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Departamento de Estadística
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (34) 91 624-98-49

PERCENTILE RESIDUAL LIFE ORDERS

Alba M. Franco-Pereira, Rosa E. Lillo, Juan Romo and Moshe Shaked

Abstract

In this paper we study a family of stochastic orders of random variables defined via the comparison of their percentile residual life functions. Some interpretations of these stochastic orders are given, and various properties of them are derived. The relationships to other stochastic orders are also studied. Finally, some applications in reliability theory and finance are described.

Keywords: Mean residual life function, hazard rate order, reversed hazard rate order, mixtures, Lehmann's alternative, proportional hazards, imperfect repair, reliability theory, value at risk, deductible.

Universidad Carlos III de Madrid, Departament de Statistics, Facultad de Ciencias Sociales y Jurídicas, C/ Madrid 126, 28903 Getafe (Madrid), e-mail: alba.franco@uc3m.es (Alba M. Franco-Pereira).

1 Introduction

Let X be a random variable, and let u_X be the right endpoint of its support. For any $t < u_X$, the *residual life* at time t, that is associated with X, is any random variable that has the conditional distribution of $X - t$ given that $X > t$. We denote it by

$$X_t = [X - t \big| X > t], \quad t < u_X. \tag{1.1}$$

If F_X denotes the distribution function of X, and $\overline{F}_X = 1 - F_X$ denotes the corresponding survival function, then the survival function of X_t is given by

$$\overline{F}_{X_t}(x) = \frac{\overline{F}_X(t+x)}{\overline{F}_X(t)}, \quad x \geq 0.$$

The residual life is of interest in many areas of applied probability and statistics such as actuarial studies, biometry, survivorship analysis, and reliability — see, for example, Lillo (2005) for a list of references.

The mean residual life function m_X that is associated with X is given by

$$m_X(t) = \begin{cases} E[X - t \big| X > t], & t < u_X; \\ 0, & t \geq u_X, \end{cases} \tag{1.2}$$

provided the expectation exists. It is a useful tool for analyzing important properties of X when it exists. However, the mean residual life function may not exist. Even when it exists it may have some practical shortcomings, especially in situations where the data are censored, or when the underlying distribution is skewed or heavy-tailed. In such cases, either the empirical mean residual life function cannot be calculated, or a single long-term survivor can have a marked effect upon it which will tend to be unstable due to its strong dependence on very long durations.

An alternative to the mean residual life function is the α-percentile residual life function $q_{X,\alpha}$, where α is some number between 0 and 1. This function is defined for any $t < u_X$ by letting $q_{X,\alpha}(t)$ be the α-percentile of X_t. A formal definition of $q_{X,\alpha}(t)$ will be given in Section 2, but here we note that such a function describes, for example, the value that will be survived, by $(1-\alpha)\%$ of items (in reliability theory) or of individuals (in biology), among those that survived up to time t. The α-percentile residual life functions were studied in some detail by Arnold and Brockett (1983), Gupta and Langford (1984), Joe and Proschan (1984a), and Joe (1985), as well as by Haines and Singpurwalla (1974). Raja Rao, Alhumoud, and Damaraju (2006) identified families of distributions for which simple expressions, for the α-percentile residual life functions, can be obtained.

A particular α-percentile residual life function of interest is the median residual life function given by $q_{X,.5}$ — this function was studied in detail by Lillo (2005). Gelfand and Kottas (2003) used it for Bayesian semiparametric modeling. In the above two papers the reader can find further references to papers that studied the α-percentile and the median residual life functions, and that used them in practical applications.

In light of the extensive use of the α-percentile residual life functions in various areas of probability and statistics, it is of interest to develop a theory that compares such functions. The purpose of this paper is to do that. We study here a family of stochastic orders indexed by $\alpha \in (0,1)$. For a fixed $\alpha \in (0,1)$ the αth order compares pointwise $q_{X,\alpha}$ with $q_{Y,\alpha}$, where the latter is the α-percentile residual life function of a random variable Y. These stochastic orders were introduced in Joe and Proschan (1984b), but their properties were not extensively studied there.

In this paper the α-percentile residual life stochastic orders are formally defined in Section 2. We also give there some equivalent ways of describing these orders; these equivalent conditions turn up to be useful in the sequel. Section 3 consists of a thorough study of the relationships among the α-percentile residual life orders and other stochastic orders in the literature. Some useful properties of the α-percentile residual life orders are given in Section 4, and some applications in reliability theory and finance are described in Section 5. Finally, in the Appendix we collect some technical counterexamples that illustrate various statements in the text.

Some conventions that we use in this paper are the following. By "increasing" and "decreasing" we mean "nondecreasing" and "nonincreasing", respectively. For any distribution function F we let function F^{-1} be the left continuous version of the inverse of F, that is

$$F^{-1}(p) = \inf\{x : F(x) \geq p\}, \quad p \in (0,1).$$

2 Definition and equivalent conditions

Let X be a random variable. The α-percentile residual life function $q_{X,\alpha}$ is defined by

$$q_{X,\alpha}(t) = \begin{cases} F_{X_t}^{-1}(\alpha), & t < u_X; \\ 0, & t \geq u_X. \end{cases} \tag{2.1}$$

A straightforward computation shows that

$$q_{X,\alpha}(t) = \overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(t)) - t, \quad t < u_X, \tag{2.2}$$

where $\overline{\alpha} = 1 - \alpha$. Alternatively,

$$q_{X,\alpha}(t) = F_X^{-1}(\alpha + \overline{\alpha}F_X(t)) - t, \quad t < u_X. \tag{2.3}$$

Similar expressions can be found in Joe and Proschan (1984b). Note that, unlike Joe and Proschan (1984a,b), we do not assume here that X is a nonnegative random variable.

Now let Y be another random variable, and let $q_{Y,\alpha}$ be its α-percentile residual life function. If

$$q_{X,\alpha}(t) \leq q_{Y,\alpha}(t) \quad \text{for all } t, \tag{2.4}$$

then we say that X is smaller than Y in the *α-percentile residual life order*, and we denote it as $X \leq_{\alpha\text{-rl}} Y$. The α-percentile residual life orders were introduced in Joe and Proschan

(1984b), but these orders were not extensively studied there. The focus of Joe and Proschan (1984b) was to test the hypothesis $H_0 : F_X = F_Y$ versus $H_1 : q_{X,\alpha} \le q_{Y,\alpha}$.

Note that (2.4) defines a family of stochastic orders, indexed by $\alpha \in (0,1)$. It follows from (2.1) and (2.4) that if $X \le_{\alpha\text{-rl}} Y$ then

$$u_X \le u_Y, \tag{2.5}$$

where u_X and u_Y are the right endpoints of corresponding supports.

The following proposition states equivalent conditions for the α-percentile residual life order to hold.

Proposition 2.1. *Let α be in $(0,1)$ and let X and Y be two random variables.*

(i) *The random variables X and Y satisfy $X \le_{\alpha\text{-rl}} Y$ if, and only if,*

$$\overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(t)) \le \overline{F}_Y^{-1}(\overline{\alpha}\overline{F}_Y(t)) \quad \textit{for all } t.$$

(ii) *The random variables X and Y satisfy $X \le_{\alpha\text{-rl}} Y$ if, and only if,*

$$F_X^{-1}(\alpha + \overline{\alpha}F_X(t)) \le F_Y^{-1}(\alpha + \overline{\alpha}F_Y(t)) \quad \textit{for all } t.$$

(iii) *Suppose that F_X and F_Y are continuous. Then $X \le_{\alpha\text{-rl}} Y$ if, and only if,*

$$\frac{\overline{F}_Y(\overline{F}_X^{-1}(u))}{u} \le \frac{\overline{F}_Y(\overline{F}_X^{-1}(\overline{\alpha}u))}{\overline{\alpha}u} \quad \textit{for all } u \in (0,1).$$

Proof. Parts (i) and (ii) follow at once from (2.2), (2.3), and (2.4). In order to prove part (iii) we note that under the stated assumptions we have that $\overline{F}_X\big(\overline{F}_X^{-1}(p)\big) = p$ and $\overline{F}_Y\big(\overline{F}_Y^{-1}(p)\big) = p$ for all $p \in (0,1)$. Now, by part (i), we have that $X \le_{\alpha\text{-rl}} Y$ is equivalent to

$$\overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(t)) \le \overline{F}_Y^{-1}(\overline{\alpha}\overline{F}_Y(t)) \quad \text{for all } t.$$

Applying $\overline{F}_Y$ to both sides of the above inequality we get that it is equivalent to

$$\overline{F}_Y\big(\overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(t))\big) \ge \overline{\alpha}\overline{F}_Y(t) \quad \text{for all } t.$$

Letting $t = \overline{F}_X^{-1}(u)$ in the latter inequality we see that it is equivalent to

$$\frac{\overline{F}_Y\big(\overline{F}_X^{-1}(\overline{\alpha}u)\big)}{\overline{\alpha}u} \ge \frac{\overline{F}_Y\big(\overline{F}_X^{-1}(u)\big)}{u} \quad \text{for all } u \in (0,1),$$

completing the proof. $\square$

The α-percentile residual life orders indicate comparisons of size or magnitude. For example, letting $t \to -\infty$ in (2.4) we see that if $X \leq_{\alpha\text{-rl}} Y$ then the α-percentile of X is smaller than (or at least not larger than) the α-percentile of Y. Inequality (2.5) is another indication of comparisons of size or magnitude. Now, let l_X and l_Y be the left endpoints of the corresponding supports. One may wonder whether $X \leq_{\alpha\text{-rl}} Y$ implies $l_X \leq l_Y$. Surprisingly, Counterexample A.1 in the Appendix shows that this is not necessarily the case.

