We will prove that $\Omega_f = \widehat{U}$ and then we will apply Proposition 2.2. The proof will be divided in several steps.

(i) For every $k, n \in \mathbb{N}$, let us consider the sets

$$U_k = \left\{z \in U : B_E\left(z, \frac{2}{k}\right) \subset U \text{ and } |f|_{B_E(z,\frac{2}{k})} \le k\right\}$$

and

$$V_n = \bigcup_{k=1}^n \left[U_k + B_E\left(0, \frac{1}{k}\right)\right].$$

The family $(V_n)_{n=1}^\infty$ is an increasing open cover of U contained in U. Indeed, it is clear that $(V_n)_{n=1}^\infty$ is an increasing sequence of open subsets. Moreover, if $n \in \mathbb{N}$ and $z \in V_n$, there are $k \in \{1,\dots,n\}$, $z_1 \in U_k$ and $z_2 \in B_E(0,\frac{1}{k})$ such that $z = z_1 + z_2$. By the definition of U_k, we have that $B_E(z_1, \frac{2}{k}) \subset U$. As $\|z - z_1\| = \|z_2\| < \frac{1}{k}$, we obtain that

$$z \in B_E\left(z_1, \frac{1}{k}\right) \subset B_E\left(z_1, \frac{2}{k}\right) \subset U$$

and then $V_n \subset U$.

Let $z \in U$. Since U is open and f is continuous on U, there exists $n \in \mathbb{N}$ such that

$$B_E\left(z, \frac{2}{n}\right) \subset U \text{ and } |f|_{B_E(z,\frac{2}{n})} \le n.$$

Then $z \in U_n \subset V_n$ and, therefore, $U = \bigcup_{n=1}^\infty V_n$.

(ii) For every $n \in \mathbb{N}$, the set $V_n + B_E(0,\frac{1}{n})$ is contained in U. Indeed, let $z \in V_n$ and $w \in B_E(0,\frac{1}{n})$. There exist $k \in \{1,\dots,n\}$, $z_1 \in U_k$ and $z_2 \in B_E(0,\frac{1}{k})$ such that $z = z_1 + z_2$. Then

$$\begin{aligned} z + w &= z_1 + z_2 + w \in z_1 + B_E\left(0,\frac{1}{k}\right) + B_E\left(0,\frac{1}{n}\right) \\ &\subset z_1 + B_E\left(0,\frac{1}{k}\right) + B_E\left(0,\frac{1}{k}\right) = B_E\left(z_1,\frac{2}{k}\right). \end{aligned}$$

Since $z_1 \in U_k$, the ball $B_E(z,\frac{2}{k})$ is contained in U. This proves that $z + w \in U$ and hence $V_n + B_E(0,\frac{1}{n}) \subset U$.

(iii) For every $n \in \mathbb{N}$ we have $|f|_{V_n + B_E(0,\frac{1}{n})} \le n$.

Let $z \in V_n$ and $w \in B_E(0,\frac{1}{n})$. In (ii) we have seen that there are $k \in \{1,\dots,n\}$ and $z_1 \in U_k$ such that $z + w \in B_E(z_1,\frac{2}{k})$. By the definition of U_k,

$$|f(z+w)| \le |f|_{B_E(z_1,\frac{2}{k})} \le k \le n.$$

(iv) Finally we prove that f admits an extension $\widehat{f} \in \mathcal{H}(\widehat{U})$. Let $\widehat{z} \in \widehat{U}$. By hypothesis there is a bounding sequence $A = \{z_n\}_{n=1}^\infty$ in U which converges to $\widehat{z}$. Then the seminorm

$$g \in \mathcal{H}(U) \mapsto |g|_A$$

is τ_δ-continuous. As $(V_n)_{n=1}^\infty$ is an increasing countable open cover of U, there are $C > 0$ and $n_0 \in \mathbb{N}$ such that

$$|g|_A \le C \cdot |g|_{V_{n_0}} \tag{2}$$

for every $g \in \mathcal{H}(U)$. As $(V_n)_{n=1}^\infty$ is increasing we may assume that n_0 is such that $B_{\widehat{E}}\left(\widehat{z}, \frac{1}{n_0}\right) \subset \widehat{U}$.

