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Seiberg-Witten maps for $SO(1,3)$ gauge invariance and deformations of gravity

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A family of diffeomorphism-invariant Seiberg-Witten deformations of gravity is constructed. In a first step Seiberg-Witten maps for an $SO(1,3)$ gauge symmetry are obtained for constant deformation parameters. This includes maps for the vierbein, the spin connection, and the Einstein-Hilbert Lagrangian. In a second step the vierbein postulate is imposed in normal coordinates and the deformation parameters are identified with the components $\theta^{\mu\nu}(x)$ of a covariantly constant bivector. This procedure gives for the classical action a power series in the bivector components which by construction is diffeomorphism invariant. Explicit contributions up to second order are obtained. For completeness a cosmological constant term is included in the analysis. Covariant constancy of $\theta^{\mu\nu}(x)$, together with the field equations, imply that, up to second order, only four-dimensional metrics which are direct sums of two two-dimensional metrics are admissible, the two-dimensional curvatures being expressed in terms of $\theta^{\mu\nu}$. These four-dimensional metrics can be viewed as a family of deformed emergent gravities.

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I. INTRODUCTION

It has been known for a long time now that measuring distance with accuracy a causes uncertainty $1/a$ in momentum, which, according to the Einstein equations, becomes a source of gravitational field. As a decreases, the gravitational field becomes stronger and thus the spacetime curvature grows larger. For a of the order of Schwarzschild radius, the gravitational field is strong enough to produce a black hole. In this case, no more information about position is available and an uncertainty relation for position coordinates is due. It has also been known for a while that position uncertainty relations can be realized in terms of noncommutative position operators, provided locality is assumed [1]. The concurrence of these two arguments has triggered an increasing interest in constructing a theory of gravity that includes noncommutative spacetime deformations. See Ref. [2] for a recent review. A first step along this direction is the formulation of an effective theory in which the gravitational field, i.e. the spacetime metric is deformed in a way consistent with the principles of general relativity.

Several proposals for such an effective theory have been made [3,4] in the recent past, including the so-called twist-deformed diffeomorphism models [5,6]. Even though these models preserve what are called twisted diffeomorphisms, they violate invariance under conventional diffeomorphisms [7]. One would like to insist on conventional diffeomorphism invariance, among other reasons, to be able to observe physics in a frame-independent way. Since everywhere constant tensors clash with invariance under general coordinate transformations, one is naturally led to consider position-dependent noncommutativity parameters $\theta^{\mu\nu}(x)$.

In this paper we consider an x-dependent deformation bivector $\theta^{\mu\nu}$ and formulate, using Seiberg-Witten maps [8], a theory of deformed gravity enjoying diffeomorphism invariance. Such a choice for $\theta^{\mu\nu}$ is also favored by string theory. In fact, in all realizations of noncommutative spacetimes in string theory [9], the noncommutativity parameters form an antisymmetric two tensor given in terms of a background two form $B_2 \neq 0$. Furthermore, the open string metric tensor turns out to be given in terms of $\theta^{\mu\nu}$ [8,10]. In what follows, we will use the term noncommutative to denote the deformed theory, a widely extended and commonly accepted abuse of language in the literature.

Our construction of diffeomorphism-invariant noncommutative (NC) deformations of gravity is inspired by the description of general relativity as the theory that results from imposing the vierbein postulate on an $SO(1,3)$ gauge theory [11]. It consists of two steps. The first one is the construction of a Seiberg-Witten gauge theory for $SO(1,3)$ with constant deformation parameters θ^{mn}. This construction is algebraic, in the sense that it is provided by the solution to a Becchi-Rouet-Stora (BRS) cohomology problem, and is metric independent. In the second step gravity is introduced along the following lines:

(i) Take normal coordinates with respect to a point x^a at which the Seiberg-Witten construction has been performed. This means that the Christoffel symbols vanish at x^a, but not in a neighborhood $\bar{x}^a$ of it. Solve the vierbein postulate in this coordinate system.

