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THE PERCENTILE RESIDUAL LIFE UP TO TIME t_0: ORDERING AND AGING PROPERTIES

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Abstract

Motivated by practical issues, a new stochastic order for random variables is introduced by comparing all their percentile residual life functions until a certain instant. Some interpretations of these stochastic orders are given, and various properties of them are derived. The relationships to other stochastic orders are studied, and also an application in Reliability Theory is described. Finally, we present some characterization results of the decreasing percentile residual life up to time t_0 aging notion.

Keywords: Aging notion, hazard rate, mean residual life, percentile residual life, reliability, stochastic ordering.

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1 Introduction

In medical research where times to an event of interest are routinely collected, it is natural to consider the remaining, or residual, life years (or months or days) as the survival outcome. For example, it is more comprehensible in interpretation if a benefit of a new drug can be evaluated in terms of how much it would extend a patient's remaining life years in a clinical study. The mean residual life function and the median residual life function -or, more generally, the percentile residual life function- are commonly used to summarize the residual survival experience of patients.

Motivated by the applicability of comparing items until a certain moment, in this paper we define and study a new stochastic order for nonnegative random variables, called **the percentile residual life up to time t_0 order**, that is based on the comparison of all the percentile residual life functions of two random variables until a certain moment t_0. The proposed ordering can be useful when the efficacy of two drugs or two treatments need to be compared in terms of prolonging patients' remaining lifetimes or delaying recurrence of an original disease in clinical trials. In these cases, the duration of the clinical trials is limited and it is only possible to compare the two alternatives until a time t_0. In Industrial Engineering, it is also useful to establish comparisons between items but only during the warranty period (fixed by a time t_0).

The percentile residual life function has been promoted by many researchers, especially for censored survival data, because it does not depend as heavily on the outliers as the mean residual life function does. The percentile residual life functions were studied in some detail by Arnold and Brockett (1983), Gupta and Langford (1984), Joe and Proschan (1984), and Joe (1985), as well as by Haines and Singpurwalla (1974). Families of distributions for which simple expressions for the percentile residual life functions can be obtained, are identified in Raja Rao, Alhumoud, and Damaraju (2006). A particular α-percentile residual life function of interest is the median residual life function given by $q_{X,0.5}$. This function was studied in detail by Lillo (2005) and Gelfand and Kottas (2003) used it for Bayesian semiparametric modeling. See the above two references for further references to papers that studied the α-percentile and the median residual life functions, and that used them in practical applications.

Franco-Pereira, Lillo, Romo, and Shaked (2010) introduced and studied a new family of stochastic orderings that are based on the comparison of percentile residual life functions in the following sense. Given any $\alpha \in (0,1)$, two random variables X and Y are ordered with respect to the α-percentile residual life order, denoted by $X \leq_{\alpha-\mathrm{rl}} Y$, if their corresponding α-percentile residual life functions are ordered in the whole support. These stochastic orders can be useful in Reliability Theory when it is of importance to compare a particular percentile (say, the median, that is, $\alpha = 0.5$) of the residual life of a series system, with the same percentile (again, say, the median) of the residual life of another series system, with different components. In this paper, we introduce two novelties with respect to these orders: (1) to compare for all $\alpha \in (0,1)$, and (2) to compare in a subset of the distribution support, which is more realistic in practical applications. As we will explain latter, the percentile residual life up to time t_0 orders, unlike the percentile residual life orders, are orders and not only preorders. Moreover, we show that the new order is stronger than the usual stochastic order and weaker than the hazard rate order.

The rest of the paper is organized as follows. The percentile residual life up to time t_0 order is formally defined in Section 2. We also give there some equivalent ways of describing this order that turn up to be useful in the sequel. Section 3 consists of a thorough study of the relationships among the percentile residual life up to time t_0 orders and other stochastic orders in the literature. Some useful properties of the percentile residual life up to time t_0 orders are given in Section 4. An application in Reliability Theory is described in Section 5, and in Section 6 some characterization results of the decreasing percentile residual life up to time t_0 aging notion are derived. A brief discussion, in Section 7 concludes the paper. We will assume that all random variables considered along this paper are nonnegative, unless stated otherwise.

2 The percentile residual life up to time t_0 orders

Let X be a random variable, and let u_X be the right endpoint of its support. For any $\alpha \in (0,1)$ and for any $t < u_X$, the α-percentile residual life function of X, denoted by $q_{X,\alpha}(t)$, is defined as the α-percentile residual life function of X_t, where $X_t = [X - t | X > t]$ is the *residual life* at time t, that is associated with X. That is,

$$q_{X,\alpha}(t) = \begin{cases} F_{X_t}^{-1}(\alpha), & t < u_X; \\ 0, & t \geq u_X. \end{cases} \tag{2.1}$$

A straightforward computation shows that

$$q_{X,\alpha}(t) = \overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(t)) - t, \quad t < u_X, \tag{2.2}$$

where $\overline{\alpha} = 1 - \alpha$. And, equivalently,

$$q_{X,\alpha}(t) = F_X^{-1}(\alpha + \overline{\alpha}F_X(t)) - t, \quad t < u_X. \tag{2.3}$$

Let Y be another random variable with α-percentile residual life function $q_{Y,\alpha}$, and let $t_0 > 0$. If

$$q_{X,\alpha}(t) \leq q_{Y,\alpha}(t), \quad \text{for all } t \leq t_0 \text{ and for all } \alpha \in (0,1), \tag{2.4}$$

then we say that X is smaller than Y in the *percentile residual life up to time* t_0 *order*, and we denote it as $X \leq_{\text{prl}}^{t_0} Y$.

As we pointed out before, Franco-Pereira, Lillo, Romo, and Shaked (2010) introduced and studied a new family of stochastic orderings that are based on the comparison, for a fixed $\alpha \in (0,1)$, of the α-percentile residual life functions of two random variables. In this paper we define and study a new stochastic order that is based on the comparison of all the percentile residual life functions of two random variables but in a subset of the distribution support, which is more realistic in practical applications.

It follows from (2.1) and (2.4) that if $X \leq_{\text{prl}}^{t_0} Y$ then

$$u_X \leq u_Y, \tag{2.5}$$

where u_X and u_Y are the right endpoints of corresponding supports.