Since $(z_n)_{n=1}^\infty$ converges to $\widehat{z}$, there exists $k \in \mathbb{N}$ such that $z_k \in B_{\widehat{E}}\left(\widehat{z}, \frac{1}{2n_0}\right)$ and

$$B_{\widehat{E}}\left(z_k, \frac{1}{2n_0}\right) \subset B_{\widehat{E}}\left(\widehat{z}, \frac{1}{n_0}\right) \subset \widehat{U}.$$

Then $B_E\left(z_k, \frac{1}{2n_0}\right) \subset U$.

Let $n \in \mathbb{N} \cup \{0\}$ and let $w \in B_{\widehat{E}}\left(0, \frac{1}{2n_0}\right)$. By Proposition 3.1, the function

$$\begin{array}{rccl} g: & U & \to & \mathbb{C} \\ & z & \mapsto & g(z) = \frac{\widehat{d^n f(z)}}{n!}(w) \end{array}$$

is holomorphic on U. As $z_k \in A$, by (2) we get that

$$\left|\frac{\widehat{d^n f(z_k)}}{n!}(w)\right| = |g(z_k)| \le C \cdot |g|_{V_{n_0}} = C \sup_{z\in V_{n_0}} \left|\frac{\widehat{d^n f(z)}}{n!}(w)\right|.$$

Therefore,

$$\left|\frac{\widehat{d^n f(z_k)}}{n!}\right|_{B_{\widehat{E}}\left(0,\frac{1}{2n_0}\right)} \le C \sup_{z\in V_{n_0}} \left(\left|\frac{\widehat{d^n f(z)}}{n!}\right|_{B_{\widehat{E}}\left(0,\frac{1}{2n_0}\right)}\right).$$

We now use that the ball $B_E\left(0, \frac{1}{2n_0}\right)$ is dense in $B_{\widehat{E}}\left(0, \frac{1}{2n_0}\right)$:

$$\begin{aligned} \left|\frac{\widehat{d^n f(z_k)}}{n!}\right|_{B_{\widehat{E}}\left(0,\frac{1}{2n_0}\right)} &\le C \sup_{z\in V_{n_0}} \left(\left|\frac{\widehat{d^n f(z)}}{n!}\right|_{B_E\left(0,\frac{1}{2n_0}\right)}\right) \\ &= \frac{C}{2^n} \sup_{z\in V_{n_0}} \left(\left|\frac{\widehat{d^n f(z)}}{n!}\right|_{B_E\left(0,\frac{1}{n_0}\right)}\right). \end{aligned}$$

By the Cauchy inequalities,

$$\left|\frac{\widehat{d^n f(z_k)}}{n!}\right|_{B_{\widehat{E}}\left(0,\frac{1}{2n_0}\right)} \le \frac{C}{2^n} \sup_{z\in V_{n_0}} \left(|f|_{B_E\left(z,\frac{1}{n_0}\right)}\right) = \frac{C}{2^n} |f|_{V_{n_0}+B_E\left(0,\frac{1}{n_0}\right)}.$$

Using (iii), we obtain

$$\left|\frac{\widehat{d^n f(z_k)}}{n!}\right|_{B_{\widehat{E}}\left(0,\frac{1}{2n_0}\right)} \le \frac{C}{2^n} n_0.$$

This inequality holds for all n, so

$$\sum_{n=0}^{\infty} \left|\frac{\widehat{d^n f(z_k)}}{n!}\right|_{B_{\widehat{E}}\left(0,\frac{1}{2n_0}\right)} \le \sum_{n=0}^{\infty} \frac{C}{2^n} n_0 < \infty.$$

This implies that $B_{\widehat{E}}\left(z_k, \frac{1}{2n_0}\right) \in A_f$ and then $B_{\widehat{E}}\left(z_k, \frac{1}{2n_0}\right) \subset \Omega_f$. Therefore,

$$\widehat{z} \in B_{\widehat{E}}\left(z_k, \frac{1}{2n_0}\right) \subset \Omega_f.$$

As $\widehat{z}$ is arbitrary in $\widehat{U}$, we get that $\widehat{U} \subset \Omega_f$. As always $\Omega_f \subset \widehat{U}$, we have $\Omega_f = \widehat{U}$. □

4. The real case

In this section we obtain the results proved in the above sections in the context of real normed spaces. Analytic functions on those spaces have been studied by Bochnak [1] and Chae [2], among others.