(ii) Identify the deformation parameters θ^{mn} with the components at x^a of a bivector. Note that since $\Gamma^a_{bc}(x) = 0$, this bivector must be covariantly constant in the neighborhood of x^a, i.e. $\bar{\nabla}_r \bar{\theta}^{mn}(\bar{x}) = 0$.

This terminates the "covariantization" process in the normal coordinate patch and allows transition to the whole four-dimensional manifold. To emphasize this last step Greek indices replace lower case Latin ones.

The result will be a Lagrangian which is a power series in $\theta^{\mu\nu}$, whose coefficients are functions of the Riemann tensor and its derivatives, and for which diffeomorphism invariance is manifest. We will find explicit expressions for the classical action up to second order in $\theta^{\mu\nu}$.

Having a prescription to construct a classical action as a power series in $\theta^{\mu\nu}$ is not enough to determine if noncommutativity may act as a source of gravity. One must elucidate whether the corresponding field equations admit solutions for the gravitational field $g_{\mu\nu}$ with nonzero $\theta^{\mu\nu}$. It is worth mentioning in this regard that all NC gravity models based on constant noncommutativity proposed so far [4–6] yield a vanishing contribution to the classical action at order one in θ.

Equation $\nabla_\rho\theta^{\mu\nu}=0$ relates the spacetime metric with the noncommutativity bivector $\theta^{\mu\nu}$. As is well known [12], it only has two solutions, pp-wave metrics with null bivectors and $(2+2)$-decomposable metrics with non-null bivectors. As we will see below, for pp-wave metrics the order-two contribution to the classical action identically vanishes, whereas for $(2+2)$-decomposable metrics it takes a very simple form in terms of two arbitrary parameters. The arbitrariness of these parameters arises from the nonuniqueness of the Seiberg-Witten maps, a fact well known for other gauge groups [13]. We are thus led to the conclusion that the only four-dimensional spacetime metrics consistent with covariantly constant deformation bivectors are $(2+2)$ decomposable. This limitation on the class of metrics compatible with the approach proposed here has its origin in that the Seiberg-Witten construction involves constant deformation parameters. To include other metrics, the Seiberg-Witten construction must be extended to also account for derivatives of the deformation parameters $\theta^{\mu\nu}$. See Sec. VII for a remark on this.

We emphasize that our approach uses the Moyal-Groenewold product with constant deformation parameters. Our motivation for this is that we are interested in setting a deformation procedure that works at any order in the deformation parameters. This requires proving BRS covariance for local $SO(1,3)$ transformations, and to do so it is essential to have associativity, a property guaranteed by this choice of Moyal-Groenewold product. The generally covariant extension along the lines explained above breaks associativity but preserves BRS covariance since by the time this is performed one already has a BRS invariant deformed theory. Note that since Moyal-Groenewold products with covariantly constant bivectors $\theta^{\mu\nu}$ are not associative [14,15], such bivectors are not of Poisson type.

The paper is organized as follows. In Sec. II, using general covariance arguments that do not rely on any particular deformation of the $SO(1,3)$ gauge symmetry, all invariants up to order two in θ that depend polynomially on the Riemann tensor and its covariant derivatives and that may contribute to the classical action are constructed for metrics satisfying $\nabla_\rho\theta^{\mu\nu}=0$. For pp metrics, all invariants of this type vanish. For $(2+2)$-decomposable metrics, the number of such invariants is 16. Out of these 16, only two of them contribute to the classical action, the contribution being characterized by two arbitrary parameters a and b. After referring to Secs. IV, V, and VI for the proof that out of this 16 invariants, only two contribute to the classical action, Sec. III takes over and presents a detailed discussion of the corresponding equations of motion for constant curvature. The solutions give the scalar curvatures of the two-dimensional metrics in the $(2+2)$-dimensional metric in terms of the NC parameters, thus providing a way to classically generate NC gravity. As anticipated, Secs. IV, V, and VI contain the construction of the classical action that was the starting point for Sec. III. This is based on the formulation of a diffeomorphism-invariant Seiberg-Witten Lagrangian for an $SO(1,3)$ gauge algebra and consists of two parts. In Sec. IV, the Seiberg-Witten equations for an $SO(1,3)$ gauge symmetry are formulated and a particular solution to all orders in θ is found. This results in a Lagrangian with no relation to the underlying spacetime metric and which is not a scalar under general coordinate transformations. Section V explains how to impose the vierbein postulate so as to end up with a diffeomorphism-invariant Lagrangian. Explicit expressions for first and second-order contributions in θ are computed also in this section. In Sec. VI, we find more general solutions to the Seiberg-Witten equations which lead to the action taken as the starting point in Sec. III. Finally, Sec. VII contains our conclusions. We also include three appendices with technical issues.