The following proposition states equivalent conditions for the percentile residual life up to time t_0 order to hold. Parts (i) and (ii) follow straightforward from (2.2), (2.3), and (2.4).

Proposition 2.1. *Let $t_0 > 0$ and let X and Y be two random variables.*

(i) *The random variables X and Y satisfy $X \le_{\mathrm{prl}}^{t_0} Y$ if, and only if,*

$$\overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(t)) \le \overline{F}_Y^{-1}(\overline{\alpha}\overline{F}_Y(t)), \quad \textit{for all } t \le t_0 \textit{ and all } \alpha \in (0,1).$$

(ii) *The random variables X and Y satisfy $X \le_{\mathrm{prl}}^{t_0} Y$ if, and only if,*

$$F_X^{-1}(\alpha + \overline{\alpha}F_X(t)) \le F_Y^{-1}(\alpha + \overline{\alpha}F_Y(t)), \quad \textit{for all } t \le t_0 \textit{ and all } \alpha \in (0,1).$$

The α-percentile residual life orders indicate comparisons of size or magnitude. For example, letting $t \to -\infty$ in (2.3) we see that if $X \le_{\alpha\text{-rl}} Y$ then the α-percentile of X is smaller than (or at least not larger than) the α-percentile of Y. Inequality (2.5) is another indication of comparisons of size or magnitude.

3 Relations between the t_0-PRL and other stochastic orders

Recall that a random variable X is said to be smaller than the random variable Y in the ordinary stochastic order (denoted as $X \le_{\mathrm{st}} Y$) if $\overline{F}_X(x) \le \overline{F}_Y(x)$ for all $x \in \mathbb{R}$. It is known that $X \le_{\mathrm{st}} Y$ if, and only if,

$$F_X^{-1}(p) \le F_Y^{-1}(p), \quad \text{for all } p \in (0,1); \tag{3.1}$$

see, for example, (1.A.12) in Shaked and Shanthikumar (2007).

Next recall that a random variable X is said to be smaller than the random variable Y in the hazard rate order (denoted as $X \le_{\mathrm{hr}} Y$) if $\frac{\overline{F}_Y(t)}{\overline{F}_X(t)}$ is increasing in t. If X_t and Y_t denote the residual lives that are associated with X and Y, it is known that $X \le_{\mathrm{hr}} Y$ if, and only if,

$$X_t \le_{\mathrm{st}} Y_t, \quad \text{for all } t < u_X; \tag{3.2}$$

see, for example, (1.B.6) in Shaked and Shanthikumar (2007).

The following proposition states an equivalent condition for the percentile residual life up to time t_0 order to hold.

Proposition 3.1. *Let $t_0 > 0$, and let X and Y be two random variables. The random variables X and Y satisfy $X \le_{prl}^{t_0} Y$ if, and only if,*

$$X_t \le_{st} Y_t, \quad \textit{for all } t \le t_0.$$

Proof. For every $t \le t_0$, by (3.1),

$$X_t \le_{st} Y_t \Leftrightarrow F_{X_t}^{-1}(\alpha) \le F_{Y_t}^{-1}(\alpha), \quad \text{for all } \alpha \in (0,1).$$

Then, by equation (2.1), the latter condition is equivalent to $q_{X,\alpha}(t) \le q_{Y,\alpha}(t)$ for all $t \le t_0$ and all $\alpha \in (0,1)$; that is, $X \le_{\text{prl}}^{t_0} Y$. □

The percentile residual life orders are not orders but preorders. The reason is that these binary relations do not verify the *antisymmetry property.* That is, the two conditions $X \le_{\alpha\text{-rl}} Y$ and $Y \le_{\alpha\text{-rl}} X$ do not necessarily imply $X =_{st} Y$. From Proposition 3.1, it is obvious that the percentile residual life up to time t_0 order implies the usual stochastic order. Therefore, this new order is, unlike the percentile residual life orders, an order and not only a preorder.

From (3.2), it follows that the hazard rate order implies the percentile residual life up to time t_0 order. However, if $t_0 \ge u_X$, then

$$\le_{hr} \Leftrightarrow \le_{\text{prl}}^{t_0} . \tag{3.3}$$

The following result is one of the most interesting properties of this order.

Remark 3.2. When $t_0 < u_X$, the percentile residual life up to time t_0 order is an order between the usual stochastic order and the hazard rate order.

Going back to the relationship between the percentile residual life up to time t_0 order and the usual stochastic order, the next counterexample shows that for any $t_0 > 0$ we have

$$\le_{st} \not\Rightarrow \le_{\text{prl}}^{t_0} .$$

Counterexample 3.3. Let $t_0 > 0$ and $0 < k < t_0$ (note that such a k exists because $t_0 > 0$). Assume that X is uniformly distributed on (0,k+2) and that Y is a random variable whose distribution is the following mixture:

$$F_Y(x) = \begin{cases} \text{uniform on } [0,k], & \text{with probability } a, \\ \text{uniform on } [k,k+1], & \text{with probability } \frac{k+1}{k+2} - a, \\ \text{uniform on } [k+1,k+2], & \text{with probability } \frac{1}{k+2}; \end{cases}$$

with $a < \frac{k}{k+2}$. Then, it is easy to verify that $X \le_{st} Y$. Now consider $t = k$, X_t is uniformly distributed on $(k, k+2)$ and the distribution function of Y_t is given by the following mixture:

$$F_{Y_k}(x) = \begin{cases} \text{uniform on } [k,k+1], & \text{with probability } \frac{1}{1-a}\left(\frac{k+1}{k+2} - a\right), \\ \text{uniform on } [k+1,k+2], & \text{with probability } \frac{1}{(k+2)(1-a)}; \end{cases}$$

so that $X_k \ge_{st} Y_k$, and therefore $X \not\le_{\text{prl}}^{t_0} Y$.

It is obvious that $\le_{\text{prl}}^{t_0}$ implies $\le_{\text{prl}}^{t_1}$, when $t_1 < t_0$. However, from the previous results, it follows that $\le_{\text{prl}}^{t_0}$ does not necessarily imply $\le_{\text{prl}}^{t_1}$, when $t_1 > t_0$.