Before we proceed to the study of the relationship of the α-percentile residual life orders to other stochastic orders, we present an example that describes a family of random variables that are ordered with respect to $\leq_{\alpha\text{-rl}}$. This family will be used later in the paper.

Example 2.2. Let X have the Pareto distribution:

$$F_X(t) = 1 - \Big(\frac{\gamma}{\gamma + t}\Big)^{\nu}, \quad t \geq 0,$$

where $\gamma > 0$ and $\nu > 0$. Then, for any $\alpha \in (0, 1)$,

$$q_{X,\alpha}(t) = \begin{cases} ((1-\alpha)^{-1/\nu} - 1)\gamma - t, & t < 0; \\ ((1-\alpha)^{-1/\nu} - 1)(\gamma + t), & t \geq 0. \end{cases}$$

Let Y have the Pareto distribution:

$$F_Y(t) = 1 - \Big(\frac{\delta}{\delta + t}\Big)^{\mu}, \quad t \geq 0,$$

where $\delta > 0$ and $\mu > 0$. Then, for any $\alpha \in (0, 1)$,

$$q_{Y,\alpha}(t) = \begin{cases} ((1-\alpha)^{-1/\mu} - 1)\delta - t, & t < 0; \\ ((1-\alpha)^{-1/\mu} - 1)(\delta + t), & t \geq 0. \end{cases}$$

It follows that

$$X \leq_{\alpha\text{-rl}} Y \Longleftrightarrow \begin{cases} \mu \leq \nu \quad \text{and} \\ \frac{(1-\alpha)^{-1/\nu}-1}{(1-\alpha)^{-1/\mu}-1} \leq \frac{\delta}{\gamma}. \end{cases} \qquad \blacktriangleleft$$

3 Relationship to other stochastic orders

Recall that a random variable X is said to be smaller than the random variable Y in the ordinary stochastic order (denoted as $X \leq_{\text{st}} Y$) if $\overline{F}_X(x) \leq \overline{F}_Y(x)$ for all $x \in \mathbb{R}$. It is known that $X \leq_{\text{st}} Y$ if, and only if,

$$F_X^{-1}(p) \leq F_Y^{-1}(p) \quad \text{for all } p \in (0, 1); \tag{3.1}$$

see, for example, (1.A.12) in Shaked and Shanthikumar (2007).

Next recall that a random variable X is said to be smaller than the random variable Y in the hazard rate order (denoted as $X \leq_{\text{hr}} Y$) if $\overline{F}_X(y)\overline{F}_Y(x) \leq \overline{F}_X(x)\overline{F}_Y(y)$ for all $x \leq y$.

Recalling from (1.1) the notation X_t and Y_t for the residual lives that are associated with X and Y, it is known that $X \le_{\rm hr} Y$ if, and only if,

$$X_t \le_{\rm st} Y_t \quad \text{for all } t < u_X; \tag{3.2}$$

see, for example, (1.B.6) in Shaked and Shanthikumar (2007). Equivalently, recalling the notation $q_{X,\alpha}$ and $q_{Y,\alpha}$ from (2.1), we can apply (3.1) to (3.2) and see that $X \le_{\rm hr} Y$ if, and only if,

$$q_{X,\alpha}(t) \le q_{Y,\alpha}(t) \quad \text{for all } t < u_X \text{ and } \alpha \in (0,1).$$

From (2.4) we thus obtain the following result (which has already been observed in Joe and Proschan (1984b)).

Theorem 3.1. *Let X and Y be two random variables. Then $X \le_{\rm hr} Y$ if, and only if,*

$$X \le_{\alpha\text{-rl}} Y \quad \textit{for all } \alpha \in (0,1). \tag{3.3}$$

In particular, for any $\alpha \in (0,1)$,

$$\le_{\rm hr} \Longrightarrow \le_{\alpha\text{-rl}} .$$

Joe and Proschan (1984b) stated, without proof, that there is no relationship between the orders $\le_{\rm st}$ and $\le_{\alpha\text{-rl}}$. The following discussion, especially Remarks 3.2 and 3.3 below, extend and formalize that observation of Joe and Proschan (1984b).

Recall from (1.2) the definition of the mean residual life function m_X of a random variable X. Similarly the mean residual life function m_Y, of another random variable Y, is defined. If

$$m_X(t) \le m_Y(t) \quad \text{for all } t \in \mathbb{R},$$

then X is said to be smaller than Y in the mean residual life order (denoted as $X \le_{\rm mrl} Y$); see Shaked and Shanthikumar (2007).

Remark 3.2. The random variables in Counterexample A.1 have expectations $E[X(\alpha)] = \frac{\alpha^2+1}{2}$ and $EY = \frac{1}{2}$. Thus, although $X(\alpha) \le_{\alpha\text{-rl}} Y$ we have $EX(\alpha) > EY$. That is, the α-percentile residual life orders do not preserve expectations. It follows that any stochastic order that preserves expectations cannot be implied by any α-percentile residual life order. In particular, for any $\alpha \in (0,1)$ we have

$$\le_{\alpha\text{-rl}} \not\Longrightarrow \le_{\rm st},$$
$$\le_{\alpha\text{-rl}} \not\Longrightarrow \le_{\rm mrl},$$

and

$$\le_{\alpha\text{-rl}} \not\Longrightarrow \le_{\rm hmrl},$$

where $\le_{\rm hmrl}$ denotes the harmonic mean residual life stochastic order; see Shaked and Shanthikumar (2007) for the definition, and for the fact that the above orders preserve expectations. $\lhd$

Recall that a random variable X is said to be smaller than the random variable Y in the reversed hazard rate order (denoted as $X \le_{\text{rh}} Y$) if $F_X(y)F_Y(x) \le F_X(x)F_Y(y)$ for all $x \le y$. Since the order $\le_{\text{rh}}$ implies the order $\le_{\text{st}}$, it follows from Remark 3.2 that, for any $\alpha \in (0,1)$,

$$\le_{\alpha\text{-rl}} \not\Rightarrow \le_{\text{rh}} .$$

It is known (see, for example, Shaked and Shanthikumar (2007)) that $\le_{\text{rh}} \not\Rightarrow \le_{\text{hr}}$. It thus follows from Theorem 3.1 that $\le_{\text{rh}} \not\Rightarrow \le_{\alpha\text{-rl}}$ for *some* $\alpha \in (0,1)$. However, in the next remark we show a much stronger result.

Remark 3.3. Note that the distribution $F_{X(\alpha)}$ of the random variable $X(\alpha)$ in Counterexample A.1 can be obtained from the distribution of the random variable Y there by shifting some of the mass of F_Y to the right. Thus Counterexample A.1 shows in a simple manner that shifting some mass of a distribution of a random variable to the right can actually *decrease* it in the α-percentile residual life order. This shows that, for any $\alpha \in (0,1)$,

$$\le_{\text{st}} \not\Rightarrow \le_{\alpha\text{-rl}} . \tag{3.4}$$

In fact, it is easy to verify that $F_{X(\alpha)}$ and F_Y in Counterexample A.1 satisfy $F_Y(y)F_{X(\alpha)}(x) \le F_Y(x)F_{X(\alpha)}(y)$ for all $x \le y$; that is, $Y \le_{\text{rh}} X(\alpha)$. It follows that, for *any* $\alpha \in (0,1)$,

$$\le_{\text{rh}} \not\Rightarrow \le_{\alpha\text{-rl}} . \tag{3.5}$$

Note that (3.5) is a stronger statement than (3.4) because the order $\le_{\text{rh}}$ implies the order $\le_{\text{st}}$. ◁

Let us now return to the consideration of the relationship between the orders $\le_{\alpha\text{-rl}}$ and $\le_{\text{hr}}$. In Theorem 3.1 it is shown that condition (3.3) implies $X \le_{\text{hr}} Y$ (actually these two conditions are equivalent). The question that now arises is whether a weaker condition, such as

$$X \le_{\alpha\text{-rl}} Y \quad \text{for all } \alpha \in (0,\beta)$$

for some $\beta \in (0,1)$, implies the same conclusion. It turns out that this is indeed the case, no matter how small β is (provided it is positive). In order to show it we need the following lemma.