II. GENERAL STRUCTURE OF SEIBERG-WITTEN DEFORMATIONS UP TO ORDER TWO

We assume that we have a set of constant NC parameters $\vartheta^{\mu\nu}$ at a point x^μ of spacetime. According to the equivalence principle, it is always possible to choose a locally inertial frame centered at that point. Since we are interested in invariance under conventional diffeomorphisms, not to be confused with twisted diffeomorphisms, the NC parameters $\vartheta^{\mu\nu}$ must be the components of a bivector $\theta^{\mu\nu}$. Recalling that every bivector constant in a locally inertial frame is covariantly constant, one concludes that

$$\nabla_\mu\theta^{\nu\rho}=0. \tag{2.1}$$

It is known [12] that the only four-dimensional spacetimes admitting covariantly constant bivectors are either pp wave or $(2+2)$ decomposable, so condition (2.1) restricts the allowed metrics to

$$pp\colon ds^2 = dudv + H(u,x,y)du^2 - dx^2 - dy^2,$$
$$2+2\colon ds^2 = h'_{\alpha'\beta'}(x^{\alpha'})dx^{\alpha'}dx^{\beta'} + h''_{\alpha''\beta''}(x^{\alpha''})dx^{\alpha''}dx^{\beta''},$$

where H is an arbitrary function of its arguments, $h'_{\alpha'\beta'}$ and $h_{\alpha''\beta''}$ are two-dimensional metrics, and α', $\beta' = 0$, 1 and α'', $\beta'' = 2$, 3. In the first case, the metric can also be written as $g_{\mu\nu} = \eta_{\mu\nu} + Hk_\mu k_\nu$, where $k_\mu = \partial_\mu u$. The bivector $\theta^{\mu\nu}$ is null and has the form $\theta^{\mu\nu} = k^\mu p^\nu - k^\nu p^\mu$, with p^μ such that $k\cdot p = 0$ and $p\cdot p = -1$. In the second case the bivector $\theta^{\mu\nu}$ is not null and hence introduces a NC scale, say $\ell_{\rm NC}$.

The problem of finding the most general θ deformation of the Einstein-Hilbert action to order N in θ can be formulated as that of constructing all possible invariants of this order using the metric and the bivector $\theta^{\mu\nu}$. Let us examine how many of these invariants there are at order one and two for both pp-wave and $(2+2)$-decomposable spacetimes. We restrict ourselves to invariants with polynomial dependence on the Riemann tensor and its derivatives.

At order one, for dimensional reasons, we can only have one bivector $\theta^{\mu\nu}$ and either two Riemann tensors $R_{\mu\nu\rho\sigma}$ or one Riemann tensor and two covariant derivatives ∇_μ. It is straightforward to check that, independently of metric considerations, all invariants of this type are identically zero. Let us move on to second order.