If E is a real normed space, then the symbol $\widetilde{E}$ will denote the Taylor complexification of E. That is, $\widetilde{E}$ is the complex normed space $E \times E$ endowed with the natural operations

$$\begin{aligned} (x,y)+(x',y') &= (x+x', y+y') && \text{for } (x,y),(x',y') \in \widetilde{E}, \\ (\alpha+i\beta)(x,y) &= (\alpha x - \beta y, \alpha y + \beta x) && \text{for } \alpha+i\beta \in \mathbb{C} \text{ and } (x,y) \in \widetilde{E} \end{aligned}$$

and with the norm

$$\|(x,y)\|_{\widetilde{E}} = \sup\left\{\sqrt{\varphi(x)^2+\varphi(y)^2} : \varphi \in E',\ \|\varphi\| = 1\right\}.$$

Note that $\|(x,y)\|_{\widetilde{E}} = \|x\|$ for every $x \in E$.

Proposition 4.1. *Let U be an open subset of a real normed space E. If $f \in \mathcal{A}(U)$, then there is an open subset $\widetilde{U}_f$ in $\widetilde{E}$ and there is $\widetilde{f} \in \mathcal{H}(\widetilde{U}_f)$ such that $U \times \{0\} \subset \widetilde{U}_f$ and $\widetilde{f}(x,0) = f(x)$ for all $x \in U$.*

Proof. For every $a \in U$ there is $r_a > 0$ and there is a sequence $(P_{a,n})_{n=0}^{\infty}$ of polynomials, where each $P_{a,n}$ belongs to $\mathcal{P}(^nE)$, such that the series

$$\sum_{n=0}^{\infty} P_{a,n}(x-a)$$

converges to $f(x)$ uniformly on $B_E(a, r_a)$. Every polynomial $P_{a,n}$ has a complex extension $\widetilde{P}_{a,n} \in \mathcal{P}(^n\widetilde{E})$ such that $\left\|\widetilde{P}_{a,n}\right\| \le 2^n \|P_{a,n}\|$ (see [7] or [9]). Hence

$$\left(\limsup \left\|\widetilde{P}_{a,n}\right\|^{\frac{1}{n}}\right)^{-1} \ge \left(\limsup 2\|P_{a,n}\|^{\frac{1}{n}}\right)^{-1} = \frac{1}{2}\left(\limsup \|P_{a,n}\|^{\frac{1}{n}}\right)^{-1} \ge \frac{1}{2} r_a,$$

so the radius of convergence of the series

$$\sum_{n=0}^{\infty} \widetilde{P}_{a,n}(z-(a,0)) \tag{3}$$

is bigger than or equal to $\frac{r_a}{2}$. Therefore, the function

$$\widetilde{f}_a(z) = \sum_{n=0}^{\infty} \widetilde{P}_{a,n}(z-(a,0))$$

is holomorphic on $B_{\widetilde{E}}\left((a,0), \frac{r_a}{2}\right)$. By definition, $\widetilde{f}_a(x,0) = f(x)$ for all $x \in B_E\left(a, \frac{r_a}{2}\right)$.

Let

$$\widetilde{U}_f = \bigcup\left\{B_{\widetilde{E}}\left((a,0), \frac{r_a}{2}\right) : a \in U\right\}.$$

Suppose that $a, b \in U$ and

$$B_{\widetilde{E}}\left((a,0), \frac{r_a}{2}\right) \cap B_{\widetilde{E}}\left((b,0), \frac{r_b}{2}\right) \neq \emptyset.$$

Then

$$V = B_E\left(a, \frac{r_a}{2}\right) \cap B_E\left(b, \frac{r_b}{2}\right)$$

is a non void open subset of E.