Lemma 3.4. *Let $\alpha \in (0,1)$ and let X and Y be two random variables with continuous distributions. If $X \le_{\alpha\text{-rl}} Y$ then*

$$X \le_{(1-\overline{\alpha}^{2^m})\text{-rl}} Y \quad \textit{for all } m = 1, 2, \ldots .$$

Proof. By Proposition 2.1(iii), if $X \le_{\alpha\text{-rl}} Y$ then

$$\frac{\overline{F}_Y(\overline{F}_X^{-1}(u))}{u} \le \frac{\overline{F}_Y(\overline{F}_X^{-1}(\overline{\alpha}u))}{\overline{\alpha}u} \quad \text{for all } u \in (0,1).$$

Replacing above u by $\overline{\alpha}u$ we get

$$\frac{\overline{F}_Y(\overline{F}_X^{-1}(\overline{\alpha}u))}{\overline{\alpha}u} \le \frac{\overline{F}_Y(\overline{F}_X^{-1}(\overline{\alpha}^2u))}{\overline{\alpha}^2u} \quad \text{for all } u \in (0,1),$$

and by induction

$$\frac{\overline{F}_Y(\overline{F}_X^{-1}(\overline{\alpha}^{2^{m-1}}u))}{\overline{\alpha}^{2^{m-1}}u} \leq \frac{\overline{F}_Y(\overline{F}_X^{-1}(\overline{\alpha}^{2^m}u))}{\overline{\alpha}^{2^m}u} \quad \text{for all } u \in (0,1) \text{ and } m = 1,2,\ldots.$$

Multiplying the above inequalities we get

$$\frac{\overline{F}_Y(\overline{F}_X^{-1}(u))}{u} \leq \frac{\overline{F}_Y(\overline{F}_X^{-1}(\overline{\alpha}^{2^m}u))}{\overline{\alpha}^{2^m}u} \quad \text{for all } u \in (0,1) \text{ and } m = 1,2,\ldots,$$

and, by Proposition 2.1(iii), this yields the stated result. □

Theorem 3.5. *Let $\beta \in (0,1)$ and let X and Y be two random variables with continuous distributions. If*

$$X \leq_{\alpha\text{-rl}} Y \quad \textit{for all } \alpha \in (0,\beta)$$

then $X \leq_{\mathrm{hr}} Y$.

Proof. For any $\alpha \in (0,\beta)$, since $X \leq_{\alpha\text{-rl}} Y$ it follows from Lemma 3.4 that

$$X \leq_{(1-\overline{\alpha}^{2^m})\text{-rl}} Y \quad \text{for all } m = 1,2,\ldots.$$

Now, let $\gamma \in [\beta,1)$, and consider

$$\alpha \stackrel{\text{def}}{=} 1-(1-\gamma)^{\frac{1}{2^m}} \quad \text{where} \quad m = \left[\frac{\log\left(\frac{\log(1-\gamma)}{\log(1-\beta)}\right)}{\log 2}\right] + 1;$$

here $[s]$ denotes the integer part of s. It is straightforward to verify that $\alpha < \beta$. Plugging this α in the inequality $X \leq_{(1-\overline{\alpha}^{2^m})\text{-rl}} Y$ we obtain $X \leq_{\gamma\text{-rl}} Y$. Since this is true for every $\gamma \in [\beta,1)$ we get $X \leq_{\mathrm{hr}} Y$ from (3.3). □

Remark 3.6. Looking at condition (3.3) and at Theorem 3.5 it is natural to wonder whether a condition such as

$$X \leq_{\alpha\text{-rl}} Y \quad \text{for all } \alpha \in (\gamma,\beta),$$

for some $0 < \gamma < \beta < 1$ (note that here we do not allow $\gamma = 0$), implies $X \leq_{\mathrm{hr}} Y$. It turns out that this is not the case. In order to see it, fix a $\gamma \in (0,1)$, and consider the random variables $X(\gamma)$ and Y from Counterexample A.1. For any $\alpha \in (\gamma,1)$ we have

$$q_{X(\gamma),\alpha}(t) = \begin{cases} \alpha - t, & t < \gamma; \\ \alpha(1-t), & \gamma \leq t < 1; \\ 0, & t \geq 1; \end{cases}$$

whereas $q_{Y,\alpha}$ is given in (A.1). It is now easy to verify that $X(\gamma) \leq_{\alpha\text{-rl}} Y$ (and this is true for all $\alpha \in (\gamma,1)$), but $X(\gamma) \not\leq_{\mathrm{hr}} Y$. ◁

In Counterexample A.2 in the Appendix it is shown that for any $\alpha \in (0,1)$ we have

$$\leq_{\text{mrl}} \not\Longrightarrow \leq_{\alpha\text{-rl}} .$$

Since $\leq_{\text{mrl}} \Longrightarrow \leq_{\text{hmrl}}$ (see Shaked and Shanthikumar (2007, page 95)), it follows from Counterexample A.2 that for any $\alpha \in (0,1)$ we have

$$\leq_{\text{hmrl}} \not\Longrightarrow \leq_{\alpha\text{-rl}} .$$

One may wonder whether the orders $\leq_{\alpha\text{-rl}}$ and $\leq_{\beta\text{-rl}}$ imply each other when $\alpha \neq \beta$. Counterexample A.3 in the Appendix shows that if $\beta < \alpha$ then $X \leq_{\alpha\text{-rl}} Y$ does not necessarily imply that $X \leq_{\beta\text{-rl}} Y$. Counterexample A.4 in the Appendix shows that also if $\beta < \alpha$ then $X \leq_{\alpha\text{-rl}} Y$ does not necessarily imply that $X \leq_{\beta\text{-rl}} Y$.

4 Closure properties

The α-percentile residual life orders satisfy some desirable closure properties. These are described and discussed in this section.

First we show that the α-percentile residual life orders are preserved under strictly increasing transformations.

Theorem 4.1. *Let X and Y be random variables, let $\alpha \in (0,1)$, and let ϕ be a strictly increasing function. Then $X \leq_{\alpha\text{-rl}} Y$ if, and only if, $\phi(X) \leq_{\alpha\text{-rl}} \phi(Y)$.*

Proof. Let $\overline{F}_{\phi(X)}$ and $\overline{F}_{\phi(Y)}$ denote the survival functions of the indicated random variables. Since ϕ is strictly increasing we have

$$\overline{F}_{\phi(X)}(t) = \overline{F}_X(\phi^{-1}(t)) \quad \text{and} \quad \overline{F}_{\phi(Y)}(t) = \overline{F}_Y(\phi^{-1}(t)) \quad \text{for all } t,$$

and

$$\overline{F}^{-1}_{\phi(X)}(u) = \phi(\overline{F}^{-1}_X(u)) \quad \text{and} \quad \overline{F}^{-1}_{\phi(Y)}(u) = \phi(\overline{F}^{-1}_Y(u)) \quad \text{for all } u \in (0,1).$$

Therefore, by Proposition 2.1(i), $\phi(X) \leq_{\alpha\text{-rl}} \phi(Y)$ if, and only if,

$$\phi(\overline{F}^{-1}_X(\overline{\alpha}\overline{F}_X(\phi^{-1}(t)))) \leq \phi(\overline{F}^{-1}_Y(\overline{\alpha}\overline{F}_Y(\phi^{-1}(t)))) \quad \text{for all } t.$$

By the strict monotonicity of ϕ, the latter condition is equivalent to

$$\overline{F}^{-1}_X(\overline{\alpha}\overline{F}_X(\phi^{-1}(t))) \leq \overline{F}^{-1}_Y(\overline{\alpha}\overline{F}_Y(\phi^{-1}(t))) \quad \text{for all } t.$$

Letting $t' = \phi^{-1}(t)$, this condition is the same as

$$\overline{F}^{-1}_X(\overline{\alpha}\overline{F}_X(t')) \leq \overline{F}^{-1}_Y(\overline{\alpha}\overline{F}_Y(t')) \quad \text{for all } t',$$

and the stated result follows from Proposition 2.1(i). $\square$

For the next result we need the following lemma from van der Vaart (1998, page 305). Note that the assumption in the following lemma, that a distribution function has interval support, means that the distribution function has no "flats" on that interval.