At order two, we must construct all invariants with two $\theta^{\mu\nu}$ and one of the following three contents: (i) three Riemann tensors, (ii) two Riemann tensors and two covariant derivatives, or (iii) one Riemann tensor and four covariant derivatives. Note that invariants without any Riemann tensor are trivially zero, since in that case covariant derivatives may only act on $\theta^{\mu\nu}$ and this gives zero. To compute the allowed invariants, we rely on the form of the allowed spacetime geometries. Let us first consider the case of pp waves. The Riemann tensor takes the form $R_{\mu\nu\rho\sigma} = -2k_{[\mu}\partial_{\nu]}\partial_{[\rho}Hk_{\sigma]}$. It follows that

$$\theta^{\mu\nu}R_{\nu\alpha\beta\gamma} = 2k^\mu k_\alpha p^\nu \partial_\nu \partial_{[\beta}Hk_{\gamma]} \neq 0,$$

which in turn implies

$$\theta^{\mu\nu}R_{\mu\nu\rho\sigma} = \theta^{\mu\nu}\theta^{\rho\sigma}R_{\mu\rho\alpha\beta} = \theta^{\mu\nu}R_{\nu\beta} = 0.$$

It is then easy to convince oneself that all invariants of type (i), (ii), and (iii) are trivially zero. In other words, there are no diffeomorphism-invariant, second-order in θ deformations of pp-wave metrics.

Consider now $(2+2)$-decomposable metrics. In this case, condition (2.1) reduces to

$$\nabla'_{\alpha'}\theta'^{\beta'\gamma''} = \nabla''_{\alpha''}\theta''^{\beta''\gamma''} = 0,$$

whose solutions are

$$\theta'^{\alpha'\beta'} = \frac{\theta'}{\sqrt{-h'}}\epsilon^{\alpha'\beta'}, \qquad \theta''^{\alpha''\beta''} = \frac{\theta''}{\sqrt{h''}}\epsilon^{\alpha''\beta''},$$

with θ' and θ'' constants. Here $\epsilon^{01} = \epsilon^{23} = 1$, $h' = \det(h'_{\alpha'\beta'})$, and $h'' = \det(h''_{\alpha''\beta''})$. The four-dimensional bivector $\theta^{\mu\nu}$ is either spacelike or timelike, so the NC scale $\ell_{\rm NC}$ is given by

$$\theta^{\mu\nu}\theta_{\mu\nu} = 2(\theta''^2 - \theta'^2) = \pm\ell^2_{\rm NC}.$$

The only nonzero components of the Riemann tensor are

$$R'_{\alpha'\beta'\gamma'\delta'} = h'\epsilon_{\alpha'\beta'}\epsilon_{\gamma'\delta'}R',$$
$$R''_{\alpha''\beta''\gamma''\delta''} = h''\epsilon_{\alpha''\beta''}\epsilon_{\gamma''\delta''}R'',$$

with R' and R'' being the Ricci scalars of the two-dimensional metrics $h'_{\alpha'\beta'}$ and $h''_{\alpha''\beta''}$. Here we have written explicitly factors h' and h'' so as to have $-\epsilon_{01} = \epsilon_{23} = 1$. In this case there are 11 different invariants. They read

Invariants without ∇'s:

$$I_1 = \theta'^2R'^3 - \theta''^2R''^3, \tag{2.2}$$

$$I_2 = (R' + R'')(\theta'^2R'^2 - \theta''^2R''^2), \tag{2.3}$$

$$I_3 = (R' + R'')^2(\theta'^2R' - \theta''^2R''), \tag{2.4}$$

$$I_4 = (\theta'^2 - \theta''^2)(R' + R'')^3. \tag{2.5}$$

Invariants with 2 ∇'s:

$$J_1 = \theta'^2\Delta'R'^2 - \theta''^2\Delta''R''^2, \tag{2.6}$$

$$J_2 = (R' + R'')(\theta'^2\Delta'R' - \theta''^2\Delta''R''), \tag{2.7}$$

$$J_3 = (\theta'^2 - \theta''^2)(\Delta'R'^2 + \Delta''R''^2), \tag{2.8}$$

$$J_4 = (\theta'^2 - \theta''^2)(\Delta' + \Delta'')(R' + R'')^2, \tag{2.9}$$

$$J_5 = (\theta'^2 - \theta''^2)(R' + R'')(\Delta'R' + \Delta''R''). \tag{2.10}$$