Recall that the mean residual life function m_X that is associated with X is given by

$$m_X(t) = \begin{cases} E[X - t \big| X > t], & t < u_X; \\ 0, & t \geq u_X, \end{cases}$$

provided the expectation exists. Let m_Y be the mean residual life function of a random variable Y. If

$$m_X(t) \leq m_Y(t) \quad \text{for all } t \in \mathbb{R},$$

then X is said to be smaller than Y in the mean residual life order (denoted as $X \leq_{\text{mrl}} Y$); see Shaked and Shanthikumar (2007).

In Counterexample A.2 in the Appendix of Franco-Pereira, Lillo, Romo, and Shaked (2010) it is shown that for any $\alpha \in (0, 1)$ we have

$$\leq_{\text{mrl}} \not\Longrightarrow \leq_{\alpha\text{-rl}} . \tag{3.4}$$

In that counterexample, X and Y are two nonnegative random variables such that $X \leq_{\text{mrl}} Y$ but $q_{X,\alpha}(0) > q_{Y,\alpha}(0)$. Therefore, the same counterexample shows that $\leq_{\text{mrl}} \not\Rightarrow \leq_{\text{prl}}^{t_0}$ for any $t_0 > 0$.

Let $\leq_{\text{hmrl}}$ denote the harmonic mean residual life stochastic order. Since $\leq_{\text{mrl}} \Longrightarrow \leq_{\text{hmrl}}$ (see Shaked and Shanthikumar (2007)), it follows from (3.4) that, for any $t_0 > 0$, we have

$$\leq_{\text{hmrl}} \not\Longrightarrow \leq_{\text{prl}}^{t_0} .$$

The following example shows that the percentile residual life up to time t_0 order does not imply the mean residual life order. Therefore, since the hazard rate order implies the mean residual life order, the same example shows that percentile residual life up to time t_0 order does not imply the hazard rate order.

Counterexample 3.4. Let $k > 0$ and $\frac{1}{2} < w < \frac{k+1}{k+2}$. Let X have the uniform distribution on $(0, k+2)$ and let Y be distributed as a mixture of a degenerate random variable at $k+1$ with probability w, and a degenerate random variable at $k+2$ with probability $1-w$.

For every $0 < t \leq k + 2 - \frac{1}{1-w}$, the variable X_t is uniformly distributed on $(t, k+2)$ and $Y_t =_{st} Y$. That is,

$$F_{X_t}(x) = \begin{cases} 0, & x < t; \\ \frac{x-t}{k+2-t}, & t \leq x < k+2; \\ 1 & x \geq k+2; \end{cases}$$

and

$$F_{Y_t}(x) = \begin{cases} 0, & x < k+1; \\ w, & k+1 \leq x < k+2; \\ 1 & x \geq k+2. \end{cases}$$

It is easy to verify that $X_t \leq_{st} Y_t$, for every $t \leq k + 2 - \frac{1}{1-w}$ (note that, since $w < \frac{k+1}{k+2}$, then $0 < k + 2 - \frac{1}{1-w} < k+1$). Therefore, $X \leq_{\text{prl}}^{t_0} Y$, for $t_0 \leq k + 2 - \frac{1}{1-w}$.

Now, take $\varepsilon > 0$ such that $\varepsilon < 2w - 1$ (note that such an ε exists because $w > \frac{1}{2}$). Then, $X_{k+1-\varepsilon}$ is uniformly distributed on $(k+1-\varepsilon, k+2)$ and $Y_{k+1-\varepsilon} =_{st} Y$. We compute,

$$k + \frac{3-\varepsilon}{2} = E(X_{k+1-\varepsilon}) = m_X(k+1-\varepsilon) > m_Y(k+1-\varepsilon) = E(Y_{k+1-\varepsilon}) = k + 2 - w.$$

Therefore, $X \nleq_{mrl} Y$.

However, the following result shows that there exists a relationship between the percentile residual life up to time t_0 order and the mean residual life order.

Theorem 3.5. *Let X and Y be two random variables and $t_0 > 0$. If $X \leq_{prl}^{t_0} Y$, then*

$$m_X(t) \leq m_Y(t), \quad \text{for all } t \leq t_0.$$

Proof. By Proposition 3.1,

$$X \leq_{\text{prl}}^{t_0} Y \Leftrightarrow X_t \leq_{st} Y_t \text{ for all } t \leq t_0.$$

Since the usual stochastic order preserve expectations, from the latter condition we have that $E(X_t) = m_X(t) \leq m_Y(t) = E(Y_t)$, for all $t \leq t_0$. $\square$

Recall that a random variable X is said to be smaller than the random variable Y in the reversed hazard rate order (denoted as $X \leq_{\text{rh}} Y$) if $F_X(y)F_Y(x) \leq F_X(x)F_Y(y)$ for all $x \leq y$. The following counterexample shows that $\leq_{\text{rh}} \nRightarrow \leq_{\text{prl}}^{t_0}$.

Counterexample 3.6. Let $t_0 > 0$, and take any $\alpha \in (0,1)$ such that $t_0 - \alpha > 0$ (note that such an α exists because $t_0 > 0$). Let $k = t_0 - \alpha$, and let X have the distribution function given by

$$F_X(x) = \begin{cases} 0, & x < k + \alpha; \\ x - k, & k + \alpha \leq x < k + 1; \\ 1, & x \geq k + 1; \end{cases}$$

that is, F_X is a mixture of a degenerate variable at $k + \alpha$ with probability α, and a uniform distribution on $(k+\alpha, k+1)$ with probability $1-\alpha$. Let Y be another random variable with uniform distribution on $(k, k+1)$. We compute,

$$q_{X,\alpha}(x) = \begin{cases} k + \alpha - x, & x < k + \alpha; \\ \alpha(k + 1 - x), & k + \alpha \leq x < k + 1; \\ 0, & x \geq k + 1; \end{cases}$$

and

$$q_{Y,\alpha}(x) = \begin{cases} k + \alpha - x, & x < k; \\ \alpha(k + 1 - x), & k \leq x < k + 1; \\ 0, & x \geq k + 1. \end{cases}$$

It is easy to verify that F_X and F_Y satisfy $F_Y(y)F_X(x) \leq F_Y(x)F_X(y)$ for all $x \leq y$; that is, $X \leq_{\text{rh}} Y$. However, $q_{X,\alpha}(t) > q_{Y,\alpha}(t)$ for all $t \in (k, k+\alpha)$ and, since $k + \alpha = t_0$, then $X \nleq_{\text{prl}}^{t_0} Y$.