Lemma 4.2. *Let $\{F_n\}$ be a sequence of distribution functions that converges in distribution to F. Suppose that F is continuous and has interval support. Then F_n^{-1} converges to F^{-1} on $(0, 1)$.*

The following result gives conditions under which the α-percentile residual life orders are closed under limits in distribution.

Theorem 4.3. *Let $\{X_n,\ n = 1, 2, \dots\}$ and $\{Y_n,\ n = 1, 2, \dots\}$ be two sequences of random variables such that $X_n \to_{\text{st}} X$ and $Y_n \to_{\text{st}} Y$ as $n \to \infty$, where "$\to_{\text{st}}$" denotes convergence in distribution. Suppose that both X and Y have continuous distribution functions with interval supports. For any $\alpha \in (0, 1)$, if $X_n \leq_{\alpha\text{-rl}} Y_n$, $n = 1, 2, \dots$, then $X \leq_{\alpha\text{-rl}} Y$.*

Proof. From Lemma 4.2 we know that

$$F_X^{-1}(\alpha + \overline{\alpha} F_X(t)) = \lim_{n\to\infty} F_{X_n}^{-1}(\alpha + \overline{\alpha} F_{X_n}(t))$$

and that

$$F_Y^{-1}(\alpha + \overline{\alpha} F_Y(t)) = \lim_{n\to\infty} F_{Y_n}^{-1}(\alpha + \overline{\alpha} F_{Y_n}(t))$$

for all t. If $X_n \leq_{\alpha\text{-rl}} Y_n$, $n = 1, 2, \dots$, then, using Proposition 2.1(ii), we have

$$F_X^{-1}(\alpha + \overline{\alpha} F_X(t)) = \lim_{n\to\infty} F_{X_n}^{-1}(\alpha + \overline{\alpha} F_{X_n}(t)) \leq \lim_{n\to\infty} F_{Y_n}^{-1}(\alpha + \overline{\alpha} F_{Y_n}(t)) = F_Y^{-1}(\alpha + \overline{\alpha} F_Y(t)),$$

and the stated result follows from Proposition 2.1(ii). □

Without the assumption of interval supports in Theorem 4.3 the conclusion of the theorem may not hold. This is shown in Counterexample A.5 in the Appendix.

The following two lemmas, that deal with simple mixtures, will yield a general closure under mixtures property of the α-percentile residual life orders.

Lemma 4.4. *Let X, Y, U, and V be random variables with continuous distribution functions, and let W be a random variable with distribution function*

$$F_W = pF_X + (1 - p)F_Y,$$

for some $p \in [0, 1]$.

(i) *If $U \leq_{\alpha\text{-rl}} X$ and $U \leq_{\alpha\text{-rl}} Y$ then $U \leq_{\alpha\text{-rl}} W$.*

(ii) *If $X \leq_{\alpha\text{-rl}} V$ and $Y \leq_{\alpha\text{-rl}} V$ then $W \leq_{\alpha\text{-rl}} V$.*

Proof. First we prove (i). From $U \leq_{\alpha\text{-rl}} X$ and $U \leq_{\alpha\text{-rl}} Y$, using Proposition 2.1(i), we obtain

$$\overline{F}_U^{-1}(\overline{\alpha}\,\overline{F}_U(t)) \leq \overline{F}_X^{-1}(\overline{\alpha}\,\overline{F}_X(t)) \quad \text{and} \quad \overline{F}_U^{-1}(\overline{\alpha}\,\overline{F}_U(t)) \leq \overline{F}_Y^{-1}(\overline{\alpha}\,\overline{F}_Y(t)) \quad \text{for all } t.$$

It follows, by the continuity of F_X and of F_Y, that

$$\overline{F}_X(\overline{F}_U^{-1}(\overline{\alpha}\,\overline{F}_U(t))) \geq \overline{\alpha}\,\overline{F}_X(t) \quad \text{and} \quad \overline{F}_Y(\overline{F}_U^{-1}(\overline{\alpha}\,\overline{F}_U(t))) \geq \overline{\alpha}\,\overline{F}_Y(t) \quad \text{for all } t.$$

Therefore,

$$p\overline{F}_X(\overline{F}_U^{-1}(\overline{\alpha}\overline{F}_U(t))) + (1-p)\overline{F}_Y(\overline{F}_U^{-1}(\overline{\alpha}\overline{F}_U(t))) \geq \overline{\alpha}p\overline{F}_X(t) + \overline{\alpha}(1-p)\overline{F}_X(t) \quad \text{for all } t;$$

that is,

$$\overline{F}_W(\overline{F}_U^{-1}(\overline{\alpha}\overline{F}_U(t))) \geq \overline{\alpha}\overline{F}_W(t) \quad \text{for all } t.$$

By the continuity of F_W we get

$$\overline{F}_U^{-1}(\overline{\alpha}\overline{F}_U(t)) \leq \overline{F}_W^{-1}(\overline{\alpha}\overline{F}_W(t)) \quad \text{for all } t;$$

that is, by Proposition 2.1(i), $U \leq_{\alpha\text{-rl}} W$.

Now we prove (ii). From $X \leq_{\alpha\text{-rl}} V$ and $Y \leq_{\alpha\text{-rl}} V$, using Proposition 2.1(i), we obtain

$$\overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(t)) \leq \overline{F}_V^{-1}(\overline{\alpha}\overline{F}_V(t)) \quad \text{and} \quad \overline{F}_Y^{-1}(\overline{\alpha}\overline{F}_Y(t)) \leq \overline{F}_V^{-1}(\overline{\alpha}\overline{F}_V(t)) \quad \text{for all } t.$$

It follows, by the continuity of F_X and of F_Y, that

$$\overline{\alpha}\overline{F}_X(t) \geq \overline{F}_X(\overline{F}_V^{-1}(\overline{\alpha}\overline{F}_V(t))) \quad \text{and} \quad \overline{\alpha}\overline{F}_Y(t) \geq \overline{F}_Y(\overline{F}_V^{-1}(\overline{\alpha}\overline{F}_V(t))) \quad \text{for all } t.$$

Therefore,

$$\overline{\alpha}p\overline{F}_X(t) + \overline{\alpha}(1-p)\overline{F}_Y(t) \geq p\overline{F}_X(\overline{F}_V^{-1}(\overline{\alpha}\overline{F}_V(t))) + (1-p)\overline{F}_Y(\overline{F}_V^{-1}(\overline{\alpha}\overline{F}_V(t))) \quad \text{for all } t;$$

that is,

$$\overline{\alpha}\overline{F}_W(t) \geq \overline{F}_W(\overline{F}_V^{-1}(\overline{\alpha}\overline{F}_V(t))) \quad \text{for all } t.$$

By the continuity of F_W we get

$$\overline{F}_W^{-1}(\overline{\alpha}\overline{F}_W(t)) \leq \overline{F}_V^{-1}(\overline{\alpha}\overline{F}_V(t)) \quad \text{for all } t;$$

that is, by Proposition 2.1(i), $W \leq_{\alpha\text{-rl}} V$. □

Lemma 4.5. *Let X_1, X_2, Y_1, and Y_2 be random variables with continuous distribution functions, and let W and Z be random variables with distribution functions*

$$F_W = pF_{X_1} + (1-p)F_{X_2} \quad \textit{and} \quad F_Z = pF_{Y_1} + (1-p)F_{Y_2},$$

for some $p \in [0,1]$. If there exists a random variable S such that

$$X_1 \leq_{\alpha\text{-rl}} S, \quad X_2 \leq_{\alpha\text{-rl}} S, \quad S \leq_{\alpha\text{-rl}} Y_1, \quad S \leq_{\alpha\text{-rl}} Y_2,$$

then $W \leq_{\alpha\text{-rl}} Z$.