Since $\widetilde{f_a}(x,0) = f(x) = \widetilde{f_b}(x,0)$ for all $x \in V$, we obtain that $\widetilde{f_a} = \widetilde{f_b}$ on $B_{\widetilde{E}}\left((a,0), \frac{r_a}{2}\right) \cap B_{\widetilde{E}}\left((b,0), \frac{r_b}{2}\right)$ (see [2, Theorem 12.11]). Therefore, we can define a holomorphic function $\widetilde{f} : \widetilde{U}_f \to \mathbb{C}$ as

$$z \in B_{\widetilde{E}}\left((a,0), \frac{r_a}{2}\right) \mapsto \widetilde{f}(z) = \widetilde{f_a}(z)$$

for every $a \in U$. □

Proposition 4.2. *Let E be a real normed space. Suppose that g is a holomorphic function in an open subset V in $\widetilde{E}$ such that $V \cap (E \times \{0\}) \neq \emptyset$ and let*

$$U = \{x \in E : (x,0) \in V\}.$$

Then:

(a) U is open in E and the functions

$$u : \begin{array}{ccc} U & \to & \mathbb{R} \\ x & \mapsto & u(x) = Re\left[g(x,0)\right] \end{array} \quad \text{and} \quad v : \begin{array}{ccc} U & \to & \mathbb{R} \\ x & \mapsto & v(x) = Im\left[g(x,0)\right] \end{array}$$

are real analytic on U.

(b) If A is a subset of U which is bounding for $\mathcal{A}(U)$, then $A \times \{0\}$ is bounding for $\mathcal{H}(V)$.

Proof. (a) Let $x_0 \in U$. There is $r > 0$ such that $B_{\widetilde{E}}((x_0,0), r) \subset V$. If $x \in B_E(x_0, r)$, then

$$\|(x,0) - (x_0,0)\|_{\widetilde{E}} = \|x - x_0\| < r,$$

so $(x,0) \in B_{\widetilde{E}}((x_0,0), r) \subset V$. Therefore, $B_E(x_0, r) \subset U$, so U is an open subset in E.

As g is holomorphic on V, there are $R > 0$ and a sequence $(P_n)_{n=0}^{\infty}$, where $P_n \in \mathcal{P}(^n\widetilde{E})$ for every n, such that $B_{\widetilde{E}}((x_0,0), R) \subset V$ and the series

$$\sum_{n=0}^{\infty} P_n((x,y) - (x_0,0)) = \sum_{n=0}^{\infty} Re\left[P_n(x - x_0, y)\right] + i \sum_{n=0}^{\infty} Im\left[P_n(x - x_0, y)\right]$$

converges to $g(x,y)$ uniformly for $(x,y) \in B_{\widetilde{E}}((x_0,0), R)$.

For every n, the function

$$x \in E \mapsto Re\left[P_n(x,0)\right] \in \mathbb{R}$$

is an n-homogeneous continuous polynomial on E. If $x \in B_E(x_0, R)$, then

$$u(x) = Re\left[g(x,0)\right] = \sum_{n=0}^{\infty} Re\left[P_n(x - x_0, 0)\right]$$

and this series converges uniformly on $B_E(x_0, R)$, so u is real analytic on U. The proof for the function v is analogous.

(b) Let g be a holomorphic function on V. By Proposition 4.2(a), the functions

$$u : x \in U \mapsto Re\left[g(x,0)\right] \quad \text{and} \quad v : x \in U \mapsto Im\left[g(x,0)\right]$$

are real analytic in U. Then

$$\sup_{(x,0) \in A \times \{0\}} |g(x,0)| \leq \sup_{x \in A} |u(x)| + \sup_{x \in A} |v(x)| < \infty.$$

□

Theorem 4.3. *Let U be an open subset of a real normed space E. The following conditions are equivalent:*

(a) Every $f \in \mathcal{A}(U)$ admits an extension $\widehat{f} \in \mathcal{A}(\widehat{U})$.

(b) If $\widehat{x} \in \widehat{U}$, then every sequence in U which converges to $\widehat{x}$ is a bounding sequence in U.

(c) Every point in $\widehat{U}$ is the limit of a bounding sequence in U.

Proof. The proof of (a)⇒(b) and (b)⇒(c) are completely similar to the corresponding case in Theorem 3.3. In order to prove that (c)⇒(a), let us take a function $f \in \mathcal{A}(U)$. By Proposition 4.1, there is an open subset $\widetilde{U}_f$ in $\widetilde{E}$ and there is $\widetilde{f} \in \mathcal{H}(\widetilde{U}_f)$ such that $U \times \{0\} \subset \widetilde{U}_f$ and $\widetilde{f}(x,0) = f(x)$ for all $x \in U$.

