LOWER BOUNDS FOR THE CONSTANTS IN THE BOHNENBLUST-HILLE INEQUALITY: THE CASE OF REAL SCALARS

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ABSTRACT. The Bohnenblust-Hille inequality was obtained in 1931 and (in the case of real scalars) asserts that for every positive integer N and every m-linear mapping $T:\ell_\infty^N\times\cdots\times\ell_\infty^N\to\mathbb{R}$ one has

$$\left(\sum_{i_1,\ldots,i_m=1}^N \left|T(e_{i_1},\ldots,e_{i_m})\right|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}} \le C_m \left\|T\right\|,$$

for some positive constant C_m. Since then, several authors obtained upper estimates for the values of C_m. However, the novelty presented in this short note is that we provide lower (and non-trivial) bounds for C_m.

1. Introduction

The Bohnenblust-Hille inequality (see [1]), for real scalars asserts that for every positive integer N and every m-linear mapping $T:\ell_\infty^N\times\cdots\times\ell_\infty^N\to\mathbb{R}$ there is a positive constant C_m such that

$$\left(\sum_{i_1,\ldots,i_m=1}^N \left|T(e_{i_1},\ldots,e_{i_m})\right|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}} \le C_m \left\|T\right\|. \tag{1.1}$$

The BohnenblustHille inequality has important applications in various fields of analysis. For details and references we mention [4]. When $m=2$ it is interesting to note that the Bohnenblust-Hille inequality is precisely the well-known Littlewood's 4/3 inequality [8].

Since the 1930's many authors have obtained estimates for upper bounds of C_m in the case of real and complex scalars (see, e.g., [1, 2, 4, 5, 7, 9, 10]). The constants of the polynomial version of the Bohnenblust-Hille inequality (complex case) were recently investigated in [3]. Until now, the more accurate upper bounds for C_m (real case) were given in [10]:

m	upper bounds for C_m
2	$\sqrt{2}\approx 1.414$
3	$2^{\frac{20}{24}}\approx 1.782$
4	$2^{\frac{32}{32}}=2$
5	$2^{\frac{48}{40}}\approx 2.298$
6	$2^{\frac{64}{48}}\approx 2.520$
7	$2^{\frac{84}{56}}\approx 2.828$
8	$2^{\frac{104}{64}}\approx 3.084$
9	$2^{\frac{128}{72}}\approx 3.429$
10	$2^{\frac{152}{80}}\approx 3.732$

Also, it has very recently been proved ([6]) that the sequence of constants $(C_m)_m$ has the best possible asymptotic behavior, that is

$$\lim_{m\to\infty}\frac{C_m}{C_{m-1}}=1.$$

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However, and to the best of our knowledge, there is absolutely no work presenting estimates for (non-trivial) lower bounds for the constants C_m. In this short note we obtain lower bounds for C_m which we believe are (specially) interesting for the cases $m = 2, 3, 4, 5$.

In the following e_k denotes the kth canonical vector in $\mathbb{R}^N$.

2. The case $m = 2$

Let $T_2 : \ell_\infty^2 \times \ell_\infty^2 \to \mathbb{R}$ be defined as

$$T_2(x, y) = x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2.$$

Note that $\|T_2\| = 2$. In fact,

$$\begin{aligned} |T_2(x, y)| &= |x_1(y_1 + y_2) + x_2(y_1 - y_2)| \\ &\le \|x\|(|y_1 + y_2| + |y_1 - y_2|) \\ &= 2\|x\| \max\{|y_1|, |y_2|\} \\ &= 2\|x\|\|y\| \end{aligned}$$

and since $T_2(e_1 + e_2, e_1 + e_2) = 2$ it follows that $\|T_2\| = 2$.

Now the inequality

$$\left(\sum_{i_1,i_2=1}^{2} \left|T_2(e_{i_1}, e_{i_2})\right|^{\frac{4}{3}} \right)^{\frac{3}{4}} \le C_2 \|T_2\|$$

can be re-written as

$$4^{3/4} \le 2C_2$$

which gives

$$C_2 \ge 2^{1/2}.$$

Since it is well-known that $C_2 \le 2^{1/2}$, we conclude that $C_2 = 2^{1/2}$, but this result seems to be already known.