Proof. Since $X_1 \leq_{\alpha\text{-rl}} S$ and $X_2 \leq_{\alpha\text{-rl}} S$, it follows from Lemma 4.4(ii) that $W \leq_{\alpha\text{-rl}} S$. Furthermore, since $S \leq_{\alpha\text{-rl}} Y_1$ and $S \leq_{\alpha\text{-rl}} Y_2$, it follows from Lemma 4.4(i) that $S \leq_{\alpha\text{-rl}} Z$. By the transitivity property of the order $\leq_{\alpha\text{-rl}}$ we get $W \leq_{\alpha\text{-rl}} Z$. □

By repeated application of Lemma 4.5, and limiting arguments, we obtain the following result.

Theorem 4.6. *Let $\{X_\theta,\ \theta \in \Theta\}$ and $\{Y_\theta,\ \theta \in \Theta\}$ be two families of random variables with continuous distribution functions. Let W and Z be random variables with distribution functions given by*

$$F_W(t) = \int_\Theta F_{X_\theta}(t)dH(\theta) \quad \textit{and} \quad F_Z(t) = \int_\Theta F_{Y_\theta}(t)dH(\theta), \quad t \in \mathbb{R},$$

where H is some distribution function on Θ. Suppose that there exists a random variable S such that

$$X_\theta \leq_{\alpha\text{-rl}} S \leq_{\alpha\text{-rl}} Y_\theta \quad \textit{for all } \theta \in \Theta. \tag{4.1}$$

then $W \leq_{\alpha\text{-rl}} Z$.

Note that condition (4.1) can be rewritten as

$$X_\theta \leq_{\alpha\text{-rl}} Y_{\theta'} \quad \text{for all } \theta, \theta' \in \Theta.$$

It is worth noting that results that are similar to Theorem 4.6 hold for the hazard rate order, the reversed hazard rate order, the likelihood ratio order, and the mean residual life order (see, respectively, Theorems 1.B.8, 1.B.46, 1.C.15, and 2.A.13 in Shaked and Shanthikumar, 2007).

A special case of Theorem 4.6 is the following result which shows that a random variable, whose distribution is a mixture of two distributions of α-percentile residual life ordered random variables, is bounded from below and from above, in the α-percentile residual life order sense, by these two random variables.

Corollary 4.7. *Let X and Y be two random variables with continuous distribution functions, and let W be a random variable with distribution function*

$$F_W = pF_X + (1-p)F_Y,$$

for some $p \in [0,1]$. If $X \leq_{\alpha\text{-rl}} Y$ then $X \leq_{\alpha\text{-rl}} W \leq_{\alpha\text{-rl}} Y$.

Again we note that similar results hold for the hazard rate order, the likelihood ratio order, and the mean residual life order (see, respectively, Theorems 1.B.22, 1.C.30, and 2.A.18 in Shaked and Shanthikumar, 2007).

The possible preservation of a stochastic order under the formation of coherent systems is a useful property that has important applications in reliability theory (see, for example, Barlow and Proschan, 1975, for the definition and the use of coherent systems). Thus it is of interest to ask whether the α-percentile residual life orders are closed under this formation. Boland, El-Neweihi, and Proschan (1994) showed that the hazard rate order is not preserved under the formation of coherent systems. It follows from Theorem 3.1 that, *for some* α, the α-percentile residual life order is not closed under this formation. However, in Counterexample A.6 in the Appendix it is shown that in fact, *for all* α, the α-percentile residual life order is not closed under this formation. This is shown by considering a parallel system of size 2 whose lifetime is the maximum of the lifetimes of its two components.

In fact, unlike the hazard rate order, for every $\alpha \in (0,1)$, the α-percentile residual life order is not even closed under the formation of series systems (that is, under the minimum operation). This is shown in Counterexample A.7 in the Appendix. We point out that some comparisons of minima in percentile residual life orders are given in Corollary 5.2 below.

In relation to Counterexample A.7, which shows that the α-percentile residual life order is not closed under the minimum operation, it is worthwhile to note that if X and Y are continuous random variables, then, for any $\alpha \in (0,1)$ we actually have

$$\min\{X,Y\} \leq_{\alpha\text{-rl}} X. \tag{4.2}$$

In order to see it we note that $\overline{F}_X(t) \geq \overline{\alpha}\overline{F}_X(t)$ for all t. Therefore $t \leq \overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(t))$ and hence $\overline{F}_Y(t) \geq \overline{F}_Y(\overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(t)))$ for all t. It follows that

$$\overline{\alpha}\overline{F}_X(t)\overline{F}_Y(t) \geq \overline{F}_X(\overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(t)))\overline{F}_Y(\overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(t))) \quad \text{for all } t.$$

Since $\overline{F}_{\min\{X,Y\}} = \overline{F}_X\overline{F}_Y$, the last inequality can be written as

$$\overline{\alpha}\overline{F}_{\min\{X,Y\}}(t) \geq \overline{F}_{\min\{X,Y\}}(\overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(t))) \quad \text{for all } t,$$

or, equivalently,

$$\overline{F}_{\min\{X,Y\}}^{-1}(\overline{\alpha}\overline{F}_{\min\{X,Y\}}(t)) \leq \overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(t)) \quad \text{for all } t.$$

Thus (4.2) follows from Proposition 2.1(i).

5 Some applications

Let X be a random variable with survival function $\overline{F}_X$. For $\theta > 0$, let $X(\theta)$ denote a random variable with survival function $\overline{F}_X^{\theta}$. In the theory of statistics, $\overline{F}_X^{\theta}$ is often referred to as the *Lehmann's alternative*. In reliability theory terminology, different $X(\theta)$'s are said to have *proportional hazards*. If $\theta < 1$ then $X(\theta)$ is the lifetime of a component with lifetime X which is subjected to *imperfect repair* procedure where θ is the probability of minimal (rather than perfect) repair (see Brown and Proschan (1983)). If $\theta = n$, where n is a positive integer, then $\overline{F}_X^{n}$ is the survival function of $\min\{X_1, X_2, \ldots, X_n\}$ where $X_1, X_2, \ldots, X_n$ are independent copies of X; that is, $\overline{F}_X^{n}$ is the survival function of a series system of size n where the component lifetimes are independent copies of X. Similarly, if Y is a random variable with survival function $\overline{F}_Y$, then denote by $Y(\theta)$ a random variable with survival function $\overline{F}_Y^{\theta}$. The following result compares $X(\theta)$ and $Y(\theta)$.

Theorem 5.1. *Let X and Y be two random variables with continuous distributions on interval supports. Let $\alpha \in (0,1)$ and $\theta > 0$. If $X \leq_{\alpha\text{-rl}} Y$ then*

$$X(\theta) \leq_{(1-\overline{\alpha}^{\theta})\text{-rl}} Y(\theta). \tag{5.1}$$

Proof. It is not hard to verify that under the continuity assumptions above we have

$$(\overline{F}_X^{\theta})^{-1}(u) = \overline{F}_X^{-1}(u^{1/\theta}) \quad \text{and} \quad (\overline{F}_Y^{\theta})^{-1}(u) = \overline{F}_Y^{-1}(u^{1/\theta}), \quad u \in (0,1),$$

or, equivalently,

$$\overline{F}_X^{-1}(u) = (\overline{F}_X^{\theta})^{-1}(u^{\theta}) \quad \text{and} \quad \overline{F}_Y^{-1}(u) = (\overline{F}_Y^{\theta})^{-1}(u^{\theta}), \quad u \in (0,1).$$

Now, by Proposition 2.1(i), $X \leq_{\alpha\text{-rl}} Y$ means

$$\overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(t)) \leq \overline{F}_Y^{-1}(\overline{\alpha}\overline{F}_Y(t)) \quad \text{for all } t,$$

that is,

$$(\overline{F}_X^{\theta})^{-1}(\overline{\alpha}^{\theta}\overline{F}_X^{\theta}(t)) \leq (\overline{F}_Y^{\theta})^{-1}(\overline{\alpha}^{\theta}\overline{F}_Y^{\theta}(t)) \quad \text{for all } t,$$

and the result follows from Proposition 2.1(i). □

As a corollary of Theorem 5.1 we have the following "preservation property" of the α-percentile residual life order under formation of series systems.

Corollary 5.2. *Let $X_1, X_2 \ldots, X_n$ be independent and identically distributed random variables with a continuous distribution function on an interval support. Also let $Y_1, Y_2 \ldots, Y_n$ be independent and identically distributed random variables with a continuous distribution function on an interval support. If $X_1 \leq_{\alpha\text{-rl}} Y_1$ then*

$$\min\{X_1, X_2 \ldots, X_n\} \leq_{(1-\overline{\alpha}^n)\text{-rl}} \min\{Y_1, Y_2 \ldots, Y_n\}. \tag{5.2}$$

It is of interest to contrast Corollary 5.2 with the result in Counterexample A.7.