Figure 1 summarizes some of the results shown in this section. Here t_0-PRL denotes the percentile residual life up to time t_0 order.

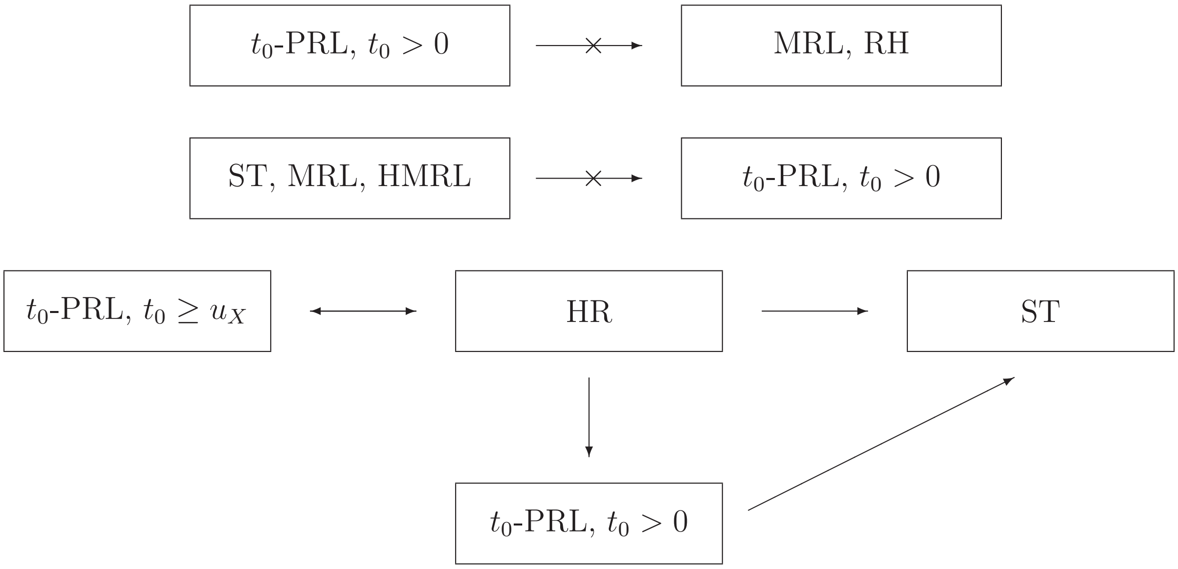

Figure 1: Diagram that shows the relationship among the percentile residual life up to time t_0 order and some common stochastic orders

4 Closure properties of the t_0-PRL

The percentile residual life up to time t_0 orders satisfy some desirable closure properties. These are described and discussed in this section. First, we show that the percentile residual life up to time t_0 orders are preserved under strictly increasing transformations.

Theorem 4.1. *Let X and Y be random variables, $t_0 > 0$, and let ϕ be a strictly increasing function. Then $X \leq_{prl}^{t_0} Y$ if, and only if, $\phi(X) \leq_{prl}^{\phi^{-1}(t_0)} \phi(Y)$.*

Proof. Let $\overline{F}_{\phi(X)}$ and $\overline{F}_{\phi(Y)}$ denote the survival functions of the indicated random variables. Since ϕ is strictly increasing, we have

$$\overline{F}_{\phi(X)}(t) = \overline{F}_X(\phi^{-1}(t)) \quad \text{and} \quad \overline{F}_{\phi(Y)}(t) = \overline{F}_Y(\phi^{-1}(t)), \quad \text{for all } t,$$

and

$$\overline{F}^{-1}_{\phi(X)}(u) = \phi(\overline{F}_X^{-1}(u)) \quad \text{and} \quad \overline{F}^{-1}_{\phi(Y)}(u) = \phi(\overline{F}_Y^{-1}(u)), \quad \text{for all } u \in (0,1).$$

Therefore, by Proposition 2.1(i), $\phi(X) \leq_{\text{prl}}^{t_0} \phi(Y)$ if, and only if,

$$\phi(\overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(\phi^{-1}(t)))) \leq \phi(\overline{F}_Y^{-1}(\overline{\alpha}\overline{F}_Y(\phi^{-1}(t)))), \quad \text{for all } t \leq t_0 \text{ and all } \alpha \in (0,1).$$

By the strict monotonicity of ϕ, the latter condition is equivalent to

$$\overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(\phi^{-1}(t))) \leq \overline{F}_Y^{-1}(\overline{\alpha}\overline{F}_Y(\phi^{-1}(t))), \quad \text{for all } t \leq t_0 \text{ and all } \alpha \in (0,1).$$

Letting $t' = \phi^{-1}(t)$, this condition is the same as

$$\overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(t')) \le \overline{F}_Y^{-1}(\overline{\alpha}\overline{F}_Y(t')), \quad \text{for all } t' \le \phi^{-1}(t_0) \text{ and all } \alpha \in (0,1),$$

and the stated result follows from Proposition 2.1(i). □

The percentile residual life up to time t_0 orders are closed under limits in distribution.

Theorem 4.2. *Let $\{X_n,\ n = 1, 2, \dots\}$ and $\{Y_n,\ n = 1, 2, \dots\}$ be two sequences of random variables such that $X_n \to_{\text{st}} X$ and $Y_n \to_{\text{st}} Y$ as $n \to \infty$, where "$\to_{\text{st}}$" denotes convergence in distribution. For any $t_0 > 0$, if $X_n \le_{prl}^{t_0} Y_n$, $n = 1, 2, \dots$, then $X \le_{prl}^{t_0} Y$.*

Proof. For every $n = 1, 2, \dots$, by Proposition 3.1,

$$X_n \le_{\text{prl}}^{t_0} Y_n \Leftrightarrow (X_n)_t \le_{st} (Y_n)_t, \text{ for all } t \le t_0,$$

where $(A_i)_t = [A_i - t | A_i > t]$, for every random variable A.