By Proposition 2.2, there is an open subset $\Omega_{\widetilde{f}}$ in $\widehat{\widetilde{E}}$ and there is $\widehat{\widetilde{f}} \in \mathcal{H}(\Omega_{\widetilde{f}})$ such that $\widetilde{U}_f \subset \Omega_{\widetilde{f}}$ and $\widehat{\widetilde{f}}|_{\widetilde{U}} = \widetilde{f}$. Then the set

$$\Omega_f = \left\{x \in \widehat{E} : (x,0) \in \Omega_{\widetilde{f}}\right\}$$

is open in $\widehat{E}$ and the function

$$\widehat{f} : x \in \Omega_f \mapsto \widehat{f}(x) = Re\left[\widehat{\widetilde{f}}(x,0)\right]$$

is analytic on Ω_f by Proposition 4.2. If $x \in U$, then $(x,0) \in \widetilde{U}_f \subset \Omega_{\widetilde{f}}$, so $x \in \Omega_f$; that is, $U \subset \Omega_f$. Also, if $x \in U$, then

$$\widehat{f}(x) = Re\left[\widehat{\widetilde{f}}(x,0)\right] = Re\left[\widetilde{f}(x,0)\right] = Re\left[f(x)\right] = f(x).$$

Let $\widehat{x} \in \widehat{U}$. By (c), there is a sequence $(x_n)_{n=1}^{\infty}$ in U which converges to $\widehat{x}$ and it is a bounding subset for $\mathcal{A}(U)$. By Proposition 4.2, the set $\{(x_n, 0) : n \in \mathbb{N}\}$ is bounding for $\mathcal{H}(\widetilde{U}_f)$. As $(x_n, 0) \to (\widehat{x}, 0)$, the proof of (c)⇒(a) in Theorem 3.3 shows that $(\widehat{x}, 0) \in \Omega_{\widetilde{f}}$. Therefore, $\widehat{x} \in \Omega_f$. □

In the proof of Theorem 4.3 we have also obtained the following result:

Proposition 4.4. *Let U be an open subset of a real normed space E. If $f \in \mathcal{A}(U)$, then there is an open subset Ω_f in $\widehat{E}$ and there is $\widehat{f} \subset \mathcal{A}(\Omega_f)$ such that $U \subset \Omega_f$ and $\widehat{f}|_U = f$.*

5. Some examples

In this section we give examples of spaces and open subsets U of them in which it is impossible to extend every function from U to $\widehat{U}$ and others in which this extension is possible. We first need to recall the definition of domain of existence of a holomorphic function and the definition of pseudoconvex set.

Definition 5.1. A connected open subset U of a complex normed space E is the domain of existence of a function $f \in \mathcal{H}(U)$ if there are no open sets V and W in E and no function $\widetilde{f} \in \mathcal{H}(V)$ with the following properties:

(i) V is connected and not contained in U.

(ii) $\emptyset \neq W \subset U \cap V$.

(iii) $\widetilde{f} = f$ on W.

Definition 5.2. A function $\theta : U \to [-\infty, +\infty)$ is plurisubharmonic on U if θ is upper semicontinuous and

$$\theta(a) \leq \frac{1}{2\pi} \int_0^{2\pi} \theta\left(a + e^{it} b\right) d\theta$$

for all $a \in U$ and $b \in E$ such that $\{a + \lambda b : |\lambda| \leq 1\} \subset U$.

Given $x \in U$, let $d_U(x)$ denote the distance from x to the boundary of U. The set U is said to be pseudoconvex if the function $-\log d_U$ is plurisubharmonic on U.

Definition 5.3. Given an open subset U of a real or complex normed space E, we define

$$U_{\mathcal{O}} = \bigcap \{\Omega_f : f \in \mathcal{A}(U)\}.$$

Note that $U \subset U_{\mathcal{O}} \subset \widehat{U}$.