3. The case $m = 3$

Now, let $T_3 : \ell_\infty^4 \times \ell_\infty^4 \times \ell_\infty^4 \to \mathbb{R}$ given by

$$T_3(x, y, z) = (z_1 + z_2)(x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2) + (z_1 - z_2)(x_3y_3 + x_3y_4 + x_4y_3 - x_4y_4).$$

We have

$$\begin{aligned} |T_3(x, y, z)| &= |(z_1 + z_2)(x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2) + (z_1 - z_2)(x_3y_3 + x_3y_4 + x_4y_3 - x_4y_4)| \\ &\le |z_1 + z_2| (|x_1||y_1 + y_2| + |x_2||y_1 - y_2|) + |z_1 - z_2| (|x_3||y_3 + y_4| + |x_4||y_3 - y_4|) \\ &\le \|x\| \Big\{ |z_1 + z_2| (|y_1 + y_2| + |y_1 - y_2|) + |z_1 - z_2| (|y_3 + y_4| + |y_3 - y_4|) \Big\} \\ &= 2\|x\| \Big\{ |z_1 + z_2| \max\{|y_1|, |y_2|\} + |z_1 - z_2|| \max\{|y_3|, |y_4|\} \Big\} \\ &\le 2\|x\|\|y\| (|z_1 + z_2| + |z_1 - z_2|) \\ &= 4\|x\|\|y\| \max\{|z_1|, |z_2|\} \\ &\le 4\|x\|\|y\|\|z\|. \end{aligned}$$

Since $T_3(e_1 + e_2 + e_3, e_1 + e_2 + e_3, e_1 + e_2 + e_3) = 4$, then $\|T_3\| = 4$. Also

$$\left(\sum_{i_1,i_2,i_3=1}^{4} \left|T_3(e_{i_1}, e_{i_2}, e_{i_3})\right|^{\frac{6}{4}} \right)^{\frac{4}{6}} \le C_3 \|T_3\|$$

becomes

$$16^{2/3} \le 4C_3$$

which gives

$$C_3 \ge 2^{2/3} \approx 1.587.$$

4. The case $m = 4$

In this case, let us consider $T_4 : \ell_\infty^8 \times \ell_\infty^8 \times \ell_\infty^8 \times \ell_\infty^8 \to \mathbb{R}$ given by

$$\begin{aligned} &T_4(x,y,z,w) = \\ &= (w_1+w_2)\begin{pmatrix} (z_1+z_2)(x_1y_1+x_1y_2+x_2y_1-x_2y_2) \\ +(z_1-z_2)(x_3y_3+x_3y_4+x_4y_3-x_4y_4) \end{pmatrix} \\ &+ (w_1-w_2)\begin{pmatrix} (z_3+z_4)(x_5y_5+x_5y_6+x_6y_5-x_6y_6) \\ +(z_3-z_4)(x_7y_7+x_7y_8+x_8y_7-x_8y_8). \end{pmatrix} \end{aligned}$$

As in Sections 2 and 3 we see that $\|T_4\| = 8$ and from (1.1) we obtain

$$64^{5/8} \leq 8C_4.$$

Hence

$$C_4 \geq 2^{3/4} \approx 1.681.$$

5. The general case

From the previous results it is not difficult to prove that in general

$$C_m \geq 2^{\frac{m-1}{m}}$$

for every $m \geq 2$. Indeed, let us define the m-linear forms $T_m : \ell_\infty^{2^{m-1}} \times \overset{m}{\cdots} \times \ell_\infty^{2^{m-1}} \to \mathbb{R}$ by induction as

$$\begin{aligned} T_2(x_1,x_2) =& x_1^1x_2^1 + x_1^1x_2^2 + x_1^2x_2^1 - x_1^2x_2^2 \\ T_m(x_1,\ldots,x_m) =& (x_m^1+x_m^2)T_{m-1}(x_1,\ldots,x_{m-1}) \\ &+ (x_m^1-x_m^2)T_{m-1}(B^{2^{m-2}}(x_1),B^{2^{m-2}}(x_2),B^{2^{m-3}}(x_3)\ldots,B^2(x_{m-1})), \end{aligned}$$

where $x_k = (x_k^n)_n \in \ell_\infty^{2^{m-1}}$ for $1 \leq k \leq m$, $1 \leq n \leq 2^{m-1}$ and B is the backward shift operator in $\ell_\infty^{2^{m-1}}$. It has been proved in Section 2 that $\|T_2\| = 2$. If we assume that $\|T_{m-1}\| = 2^{m-2}$, then