Since the usual stochastic order is closed with respect to weak convergence, then

$$\lim (X_n)_t \le_{st} \lim (Y_n)_t.$$

On the other hand, for every $n = 1, 2, \dots$, it holds that $(X_n)_t \to_{\text{st}} X_t$ and $(Y_n)_t \to_{\text{st}} Y_t$. Then

$$X_t \le_{st} Y_t$$

for all $t \le t_0$ and, by Proposition 3.1 the claim is true. □

The following two lemmas, that deal with simple mixtures, will yield a general closure under mixtures property of the percentile residual life up to time t_0 orders.

Lemma 4.3. *Let X, Y, U, and V be random variables with continuous distribution functions, and let W be a random variable with distribution function*

$$F_W = pF_X + (1-p)F_Y,$$

for some $p \in [0,1]$.

(i) *If $U \le_{prl}^{t_0} X$ and $U \le_{prl}^{t_0} Y$ then $U \le_{prl}^{t_0} W$.*

(ii) *If $X \le_{prl}^{t_0} V$ and $Y \le_{prl}^{t_0} V$ then $W \le_{prl}^{t_0} V$.*

Proof. First we prove (i). From $U \le_{\text{prl}}^{t_0} X$ and $U \le_{\text{prl}}^{t_0} Y$, using Proposition 2.1(i), we obtain

$$\overline{F}_U^{-1}(\overline{\alpha}\overline{F}_U(t)) \le \overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(t)) \quad \text{and} \quad \overline{F}_U^{-1}(\overline{\alpha}\overline{F}_U(t)) \le \overline{F}_Y^{-1}(\overline{\alpha}\overline{F}_Y(t)), \quad \text{for all } t \le t_0 \text{ and all } \alpha \in (0,1).$$

It follows, by the continuity of F_X and F_Y, that

$$\overline{F}_X(\overline{F}_U^{-1}(\overline{\alpha}\overline{F}_U(t))) \ge \overline{\alpha}\overline{F}_X(t) \quad \text{and} \quad \overline{F}_Y(\overline{F}_U^{-1}(\overline{\alpha}\overline{F}_U(t))) \ge \overline{\alpha}\overline{F}_Y(t), \quad \text{for all } t \le t_0 \text{ and all } \alpha \in (0,1).$$

Therefore, for every $\alpha \in (0,1)$,

$$p\overline{F}_X(\overline{F}_U^{-1}(\overline{\alpha}\overline{F}_U(t))) + (1-p)\overline{F}_Y(\overline{F}_U^{-1}(\overline{\alpha}\overline{F}_U(t))) \geq \overline{\alpha}p\overline{F}_X(t) + \overline{\alpha}(1-p)\overline{F}_X(t), \quad \text{for all } t \leq t_0;$$

that is,

$$\overline{F}_W(\overline{F}_U^{-1}(\overline{\alpha}\overline{F}_U(t))) \geq \overline{\alpha}\overline{F}_W(t), \quad \text{for all } t \leq t_0 \text{ and all } \alpha \in (0,1).$$

By the continuity of F_W, we get

$$\overline{F}_U^{-1}(\overline{\alpha}\overline{F}_U(t)) \leq \overline{F}_W^{-1}(\overline{\alpha}\overline{F}_W(t)), \quad \text{for all } t \leq t_0 \text{ and all } \alpha \in (0,1);$$

that is, by Proposition 2.1(i), $U \leq_{\text{prl}}^{t_0} W$.

Now we prove (ii). From $X \leq_{\text{prl}}^{t_0} V$ and $Y \leq_{\text{prl}}^{t_0} V$, using Proposition 2.1(i), we obtain

$$\overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(t)) \leq \overline{F}_V^{-1}(\overline{\alpha}\overline{F}_V(t)) \quad \text{and} \quad \overline{F}_Y^{-1}(\overline{\alpha}\overline{F}_Y(t)) \leq \overline{F}_V^{-1}(\overline{\alpha}\overline{F}_V(t)), \quad \text{for all } t \leq t_0 \text{ and all } \alpha \in (0,1).$$

It follows, by the continuity of F_X and F_Y, that

$$\overline{\alpha}\overline{F}_X(t) \geq \overline{F}_X(\overline{F}_V^{-1}(\overline{\alpha}\overline{F}_V(t))) \quad \text{and} \quad \overline{\alpha}\overline{F}_Y(t) \geq \overline{F}_Y(\overline{F}_V^{-1}(\overline{\alpha}\overline{F}_V(t))), \quad \text{for all } t \leq t_0 \text{ and all } \alpha \in (0,1).$$

Therefore, for every $\alpha \in (0,1)$,

$$\overline{\alpha}p\overline{F}_X(t) + \overline{\alpha}(1-p)\overline{F}_Y(t) \geq p\overline{F}_X(\overline{F}_V^{-1}(\overline{\alpha}\overline{F}_V(t))) + (1-p)\overline{F}_Y(\overline{F}_V^{-1}(\overline{\alpha}\overline{F}_V(t))), \quad \text{for all } t \leq t_0;$$

that is,

$$\overline{\alpha}\overline{F}_W(t) \geq \overline{F}_W(\overline{F}_V^{-1}(\overline{\alpha}\overline{F}_V(t))), \quad \text{for all } t \leq t_0 \text{ and all } \alpha \in (0,1).$$

By the continuity of F_W we get

$$\overline{F}_W^{-1}(\overline{\alpha}\overline{F}_W(t)) \leq \overline{F}_V^{-1}(\overline{\alpha}\overline{F}_V(t)), \quad \text{for all } t \leq t_0 \text{ and all } \alpha \in (0,1);$$

that is, by Proposition 2.1(i), $W \leq_{\text{prl}}^{t_0} V$. □

Lemma 4.4. *Let X_1, X_2, Y_1, and Y_2 be random variables with continuous distribution functions, and let W and Z be random variables with distribution functions*

$$F_W = pF_{X_1} + (1-p)F_{X_2} \quad \textit{and} \quad F_Z = pF_{Y_1} + (1-p)F_{Y_2},$$

for some $p \in [0,1]$. If there exists a random variable S such that

$$X_1 \leq_{prl}^{t_0} S, \quad X_2 \leq_{prl}^{t_0} S, \quad S \leq_{prl}^{t_0} Y_1, \quad S \leq_{prl}^{t_0} Y_2,$$

then $W \leq_{prl}^{t_0} Z$.