Proposition 5.4. *If U is an open subset of a real or complex normed space E, then*

$$U_{\mathcal{O}} = \bigcup \left\{ \overline{A}^{\widehat{U}} : A \text{ is a bounding subset of } U \right\}.$$

Proof. Let us take $z \in U_{\mathcal{O}}$ and let $\{z_n\}_{n=1}^{\infty}$ be a sequence in U such that $z_n \to z$. If $f \in \mathcal{A}(U)$, then, by Proposition 2.2 in the complex case, and by Proposition 4.4 in the real one, there is $\widehat{f} \in \mathcal{A}(\Omega_f)$ such that $\widehat{f}|_U = f$. As $\{z_n\}_{n=1}^{\infty}$ and z are in $U_{\mathcal{O}}$, then $\{z_n : n \in \mathbb{N}\} \cup \{z\} \subset \Omega_f$. Therefore, $f(z_n) \to \widehat{f}(z)$, so $\{f(z_n)\}_{n=1}^{\infty}$ is bounded. This implies that $\{z_n\}_{n=1}^{\infty}$ is a bounding subset of U and z belongs to its closure in $\widehat{U}$.

Now suppose that $z_0 \in \overline{A}^{\widehat{U}}$, where A is a bounding subset of U. Let $f \in \mathcal{A}(U)$. The proof of (c)⇒(a) in Theorem 3.3 in the complex case and Theorem 4.3 in the real one, shows that $z_0 \in \Omega_f$. As f is arbitrary in $\mathcal{A}(U)$, we deduce that $z_0 \in U_{\mathcal{O}}$. □

Proposition 5.5. *If E is a real or complex normed space of countable dimension and U is an open subset of E, then $U = U_{\mathcal{O}}$. Therefore, there is at least one analytic function f on U such that $\Omega_f \subsetneq \widehat{U}$; that is, f cannot be extended to $\widehat{U}$.*

Proof. We first assume that E is a complex space. The symbol ℓ_1 will denote the space of absolutely summing sequences of complex numbers.

Suppose there is $z_0 \in U_{\mathcal{O}} \backslash U$. By Proposition 5.4, there is a bounding subset A in U such that $z_0 \in \overline{A}^{\widehat{U}}$. Let $(z_n)_{n=1}^{\infty}$ be a sequence in A such that $z_n \to z_0$. The construction in [4, Exercise 6.6(c)] gives a continuous linear mapping $T : \widehat{E} \to \ell_1$ and a plurisubharmonic function $\theta : \ell_1 \to [-\infty, +\infty)$ such that

$$V = \{z \in \ell_1 : \theta(z) < \theta(T(z_0))\}$$

is a pseudoconvex open subset of ℓ_1 and $T(E) \subset V$. By [8, Theorem 45.8], V is a domain of existence.

The sequence $(T(z_n))_{n=1}^{\infty}$ converges to $T(z_0)$, $T(z_n) \in V$ for all n and $T(z_0) \notin V$. Since V is a domain of existence, by [8, Theorem 11.4] there is $g \in \mathcal{H}(V)$ such that

$$\sup\{|g(T(z_n))| : n \in \mathbb{N}\} = \infty.$$

However, as $g \circ T|_U$ is holomorphic on U and A is bounding in U,

$$\sup\{|g \circ T(z_n))| : n \in \mathbb{N}\} \leq \sup\{|g \circ T(z))| : z \in A\} < \infty.$$

We have obtained a contradiction. Therefore, such $z_0 \in U_{\mathcal{O}} \backslash U$ does not exist; that is, $U = U_{\mathcal{O}}$.

Now let us assume that E is a real space. Suppose there is $x_0 \in U_{\mathcal{O}} \backslash U$. Again by Proposition 5.4, there is a bounding subset A in U such that $x_0 \in \overline{A}^{\widetilde{U}}$. Let $(x_n)_{n=1}^{\infty}$ be a sequence in A such that $x_n \to x_0$. As we have seen before, there are a continuous linear mapping $T : \widehat{\widetilde{E}} \to \ell_1$, an open subset $V \subset \ell_1$ and a holomorphic function $g \in \mathcal{H}(V)$ such that

$$\sup\{|g(T(x_n))| : n \in \mathbb{N}\} = \infty.$$

The functions $Re(g \circ T|_U)$ and $Im(g \circ T|_U)$ are analytic on U. As A is bounding in U,

$$\sup\{|Re(g \circ T(x))| : x \in A\} < \infty \quad \text{and} \quad \sup\{|Im(g \circ T(x))| : x \in A\} < \infty,$$

so

$$\sup\{|g(T(x_n))| : n \in \mathbb{N}\} < \infty.$$

This contradiction gives that $U = U_{\mathcal{O}}$. □

Remark 5.6. An example of a normed space of countable dimension is c_{00}, the space of all eventually null sequences, with any of the p norms, $1 \leq p < \infty$. We note that the above proposition is an adaptation to open sets of [10, Theorem 5.3.7]. Our proof follows the arguments in [4, Example 6.6(d)].