It is worthwhile to remark that each of the conclusions of Theorem 5.1 and Corollary 5.2 (that is, (5.1) with $\theta > 0$, or (5.2) with $n \geq 1$) is sufficient for $X \leq_{\alpha\text{-rl}} Y$ or $X_1 \leq_{\alpha\text{-rl}} Y_1$, respectively.

Corollary 5.2 can be useful in reliability theory when it is of importance to compare a particular percentile (say the median; that is, $\alpha = .5$) of the residual life of a series system that survived up to time t_0, with the same percentile (again, say the median) of the residual life of another series system, with different components, that survived up to time t_0. This can be useful, for instance, when t_0 is the time at which the initial warranty of the system expires. For example, if the series systems consist of $n = 4$ components, then the second one will be preferable to the first one, in the median residual life order, if the lifetimes of the components of the first system are smaller than the lifetimes of the components of the second system with respect to the order $\leq_{.169\text{-rl}}$ (since $(1 - .169)^4 \approx .5$). An engineer who is familiar with the possible components of these systems can usually tell whether the two types of components have lifetimes that are ordered with respect to $\leq_{.169\text{-rl}}$.

Similar applications can be described in biometry and in statistics.

We now describe an application in the area of risk management. Consider a firm confronted with a risky business over some time period, and let the random variable X represent the loss that the firm incurs at the end of the period. A common measurement of the risk is the *value at risk*, or VaR for short, which is defined as the α-percentile of the loss distribution for some prescribed confidence level $\alpha \in (0,1)$; see, for example, Hürlimann (2002, 2003). Suppose that the firm insures itself against heavy losses, that is, against losses above some deductible t. Then the loss that the reinsurer experiences (if it does) is $X_t = [X - t \big| X > t]$. Its corresponding VaR is $q_{X,\alpha}(t)$. The order $\le_{\alpha\text{-rl}}$ yields comparisons of such VaRs, and thus it can be useful in the area of reinsurance. This order can be verified through Theorem 3.1, or, using empirical data, through Theorem 4.3.

The order $\le_{\alpha\text{-rl}}$ can also be useful in a market of used items. Suppose that an engineer (or any individual) is considering a purchase of a used machine (or a car, say). Suppose that he has a choice among a few equally aged machines (or cars). If the original machine lifetimes are ordered with respect to the hazard rate order, and if the engineer wishes to maximize a certain α-percentile of the remaining life of the purchased machine, then, obviously (for example, by Theorem 3.1), he should select the machine whose lifetime is the highest with respect to the order $\le_{\text{hr}}$. Note that the requirement that the machine lifetimes are ordered with respect to $\le_{\text{hr}}$ is a very strong requirement that may be hard to verify (or that may not hold) in practice. On the other hand, verification of the order $\le_{\alpha\text{-rl}}$ may be a simpler matter — and it yields the same decision. Moreover, if the above engineer (or individual) has a choice between two markets that have different mixtures of aged machines, and if the original machine lifetimes in these markets satisfy (4.1) [here X_θ and Y_θ, $\theta \in \Theta$, are the original machine lifetimes that are mixed in the two markets], then Theorem 4.6 can determine which market is preferable.

A Technical Counterexamples

In this appendix we give the details of the counterexamples that were mentioned in the text.

Counterexample A.1. For some $\alpha \in (0,1)$, let $X(\alpha)$ have the distribution function given by

$$F_{X(\alpha)}(t) = \begin{cases} 0, & t < \alpha; \\ t, & \alpha \le t < 1; \\ 1, & t \ge 1; \end{cases}$$

that is, $F_{X(\alpha)}$ is a mixture of a uniform distribution on $(\alpha, 1)$ with probability $1-\alpha$, and a degenerate variable at α with probability α. Let Y have the uniform distribution on $(0,1)$. We compute

$$q_{X(\alpha),\alpha}(t) = \begin{cases} \alpha - t, & t < \alpha; \\ \alpha(1-t), & \alpha \le t < 1; \\ 0, & t \ge 1; \end{cases}$$

and

$$q_{Y,\alpha}(t) = \begin{cases} \alpha - t, & t < 0; \\ \alpha(1-t), & 0 \le t < 1; \\ 0, & t \ge 1. \end{cases} \tag{A.1}$$

It is easy to check that $X(\alpha) \le_{\alpha\text{-rl}} Y$ but $l_{X(\alpha)} = \alpha \not\le 0 = l_Y$. ◀

Counterexample A.2. For $(\omega, \gamma, \lambda) \in (0,1)^3$, let X have the survival function given by

$$\overline{F}_X(t) = \begin{cases} 1, & t < 0; \\ 1 - \omega t, & 0 \le t < \gamma; \\ \omega(1-t), & \gamma \le t < 1; \\ 0, & t \ge 1; \end{cases} \tag{A.2}$$

that is, F_X is a mixture of a uniform distribution on $(0,1)$ with probability ω, and a degenerate variable at γ with probability $1-\omega$. Let Y have the survival function given by

$$\overline{F}_Y(t) = \begin{cases} 1, & t < 0; \\ 1 - \lambda t, & 0 \le t < 1; \\ 0, & t \ge 1; \end{cases} \tag{A.3}$$

that is, F_Y is a mixture of a uniform distribution on $(0,1)$ with probability λ, and a degenerate variable at 1 with probability $1-\lambda$. Lengthy computations show that the mean residual life functions of X and Y, respectively, are given by

$$m_X(t) = \begin{cases} \frac{\omega}{2} + \gamma(1-\omega) - t, & t < 0; \\ \frac{\omega(1-t^2)+2\gamma(1-\omega)}{2(1-\omega t)} - t, & 0 \le t < \gamma; \\ \frac{1-t}{2}, & \gamma \le t < 1; \\ 0, & t \ge 1, \end{cases}$$

and

$$m_Y(t) = \begin{cases} 1 - \frac{\lambda}{2} - t, & t < 0; \\ \frac{2-\lambda-\lambda t^2}{2(1-\lambda t)} - t, & 0 \le t < 1; \\ 0, & t \ge 1. \end{cases}$$

Now let ω and λ be such that

$$0 < \omega < \lambda < 1, \tag{A.4}$$

and set

$$\gamma = \frac{1-\lambda}{1-\omega}; \tag{A.5}$$

from (A.4) it follows that $0 < \gamma < 1$.

For $t < 0$ we see that

$$m_X(t) = \frac{\omega}{2} + \gamma(1-\omega) - t = \frac{\omega}{2} + 1 - \lambda - t \le 1 - \frac{\lambda}{2} - t = m_Y(t),$$

where the second equality follows from (A.5), and the inequality follows from (A.4).

For $0 \le t < \gamma$ note, by (A.4) and (A.5), that $2(1-\omega t) \ge 2(1-\lambda t)$ and that

$$\omega(1-t^2) + 2\gamma(1-\omega) = \omega(1-t^2) + 2(1-\lambda) \le \lambda(1-t^2) + 2(1-\lambda) = 2 - \lambda - \lambda t^2.$$

Thus,

$$m_X(t) = \frac{\omega(1-t^2) + 2\gamma(1-\omega)}{2(1-\omega t)} - t \le \frac{2-\lambda-\lambda t^2}{2(1-\lambda t)} - t = m_Y(t).$$

Finally, for $\gamma \le t < 1$ we have

$$m_X(t) = \frac{1-t}{2} \le \frac{2-\lambda-\lambda t^2}{2(1-\lambda t)} - t = m_Y(t),$$

where the inequality follows (after some straightforward manipulations) from $0 < \lambda < 1$ and $0 \le t \le 1$. Thus

$$X \le_{\text{mrl}} Y.$$

Now consider an $\alpha \in (0,1)$. If

$$\lambda > \alpha, \tag{A.6}$$

then the α-percentile of the random variable Y (with survival function given in (A.3)) is easily seen to be

$$q_{Y,\alpha}(0) = \frac{\alpha}{\lambda}.$$

If

$$\omega\gamma > \alpha, \tag{A.7}$$

then the α-percentile of the random variable X (with survival function given in (A.2)) is easily seen to be

$$q_{X,\alpha}(0) = \frac{\alpha}{\omega}.$$

Note that if (A.4) holds then

$$q_{X,\alpha}(0) > q_{Y,\alpha}(0),$$

and therefore $X \not\le_{\alpha\text{-rl}} Y$. For the γ in (A.5) we can rewrite the inequality (A.7) as

$$\frac{\omega(1-\lambda)}{1-\omega} > \alpha. \tag{A.8}$$

In summary, consider the following task:

$$\boxed{\text{\textit{For an} } \alpha \in (0,1)\text{\textit{, find} } (\omega,\lambda) \in (0,1)^2 \text{ \textit{that satisfy the inequalities} (A.4), (A.6), \textit{and} (A.8).}} \tag{A.9}$$

If we can find a solution to the task (A.9), then the corresponding X and Y, with survival functions given in (A.2) and (A.3), will satisfy $X \le_{\text{mrl}} Y$ and $X \not\le_{\alpha\text{-rl}} Y$.