Proof. Since $X_1 \leq_{\text{prl}}^{t_0} S$ and $X_2 \leq_{\text{prl}}^{t_0} S$, it follows from Lemma 4.3(ii) that $W \leq_{\text{prl}}^{t_0} S$. Furthermore, since $S \leq_{\text{prl}}^{t_0} Y_1$ and $S \leq_{\text{prl}}^{t_0} Y_2$, it follows from Lemma 4.3(i) that $S \leq_{\text{prl}}^{t_0} Z$. By the transitivity property of the order $\leq_{\text{prl}}^{t_0}$ we get $W \leq_{\text{prl}}^{t_0} Z$. □

By repeated application of Lemma 4.4, and convergence arguments, we obtain the following result.

Theorem 4.5. *Let $\{X_\theta,\ \theta \in \Theta\}$ and $\{Y_\theta,\ \theta \in \Theta\}$ be two families of random variables with continuous distribution functions. Let W and Z be random variables with distribution functions given by*

$$F_W(t) = \int_\Theta F_{X_\theta}(t)dH(\theta) \quad \textit{and} \quad F_Z(t) = \int_\Theta F_{Y_\theta}(t)dH(\theta), \quad t \in \mathbb{R},$$

where H is some distribution function on Θ. Suppose that there exists a random variable S such that

$$X_\theta \leq_{prl}^{t_0} S \leq_{prl}^{t_0} Y_\theta \quad \textit{for all } \theta \in \Theta. \tag{4.1}$$

Then $W \leq_{prl}^{t_0} Z$.

Note that condition (4.1) can be rewritten as

$$X_\theta \leq_{\text{prl}}^{t_0} Y_{\theta'} \quad \text{for all } \theta, \theta' \in \Theta.$$

It is worth noting that results that are similar to Theorem 4.5 hold for the hazard rate order, the reversed hazard rate order, the likelihood ratio order, and the mean residual life order (see, respectively, Theorems 1.B.8, 1.B.46, 1.C.15, and 2.A.13 in Shaked and Shanthikumar, 2007).

A special case of Theorem 4.5 is the following result which shows that a random variable, whose distribution is a mixture of two distributions of random variables which are ordered in the sense of the percentile residual life up to time t_0 order, is bounded from below and from above, in the percentile residual life up to time t_0 order sense, by these two random variables.

Corollary 4.6. *Let X and Y be two random variables with continuous distribution functions, and let W be a random variable with distribution function*

$$F_W = pF_X + (1-p)F_Y,$$

for some $p \in [0,1]$. If $X \leq_{prl}^{t_0} Y$ then $X \leq_{prl}^{t_0} W \leq_{prl}^{t_0} Y$.

Again, note that similar results hold for the hazard rate order, the likelihood ratio order, and the mean residual life order (see, respectively, Theorems 1.B.22, 1.C.30, and 2.A.18 in Shaked and Shanthikumar, 2007).

The possible preservation of a stochastic order under the formation of coherent systems is a useful property that has important applications in Reliability Theory (see, for example, Barlow and Proschan (1975) for the definition and the use of coherent systems). Thus it is of interest to ask whether the percentile residual life up to time t_0 orders are closed under this formation. Boland, El-Neweihi, and Proschan (1994) showed that the hazard rate order is not preserved under the formation of coherent systems. In the next counterexample it is shown that, *for all $t_0 > 0$*, the percentile residual life up to time t_0 order is not closed under this formation either. This is shown by considering a parallel system of size 2 whose lifetime is the maximum of the lifetimes of its two components.

Counterexample 4.7. Let X be an exponential random variable with rate $\lambda > 0$. That is,

$$F_X(t) = \begin{cases} 0, & t < 0; \\ 1 - e^{-\lambda t}, & t \geq 0. \end{cases}$$

Let Y be a random variable that is degenerate at 0, and let Z be a random variable that is degenerate at 1. Note that $\max\{X, Y\} =_{\text{st}} X$. Note also that for every $t_0 > 0$, $Y \leq_{\text{prl}}^{t_0} Z$ ($Y \leq_{hr} Z$), and, of course, $X \leq_{\text{prl}}^{t_0} X$ ($X \leq_{hr} X$). Now we compute

$$q_{\max\{X,Y\},\alpha}(t) = q_{X,\alpha}(t) = \begin{cases} \frac{-\log(1-\alpha)}{\lambda} - t, & t < 0; \\ \frac{-\log(1-\alpha)}{\lambda}, & t \geq 0, \end{cases}$$

and

$$q_{\max\{X,Z\},\alpha}(t) = \begin{cases} \frac{-\log(1-\alpha)}{\lambda} - t, & t < 1; \\ \frac{-\log(1-\alpha)}{\lambda}, & t \geq 1. \end{cases}$$

It is seen that $\max\{X, Y\} \not\leq_{\text{prl}}^{t_0} \max\{X, Z\}$ (in fact, $\max\{X, Y\} \geq_{\text{prl}}^{t_0} \max\{X, Z\}$ for every $t_0 > 0$ because $\max\{X, Y\} \geq_{hr} \max\{X, Z\}$). Thus the percentile residual life up to time t_0 order is not closed under the maximum operation. ◀

For every $t_0 > 0$, the percentile residual life up to time t_0 order is closed under the formation of series systems (that is, under the minimum operation). This is shown in the next theorem.

Theorem 4.8. *Let $X_1, X_2 \ldots, X_n$ and $Y_1, Y_2 \ldots, Y_n$ be independent random variables with $X_i \leq_{prl}^{t_0} Y_i$, for $i = 1, \ldots, n$. Then*

$$\min\{X_1, X_2 \ldots, X_n\} \leq_{prl}^{t_0} \min\{Y_1, Y_2 \ldots, Y_n\}. \tag{4.2}$$

Proof. For every $i = 1, \ldots, n$, by Proposition 3.1,

$$X_i \leq_{\text{prl}}^{t_0} Y_i \Leftrightarrow (X_i)_t \leq_{st} (Y_i)_t \text{ for all } t \leq t_0,$$

where $(A_i)_t = [A_i - t | A_i > t]$, for every random variable A.