$$\begin{aligned} |T_m(x_1,\ldots,x_m) \leq& |x_m^1+x_m^2||T_{m-1}(x_1,\ldots,x_{m-1})| \\ &+ |x_m^1-x_m^2||T_{m-1}(B^{2^{m-2}}(x_1),B^{2^{m-2}}(x_2),B^{2^{m-3}}(x_3)\ldots,B^2(x_{m-1}))| \\ \leq& 2^{m-2}[|x_m^1+x_m^2|\|x_1\|\cdots\|x_{m-1}\| \\ &+ |x_m^1-x_m^2|\|B^{2^{m-2}}(x_1)\|\|B^{2^{m-2}}(x_2)\|\|B^{2^{m-1}}(x_3)\|\cdots\|B^2(x_{m-1})\|] \\ \leq& 2^{m-2}[|x_m^1+x_m^2|+|x_m^1-x_m^2|]\|x_1\|\cdots\|x_{m-1}\| \\ =& 2^{m-1}\|x_1\|\cdots\|x_{m-1}\|\max\{|x_m^1|,|x_m^2|\} \\ \leq& 2^{m-1}\|x_1\|\cdots\|x_m\|. \end{aligned}$$

This induction argument shows that $\|T_m\| \leq 2^{m-1}$ for all $m \in \mathbb{N}$. Using a similar induction argument it is easy to prove that $T_m(x_1,\ldots,x_m) = 2^{m-1}$ for $x_1,\ldots,x_m$ such that $\|x_1\| = \ldots = \|x_m\| = 1$ and $x_j^k = 1$ with $1 \leq j,k \leq m$, which proves that $\|T_m\| = 2^{m-1}$ for all $m \in \mathbb{N}$.

On the other hand from (1.1) we have

$$\left(\sum_{i_1,\ldots,i_m=1}^{2^{m-1}} \left|T_m(e_{i_1},...,e_{i_m})\right|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C_m \|T_m\| = 2^{m-1}C_m.$$

To finish we shall prove that $|T_m(e_{i_1},...,e_{i_m})|$ is either 0 or 1 and that $|T_m(e_{i_1},...,e_{i_m})| = 1$ for exactly 4^{m-1} choices of the vectors $e_{i_1},\ldots,e_{i_m}$. Working again by induction, the reader can easily check that the latter is true for $m = 2$ (see Section 2). If we assume that the result is true for $m-1$ and $e_{i_1},\ldots,e_{i_{m-1}}$ is one of the 4^{m-2} choices of the unit vectors such that $|T_{m-1}(e_{i_1},\ldots,e_{i_{m-1}})| = 1$, then using the definition of T_m

$$\begin{aligned} |T_m(e_{i_1},\ldots,e_{i_{m-1}},e_k)| &= |T_{m-1}(e_{i_1},\ldots,e_{i_{m-1}})| = 1, \\ |T_m(e_{i_1+2^{m-2}},e_{i_2+2^{m-2}},e_{i_3+2^{m-3}}\ldots,e_{i_{m-1}+2},e_k)| &= |T_{m-1}(e_{i_1},\ldots,e_{i_{m-1}})| = 1, \end{aligned}$$

for $k = 1, 2$. Hence we have found $4 \cdot 4^{m-2} = 4^{m-1}$ choices of unit vector at which $|T_m|$ takes the value 1. A simple inspection of the problem shows that $|T_m|$ vanishes at any other choice of canonical vectors.

6. Final remarks

Notice that our estimate $2^{\frac{m-1}{m}}$ seems inaccurate as $m \to \infty$ since it is a common feeling that the optimal values for the constants C_m should tend to infinity as $m \to \infty$. However, and as a matter of fact, we must say that this common feeling seems to be just supported by the estimates of the upper bounds for C_m obtained throughout the last decades, but there seems to be not any particular result supporting this "fact". In any case (and at least for $m = 2, 3, 4, 5$) our estimates are clearly interesting. Summarizing:

$$\begin{array}{rcl} & C_2 & = \sqrt{2} \\ 1.587 \leq & C_3 & \leq 1.782 \\ 1.681 \leq & C_4 & \leq 2 \\ 1.741 \leq & C_5 & \leq 2.298. \end{array}$$

We also conclude that $C_3 > C_2$, which seems to be not known until now.

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