In order to find a solution to the task (A.9) (for any fixed α), let $b > 1$ be a number such that

$$b^{-1} > \alpha.$$

For a small positive ε (that will be shown below to exist), define

$$\omega = \alpha + \varepsilon \quad \text{and}$$
$$\lambda = \alpha + b\,\varepsilon;$$

of course, ε should be small enough so that $\lambda < 1$. Then (A.4) and (A.6) hold. To see that (A.8) also holds, we rewrite it as

$$\frac{(\alpha+\varepsilon)(1-\alpha-b\,\varepsilon)}{1-\alpha-\varepsilon} > \alpha.$$

This simplifies to

$$1 - b\,\alpha - b\,\varepsilon > 0.$$

Since $\alpha < 1/b$ we can find such an $\varepsilon > 0$, and the resulting ω and λ will satisfy (A.4), (A.6), and (A.8). ◀

Counterexample A.3. Let $0 < \beta < \alpha < 1$. Let X and Y have the Pareto distributions, given in Example 2.2, with $\mu = 1$ and $\nu = 2$. Choose γ and δ such that $\frac{(1-\alpha)^{-1/2}-1}{(1-\alpha)^{-1}-1} = \frac{\delta}{\gamma}$. Then, by Example 2.2, $X \leq_{\alpha\text{-rl}} Y$. It is not hard to verify that $\frac{(1-\alpha)^{-1/2}-1}{(1-\alpha)^{-1}-1}$ is strictly decreasing in $\alpha \in (0,1)$. Therefore $\frac{(1-\beta)^{-1/2}-1}{(1-\beta)^{-1}-1} > \frac{\delta}{\gamma}$. It follows from Example 2.2 that $X \not\leq_{\beta\text{-rl}} Y$. ◀

The basic idea in the following counterexample has been inspired by a study of Gupta and Langford (1984). For simplicity we consider a special case of their study (that is, their a and b are taken here to be both equal to 1) that still provides us with our objective – that is, that for $\beta > \alpha$, the inequality $X \leq_{\alpha\text{-rl}} Y$ does not necessarily imply that $X \leq_{\beta\text{-rl}} Y$.

Counterexample A.4. For $\alpha \in (0,1)$, let X has the Pareto distribution with survival function

$$\overline{F}_X(t) = \left(\frac{1}{1+t}\right)^{\frac{-\log(1-\alpha)}{\log 2}}, \quad t \geq 0. \tag{A.10}$$

Now define

$$k_\varepsilon(x) = 1 + \varepsilon \sin\left(\frac{2\pi x}{\log 2}\right), \quad x \in \mathbb{R},$$

where $\varepsilon > 0$, and consider the function H_ε given by

$$H_\varepsilon(t) = \left(\frac{1}{1+t}\right)^{\frac{-\log(1-\alpha)}{\log 2}} \cdot k_\varepsilon(\log(1+t)), \quad t \geq 0.$$

Obviously, $H_\varepsilon(0) = 1$ and $\lim_{t\to\infty} H_\varepsilon(t) = 0$. If we can find an $\varepsilon > 0$ such that $H_\varepsilon(t)$ is decreasing in $t \geq 0$, then it would follow that H_ε is a survival function. In order to identify such an ε, we note that the derivative of k_ε is given by

$$k'_\varepsilon(x) = \varepsilon \cos\left(\frac{2\pi x}{\log 2}\right) \cdot \frac{2\pi}{\log 2}, \quad x \in \mathbb{R},$$

and thus the derivative of H_ε is given by

$$\begin{aligned} H'_\varepsilon(t) = \frac{\log(1-\alpha)}{\log 2} \cdot \left(\frac{1}{1+t}\right)^{\frac{-\log(1-\alpha)}{\log 2}} \cdot \frac{1}{1+t}\left[1 + \varepsilon \sin\left(\frac{2\pi \log(1+t)}{\log 2}\right)\right] \\ + \frac{2\pi}{\log 2} \cdot \frac{1}{1+t} \cdot \left(\frac{1}{1+t}\right)^{\frac{-\log(1-\alpha)}{\log 2}} \cdot \varepsilon \cos\left(\frac{2\pi \log(1+t)}{\log 2}\right), \quad t \geq 0. \end{aligned}$$

Therefore H_ε is decreasing if, and only if,

$$\varepsilon\Big[\log(1-\alpha)\sin\Big(\frac{2\pi\log(1+t)}{\log 2}\Big)+2\pi\cos\Big(\frac{2\pi\log(1+t)}{\log 2}\Big)\Big]\leq-\log(1-\alpha),\quad t\geq 0. \tag{A.11}$$

Since

$$\varepsilon\Big[\log(1-\alpha)\sin\Big(\frac{2\pi\log(1+t)}{\log 2}\Big)+2\pi\cos\Big(\frac{2\pi\log(1+t)}{\log 2}\Big)\Big]\leq\varepsilon(-\log(1-\alpha)+2\pi),\quad t\geq 0,$$

we see that if

$$\varepsilon\leq\frac{-\log(1-\alpha)}{-\log(1-\alpha)+2\pi} \tag{A.12}$$

then (A.11) holds. Thus, for such an ε the function H_ε is a survival function.

Let Y be a random variable with the survival function H_ε, namely,

$$\overline{F}_Y(t)=\Big(\frac{1}{1+t}\Big)^{\frac{-\log(1-\alpha)}{\log 2}}\cdot k_\varepsilon(\log(1+t)),\quad t\geq 0.$$

Recall the random variable X with the survival function given in (A.10). From Gupta and Langford (1984) we know that $q_{X,\alpha}(t)=q_{Y,\alpha}(t)$ for all t. So,

$$X\leq_{\alpha\text{-rl}}Y.$$

Let $\beta>\alpha$ (such that $\beta<1$). We are going to identify a $t_0>0$ such that

$$q_{X,\beta}(t_0)>q_{Y,\beta}(t_0). \tag{A.13}$$

(It would then follow that $X\not\leq_{\beta\text{-rl}}Y$.) Rewriting (A.13) it is seen to be equivalent to

$$\overline{F}_Y(\overline{F}_X^{-1}(\overline{\beta}\,\overline{F}_X(t_0)))<\overline{\beta}\,\overline{F}_Y(t_0).$$

Setting $u_0=\overline{F}_X(t_0)$ it is seen that rather than identifying a t_0 that satisfies (A.13) we may as well identify a $u_0\in(0,1)$ such that

$$\overline{F}_Y(\overline{F}_X^{-1}(\overline{\beta}u_0))<\overline{\beta}\,\overline{F}_Y(\overline{F}_X^{-1}(u_0)). \tag{A.14}$$

We now compute

$$\overline{F}_X^{-1}(u)=u^{\frac{\log 2}{\log(1-\alpha)}}-1,\quad u\in(0,1),$$

and

$$\overline{F}_Y(\overline{F}_X^{-1}(u))=uk_\varepsilon(\log(1+\overline{F}_X^{-1}(u)))=uk_\varepsilon\Big(\frac{\log 2}{\log(1-\alpha)}\cdot\log u\Big).$$

So (A.14) is the same as

$$k_\varepsilon\Big(\frac{\log 2}{\log(1-\alpha)}\cdot\log(\overline{\beta}u_0)\Big)<k_\varepsilon\Big(\frac{\log 2}{\log(1-\alpha)}\cdot\log u_0\Big),$$

which is the same as

$$\sin\left(2\pi \cdot \frac{\log \overline{\beta} + \log u_0}{\log(1-\alpha)}\right) < \sin\left(2\pi \cdot \frac{\log u_0}{\log(1-\alpha)}\right). \tag{A.15}$$

Now take $u_0 = \exp\left\{\frac{\log(1-\alpha)}{4}\right\}$. Then $u_0 \in (0,1)$, as well as $\sin\left(2\pi \cdot \frac{\log \overline{\beta}+\log u_0}{\log(1-\alpha)}\right) < 1$ and $\sin\left(2\pi \cdot \frac{\log u_0}{\log(1-\alpha)}\right) = 1$. So (A.15), and therefore also (A.14), hold for this u_0. It follows that $X \not\leq_{\beta\text{-rl}} Y$. ◀