Since the usual stochastic order is closed under the minimum operation, we have that

$$\min\{(X_1)_t, (X_2)_t \ldots, (X_n)_t\} \leq_{st} \min\{(Y_1)_t, (Y_2)_t \ldots, (Y_n)_t\}.$$

On the other hand, for every $n = 1, 2, \ldots$ it holds that $\min\{(X_1)_t, (X_2)_t \ldots, (X_n)_t\} =_{st} (\min\{X_1, X_2 \ldots, X_n\})_t$ and $\min\{(Y_1)_t, (Y_2)_t \ldots, (Y_n)_t\} =_{st} (\min\{Y_1, Y_2 \ldots, Y_n\})_t$. Then,

$$(\min\{X_1, X_2 \ldots, X_n\})_t \leq_{st} (\min\{Y_1, Y_2 \ldots, Y_n\})_t$$

for all $t \leq t_0$ and, by Proposition 3.1, the claim is true. □

5 An Application of the t_0-PRL in Reliability Theory

Besides the practical applications we have enumerated in the Introduction, to compare the efficacy of two drugs in terms of prolonging patients' remaining lifetimes in clinical trials or to compare items during the warranty period, here we show an application in Reliability Theory.

Theorem 5.1. *Let X and Y be two random variables with continuous survival functions $\overline{F}_X$ and $\overline{F}_Y$ on interval supports. Let $t_0 > 0$ and $\theta > 0$. If $X \leq_{prl}^{t_0} Y$ then*

$$X(\theta) \leq_{prl}^{t_0} Y(\theta), \tag{5.1}$$

where $X(\theta)$ and $Y(\theta)$ denote the random variables with survival function $\overline{F}_X^{\theta}$ and $\overline{F}_Y^{\theta}$, respectively.

Proof. It is not hard to verify that under the continuity assumptions above we have

$$(\overline{F}_X^{\theta})^{-1}(u) = \overline{F}_X^{-1}(u^{1/\theta}) \quad \text{and} \quad (\overline{F}_Y^{\theta})^{-1}(u) = \overline{F}_Y^{-1}(u^{1/\theta}), \quad u \in (0,1),$$

or, equivalently,

$$\overline{F}_X^{-1}(u) = (\overline{F}_X^{\theta})^{-1}(u^{\theta}) \quad \text{and} \quad \overline{F}_Y^{-1}(u) = (\overline{F}_Y^{\theta})^{-1}(u^{\theta}), \quad u \in (0,1).$$

Now, by Proposition 2.1(i), $X \leq_{\text{prl}}^{t_0} Y$ means

$$\overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(t)) \leq \overline{F}_Y^{-1}(\overline{\alpha}\overline{F}_Y(t)), \quad \text{for all } t \leq t_0 \text{ and all } \alpha \in (0,1),$$

that is,

$$(\overline{F}_X^{\theta})^{-1}(\overline{\alpha}^{\theta}\overline{F}_X^{\theta}(t)) \leq (\overline{F}_Y^{\theta})^{-1}(\overline{\alpha}^{\theta}\overline{F}_Y^{\theta}(t)), \quad \text{for all } t \leq t_0 \text{ and all } \alpha \in (0,1),$$

and the result follows from Proposition 2.1(i). □

In the theory of statistics, $\overline{F}_X^{\theta}$ is often referred to as the *Lehmann's alternative*. In Reliability Theory terminology, different $X(\theta)$'s are said to have *proportional hazards*. If $\theta < 1$ then $X(\theta)$ is the lifetime of a component with lifetime X which is subjected to *imperfect repair* procedure where θ is the probability of minimal (rather than perfect) repair (see Brown and Proschan (1983)). If $\theta = n$, where n is a positive integer, then $\overline{F}_X^{n}$ is the survival function of $\min\{X_1, X_2, \ldots, X_n\}$ where $X_1, X_2, \ldots, X_n$ are independent copies of X; that is, $\overline{F}_X^{n}$ is the survival function of a series system of size n where the component lifetimes are independent copies of X. Similarly, if Y is a random variable with survival function $\overline{F}_Y$, then denote by $Y(\theta)$ a random variable with survival function $\overline{F}_Y^{\theta}$.

Note that Theorem 4.8 is a particular case of the previous theorem that can be useful in Reliability Theory when it is of importance to compare all the percentiles of the residual life of a series system with all the percentiles of the residual life of another series system, with different components, until a certain instant t_0. This can be useful, for instance, when t_0

is the time at which the initial warranty of the system expires. Similar applications can be described in biometry and in statistics.

The order $\leq_{\rm prl}^{t_0}$ can also be useful in a market of perishable goods or short-dated products. Suppose that an engineer (or any individual) is considering a purchase of a machine (or a car, say) to sell it x months later. Suppose that she has a choice among a few machines (or cars). If the original machine lifetimes are ordered with respect to the hazard rate order, and if the engineer wishes to maximize all the α-percentiles of the remaining life of the purchased machine until the selling time, then, obviously (for example, by equivalence (3.3)), she should select the machine whose lifetime is the highest with respect to the order $\leq_{\rm hr}$.

Note, however, that the requirement that the machine lifetimes are ordered with respect to $\leq_{\rm hr}$ is a very strong requirement that may be hard to verify (or that actually may not even hold) in practice. On the other hand, verification of the order $\leq_{\rm prl}^{t_0}$ may be a simpler matter — and it yields the same decision!

Moreover, if the above engineer (or individual) has a choice between two markets that have different mixtures of machines with the intention of selling them x months later, and if the original machine lifetimes in these markets satisfy (4.1) [here X_θ and Y_θ, $\theta \in \Theta$, are the original machine lifetimes that are mixed in the two markets], then Theorem 4.5 determines which market is preferable.

6 The t_0-DPRL Aging Notion

From the definitions of the life distribution classes, results may be derived concerning such things as properties of systems (based upon properties of components), bounds for survival functions, moment inequalities, and algorithms for use in maintenance policies (Hollander and Proschan, 1984). In Launer (1993) some results relating the monotonous behavior of the hazard rate function and the percentile residual life function are given. He states and illustrates how those relationships can be useful for studying the behavior of the empirical hazard rate function. This set of relationships helps to determine or to approximate the time at which the γ-percentile residual life function is a maximum can be important in fixing product warranty. For example, product burn-in could be used to eliminate the units which fail early, and thus, maximize the reliability of the remaining product. Franco-Pereira, Lillo, and Shaked studied the family of aging notions known as the decreasing α-percentile residual life (DPRL(α)), $\alpha \in (0,1)$, characterize it through the percentile residual life orders and propose an estimator of the percentile residual life function under monotone restrictions. In this section, we give some characterization results of the classes of distribution functions with decreasing percentile residual life up to time t_0 (t_0-DPRL)) introduced in Franco-Pereira, Lillo, and Romo (2010a), in terms of the percentile residual life orders.