Proposition 5.7. [5, Corollary 11] *If F is an infinite dimensional complex Banach space, then there is a dense hyperplane H in F such that $H_{\mathcal{O}} = F$. Therefore, every holomorphic function on H can be extended to a holomorphic function on F.*

Remark 5.8. We note that the argument in [5, Corollary 11] really proves that there are infinitely many dense hyperplanes with this property.

References

[1] J. Bochnak, Analytic functions in Banach spaces, Studia Math., **XXXV** (1970), 272–292.

[2] S.B. Chae, Holomorphy and Calculus in Normed Spaces, New York and Basel, Marcel Dekker Inc., 1985.

[3] S. Dineen, Holomorphically complete locally convex topological vector spaces, Séminaire P. Lelong (Analyse), 1971–1972, Lecture Notes in Math., **332** (1973), 77–111.

[4] S. Dineen, Complex Analysis on Infinite Dimensional Spaces, Springer Monographs in Mathematics, 1999.

[5] S. Dineen, Ph. Noverraz, Gaussian Measures and Polar Sets in Locally Convex Spaces, Ark. Mat., **17** (1979), 217–223.

[6] A. Hirschowitz, Prolongement des fonctions analytiques, Ann. Inst. Fourier (Grenoble), **22** (1972), 255–292.

[7] P. Kirwan, Complexification of multilinear mappings and polynomials, Math. Nachr., **231** (2001), 39 68.

[8] J. Mujica, Complex Analysis in Banach Spaces, Dover Publications, 2010.

[9] G. Muñoz, Y. Sarantopoulos, A. Tonge, Complexifications of real Banach spaces, polynomials and multilinear maps, Studia Math., **134** (1999), 1–33.

[10] Ph. Noverraz, Pseudo-Convexité, Convexité Polynomiale et Domains d'Holomorphie en Dimension Infinie, Notas de Matemática, **48**, North-Holland Math. Stud., Amsterdam, 1973.

Topological classification of finite groups acting on compact surfaces

Emilio BUJALANCE and Francisco Javier CIRRE

Departamento de Matemáticas Fundamentales
Facultad de Ciencias, UNED
c/ Senda del Rey, 9, E-28040 Madrid, Spain
eb@mat.uned.es
jcirre@mat.uned.es

Dedicado a nuestro amigo Juan, con ocasión de su jubilación.

ABSTRACT

This is a survey on results about topological classification of finite groups acting as orientation-preserving homeomorphisms on compact connected oriented two dimensional manifolds.

Key words: finite group action, topological equivalence, Riemann surface.

2010 Mathematics Subject Classification: Primary: 57M60; Secondary: 30F10

1. Introducción

The goal of this note is to survey known results on the classification up to topological equivalence of finite group actions on surfaces of genus $g \geq 2$. By a surface here we mean a compact connected oriented two dimensional manifold. These surfaces have the hyperbolic plane $\mathcal{H}$ as their universal covering and therefore they are orbit spaces $\mathcal{H}/\Gamma$ of $\mathcal{H}$ under the action of a Fuchsian group Γ. This endows the surface $\mathcal{H}/\Gamma$ with an analytic structure and yields the concept of Riemann surface. It follows from the positive answer to the Nielsen realization problem, see [25], that *finite groups acting on surfaces can be seen as groups of analytic automorphisms of Riemann surfaces.*

There is a vast literature on the study of groups of automorphisms of Riemann surfaces, see for instance the introductory chapter in [9] or the recent survey [7]. Much effort has been devoted to find lists of non-isomorphic abstract groups acting on prescribed families of surfaces. The problem we deal with here, that is, the classification of inequivalent topological group actions, is a finer classification since the same abstract group may act on the same surface in different topological ways.

There are important motivations for classifying topological group actions rather than just the groups. One of them is the existence of a one-to-one correspondence between equivalence classes of group actions on a surface and conjugacy classes of

Both authors partially supported by MTM2011-23092.