Counterexample A.5. Let $\alpha \in (0,1)$. For every $n > \frac{1}{\alpha}$ let X_n be a random variable whose distribution is the following mixture:

$$\begin{cases} \text{uniform on } [1.5, 2.5] & \text{with probability } \frac{\alpha n}{n+1}, \\ \text{uniform on } [2.5, 3.5] & \text{with probability } \frac{1}{n+1}, \\ \text{standard exponential with shift } 4.5 & \text{with probability } \frac{(1-\alpha)n}{n+1}; \end{cases}$$

that is

$$F_{X_n}(t) = \begin{cases} 0, & t < 1.5; \\ \frac{\alpha n(t-1.5)}{n+1}, & 1.5 \leq t < 2.5; \\ \frac{\alpha n + t - 2.5}{n+1}, & 2.5 \leq t < 3.5; \\ \frac{\alpha n+1}{n+1}, & 3.5 \leq t < 4.5; \\ 1 - \frac{(1-\alpha)ne^{-(t-4.5)}}{n+1}, & t \geq 4.5. \end{cases}$$

It is easy to see that X_n converges in distribution to X whose distribution is

$$\begin{cases} \text{uniform on } [1.5, 2.5] & \text{with probability } \alpha, \\ \text{standard exponential with shift } 4.5 & \text{with probability } (1-\alpha); \end{cases}$$

that is

$$F_X(t) = \begin{cases} 0, & t < 1.5; \\ \alpha(t-1.5), & 1.5 \leq t < 2.5; \\ \alpha, & 2.5 \leq t < 4.5; \\ 1-(1-\alpha)e^{-(t-4.5)}, & t \geq 4.5. \end{cases}$$

Next, for every $n > \frac{1}{\alpha}$ let Y_n be a random variable whose distribution is the following mixture:

$$\begin{cases} \text{uniform on } [0.5, 1.5] & \text{with probability } \frac{\alpha n}{n+1}, \\ \text{uniform on } [2.5, 3.5] & \text{with probability } \frac{1}{n+1}, \\ \text{standard exponential with shift } 4.5 & \text{with probability } \frac{(1-\alpha)n}{n+1}; \end{cases}$$

that is

$$F_{Y_n}(t) = \begin{cases} 0, & t < 0.5; \\ \frac{\alpha n(t-0.5)}{n+1}, & 0.5 \leq t < 1.5; \\ \frac{\alpha n}{n+1}, & 1.5 \leq t < 2.5; \\ \frac{\alpha n + t - 2.5}{n+1}, & 2.5 \leq t < 3.5; \\ \frac{\alpha n+1}{n+1}, & 3.5 \leq t < 4.5; \\ 1 - \frac{(1-\alpha)ne^{-(t-4.5)}}{n+1}, & t \geq 4.5. \end{cases}$$

It is easy to see that Y_n converges in distribution to Y whose distribution is

$$\begin{cases} \text{uniform on } [0.5, 1.5] & \text{with probability } \alpha, \\ \text{standard exponential with shift } 4.5 & \text{with probability } (1-\alpha); \end{cases}$$

that is

$$F_Y(t) = \begin{cases} 0, & t < 0.5; \\ \alpha(t-0.5), & 0.5 \le t < 1.5; \\ \alpha, & 1.5 \le t < 4.5; \\ 1-(1-\alpha)e^{-(t-4.5)}, & t \ge 4.5. \end{cases}$$

Computing the α-percentile residual life functions that are associated with X_n and with Y_n we get

$$q_{X_n,\alpha}(t) = \begin{cases} 2.5+\alpha-t, & t < 1.5; \\ 2.5+\alpha+n\alpha(1-\alpha)(t-1.5)-t, & 1.5 \le t < 1.5+\frac{1}{\alpha n}; \\ 4.5-\log\left(\frac{n+1-\alpha n(t-1.5)}{n}\right)-t, & 1.5+\frac{1}{\alpha n} \le t < 2.5; \\ 4.5-\log\left(\frac{n-\alpha n-t+3.5}{n}\right)-t, & 2.5 \le t < 3.5; \\ 4.5-\log(1-\alpha)-t, & 3.5 \le t < 4.5; \\ -\log(1-\alpha), & t \ge 4.5; \end{cases}$$

and

$$q_{Y_n,\alpha}(t) = \begin{cases} 2.5+\alpha-t, & t < 0.5; \\ 2.5+\alpha+n\alpha(1-\alpha)(t-0.5)-t, & 0.5 \le t < 0.5+\frac{1}{\alpha n}; \\ 4.5-\log\left(\frac{n+1-\alpha n(t-0.5)}{n}\right)-t, & 0.5+\frac{1}{\alpha n} \le t < 1.5; \\ 4.5-\log\left(\frac{n-\alpha n+1}{n}\right)-t, & 1.5 \le t < 2.5; \\ 4.5-\log\left(\frac{n-\alpha n-t+3.5}{n}\right)-t, & 2.5 \le t < 3.5; \\ 4.5-\log(1-\alpha)-t, & 3.5 \le t < 4.5; \\ -\log(1-\alpha), & t \ge 4.5. \end{cases}$$

It is straightforward to verify that $q_{X_n,\alpha}(t) \le q_{Y_n,\alpha}(t)$ for all t. Thus $X_n \le_{\alpha\text{-rl}} Y_n$, $n > \frac{1}{\alpha}$. On the other hand, by our convention that the inverse distribution function is the left continuous version of it, we see that the α-percentile of X is 2.5 while the α-percentile of Y is 1.5. So $X \not\le_{\alpha\text{-rl}} Y$. ◀

Counterexample A.6. For any $\alpha \in (0,1)$, let X be an exponential random variable with rate $-\log(1-\alpha)$. That is,

$$F_X(t) = \begin{cases} 0, & t < 0; \\ 1-e^{(\log(1-\alpha))t}, & t \ge 0. \end{cases}$$

Let Y be a random variable that is degenerate at 0, and let Z be a random variable that is degenerate at 1. Note that $\max\{X,Y\} =_{\text{st}} X$. Note also that $Y \le_{\alpha\text{-rl}} Z$, and, of course, $X \le_{\alpha\text{-rl}} X$. Now we compute

$$q_{\max\{X,Y\},\alpha}(t) = q_{X,\alpha}(t) = \begin{cases} 1-t, & t < 0; \\ 1, & t \ge 0, \end{cases}$$

and

$$q_{\max\{X,Z\},\alpha}(t) = \begin{cases} 1-t, & t<1; \\ 1, & t \geq 1. \end{cases}$$

It is seen that $\max\{X,Y\} \not\leq_{\alpha\text{-rl}} \max\{X,Z\}$ (in fact, $\max\{X,Y\} \geq_{\alpha\text{-rl}} \max\{X,Z\}$ strictly). Thus the α-percentile residual life order is not closed under the maximum operation. ◀

Counterexample A.7. Let X_1 and X_2 be two random variables that are degenerate at 1. For any $\alpha \in (0,1)$, let Y_1 and Y_2 be two independent exponential random variables, each with rate $-\log(1-\alpha)$. The corresponding α-percentile residual life functions are

$$q_{X_1,\alpha}(t) = q_{X_2,\alpha}(t) = \begin{cases} 1-t, & t<1; \\ 0, & t \geq 1; \end{cases}$$

and

$$q_{Y_1,\alpha}(t) = q_{Y_2,\alpha}(t) = \begin{cases} 1-t, & t<0; \\ 1, & t \geq 0. \end{cases}$$

It is easy to see that $X_1 \leq_{\alpha\text{-rl}} Y_1$ and $X_2 \leq_{\alpha\text{-rl}} Y_2$. Now we compute

$$q_{\min\{X_1,X_2\},\alpha}(t) = q_{X_1,\alpha}(t) = \begin{cases} 1-t, & t<1; \\ 0, & t \geq 1; \end{cases}$$

and (note that $\min\{Y_1,Y_2\}$ is an exponential random variable with rate 2)

$$q_{\min\{Y_1,Y_2\},\alpha}(t) = \begin{cases} 1/2-t, & t<0; \\ 1/2, & t \geq 0. \end{cases}$$

It is seen that $\min\{X_1,X_2\} \not\leq_{\alpha\text{-rl}} \min\{Y_1,Y_2\}$. Thus the α-percentile residual life order is not closed under the minimum operation. ◀

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