We recall the following aging notion that was recently studied in Franco-Pereira, Lillo, and Romo (2010a) with the purpose of characterizing distributions with bathtub hazard rate. Let $t_0 > 0$. A random variable X is said to be decreasing percentile residual life up to time t_0, denoted t_0-DPRL, if its α-percentile residual life function is decreasing for every $\alpha \in (0,1)$ and for every $t \leq t_0$. That is,

$$q_{X,\alpha}(t) \geq q_{X,\alpha}(t'), \qquad \text{for all } t < t' \leq t_0, \text{ and for all } \alpha \in (0,1).$$

Franco-Pereira, Lillo, and Romo (2010a) established some useful equivalent conditions for the t_0-DPRL aging notion in terms of the density, the survival, and the hazard rate functions. In the following result we provide some characterizations of the t_0-DPRL aging notion in terms of the percentile residual life orders. Recall that $X_t = [X - t | X > t]$.

Theorem 6.1. *Let X be an absolutely continuous random variable with interval support. Then X is t_0-DPRL if, and only if, any of the following equivalent conditions holds:*

(i) $X_t \geq_{\alpha\text{-rl}} X_{t'}$ *whenever* $t \leq t' < t_0$, *and for all* $\alpha \in (0,1)$;

(ii) $X \geq_{\alpha\text{-rl}} X_t$ *whenever* $0 \leq t < t_0$, *and for all* $\alpha \in (0,1)$ (*when X is a nonnegative random variable*);

Proof. From (2.2) it is easy to verify that

$$q_{X_t,\alpha}(x) = \overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(t+x)) - (t+x) \quad \text{for all } 0 < x < u_X - t.$$

Now, let $t \leq t' < t_0$. Then $X_t \geq_{\alpha\text{-rl}} X_{t'}$ for all $\alpha \in (0,1)$ if, and only if, for all $\alpha \in (0,1)$

$$\overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(t+x)) - (t+x) \geq \overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(t'+x)) - (t'+x) \quad \text{for all } x < t_0 - t';$$

that is (by (2.4)), $q_{X,\alpha}(t+x) \geq q_{X,\alpha}(t'+x)$ whenever $t + x \leq t' + x < t_0$ for all $\alpha \in (0,1)$; that is, $q_{X,\alpha}$ is decreasing up to t_0. This proves the equivalence of t_0-DPRL and (i).

Next, let $0 \leq t < t_0$. Then $X \geq_{\alpha\text{-rl}} X_t$ for all $\alpha \in (0,1)$ if, and only if, for all $\alpha \in (0,1)$

$$\overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(x)) - x \geq \overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(t+x)) - (t+x) \quad \text{for all } x < t_0 - t;$$

that is (by (2.4)), $q_{X,\alpha}(x) \geq q_{X,\alpha}(t+x)$ whenever $t + x \leq t_0$ for all $\alpha \in (0,1)$; that is, $q_{X,\alpha}$ is decreasing up to t_0. This proves the equivalence of t_0-DPRL and (ii). □

In the literature there are results that are similar to Theorem 6.1, but which involve aging notions other than t_0-DPRL. For example, Theorems 1.A.30, 1.B.38, 3.B.24, 3.B.25, and 4.A.53 in Shaked and Shanthikumar (2007) give similar characterizations for the IFR aging notion. Theorems 2.A.23, 2.B.17, 3.A.56, 3.C.13, and 4.A.51 in Shaked and Shanthikumar (2007), as well as a result in Belzunce, Gao, Hu, and Pellerey (2004), give similar characterizations for the decreasing mean residual life (DMRL) aging notion and a result in Franco-Pereira, Lillo, and Shaked (2010) for the decreasing percentile residual life (DPRL).

Recall that, by Theorem 1.B.38 in Shaked and Shanthikumar (2007),

$$X \text{ is } IFR \Leftrightarrow X_t \geq_{\text{hr}} X_{t'} \text{ whenever } t \leq t'.$$

Since the hazard rate order is equivalent to the α-percentile residual life order for all $\alpha \in (0,1)$, Part (i) in Theorem 6.1 is equivalent to

$$X_t \geq_{\alpha\text{-rl}} X_{t'} \text{ whenever } t \leq t' \leq t_0, \text{ and for all } \alpha \in (0,1).$$

Therefore, it is shown that IFR implies t_0-DPRL, for any $t_0 > 0$.

In Franco-Pereira, Lillo, and Romo (2010a) the relationship between the t_0-DPRL aging notion and the monotonous behaviour of the hazard rate function was studied. Since increasing hazard rate implies decreasing mean residual life, it is straightforward than the t_0-DPRL aging notion implies that the mean residual life function is decreasing up to t_0.

7 Discussion

Motivated by the practical issues given in the Introduction, in this paper we introduce and study a new stochastic order for random variables by comparing all the percentile residual life functions of two random variables until a certain instant. The relationships to other stochastic orders and some properties are studied. Among the properties that we study, we prove that the new order is stronger than the usual stochastic order and weaker than the hazard rate order. However, there are still some questions with important applications to be answered. Since the verification of (2.4) or Parts (i) and (ii) in Proposition 2.1 may be hard to check in practice, it is of importance to find conditions under which property (2.4) has to be verified only by a set of α's $\in (0,1)$ (instead of every $\alpha \in (0,1)$). Besides, it would be convenient to develop statistical tools to check whether two random variables are ordered or not with respect to the percentile residual life up to t_0 order, as Franco-Pereira, Lillo, and Romo (2010b) design for comparing percentile residual life functions.

It is also interesting to study under which conditions the monotonous behaviour of the mean residual life function implies a monotonous behaviour of the percentile residual life function. This type of relationships may help to fix product warranty.

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