Invariants with 4 ∇'s:

$$K_1 = \theta'^2\Delta'^2R' - \theta''^2\Delta''^2R'', \tag{2.11}$$

$$K_2 = (\theta'^2 - \theta''^2)(\Delta' + \Delta'')(\Delta'R' + \Delta''R''), \tag{2.12}$$

where $\Delta' = h'^{\alpha'\beta'}\nabla'_{\alpha'}\nabla'_{\beta'}$ and similarly for Δ''.

If a cosmological constant term is included in the undeformed action, some other invariants are possible. On dimensional reasons, the presence of Λ decreases either the number of Riemann tensors by one or the number of covariant derivatives by two. At first order in $\theta^{\mu\nu}$, the only invariant that may be constructed is $\theta^{\mu\nu}R_{\mu\nu}$, which is identically zero. At order two we may have either (i) two Riemann tensors without covariant derivatives or (ii) one Riemann tensor and two covariant derivatives. For pp-wave metrics, it is very easy to check that all invariants of these types are identically zero. For $(2+2)$-decomposable metrics, the list (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), (2.8), (2.9), (2.10), (2.11), and (2.12) is enlarged with the invariants

$$\text{Invariants for } \Lambda\text{-term: } I_5 = \theta'^2 R'^2 - \theta''^2 R''^2, \tag{2.13}$$

$$I_6 = (R' + R'')(\theta'^2 R' - \theta''^2 R''), \tag{2.14}$$

$$I_7 = (\theta'^2 - \theta''^2)(R' + R'')^2, \tag{2.15}$$

$$J_6 = \theta'^2 \Delta' R' - \theta''^2 \Delta'' R'', \tag{2.16}$$

$$J_7 = (\theta'^2 - \theta''^2)(\Delta' R' + \Delta'' R''). \tag{2.17}$$

We conclude that, for pp-wave metrics, there are neither first-order, nor second-order polynomial deformations in θ of the Einstein-Hilbert action. For $(2 + 2)$-decomposable metrics, the most general deformed Lagrangian up to second order in θ is an arbitrary linear combination of all invariants in (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), (2.8), (2.9), (2.10), (2.11), (2.12), (2.13), (2.14), (2.15), (2.16), and (2.17). This is as far as one can go using general invariance arguments. In Secs. IV, V, and VI we use the Seiberg-Witten formalism and the vierbein postulate to construct a diffeomorphism-invariant NC deformation of the Einstein-Hilbert action. The method yields for $(2 + 2)$-decomposable metrics the following deformed action up to order two:

$$\begin{aligned} S_{2+2} = \frac{1}{\kappa^2} \int d^2x' d^2x'' \sqrt{-h'h''} \bigg\{ \Big(R' + R'' - \frac{\Lambda}{2} \Big) \\ \times \Big[1 - \frac{b}{8} (\theta'^2 R'^2 - \theta''^2 R''^2) \Big] \\ + \frac{a}{8} (\theta'^2 R'^3 - \theta''^2 R''^3) \bigg\} + O(\theta^3). \end{aligned} \tag{2.18}$$

Here a and b are arbitrary real coefficients, their arbitrariness being due to the fact that the solutions to the Seiberg-Witten equations are not unique.

III. FIELD EQUATIONS FOR DEFORMED GRAVITY AND SOLUTIONS

The purpose of this section is to show that the equations of motion for the model described by the classical action (2.18) have nontrivial solutions. For this purpose, we restrict ourselves to solutions with constant curvatures R' and R''. The field equations then become algebraic and have the form

$$\begin{aligned} -R' + \frac{\Lambda}{2} = \frac{a-b}{8} (\theta'^2 R'^3 + 2\theta''^2 R''^3) \\ + \frac{b}{8} \Big[\frac{\Lambda}{2} \theta'^2 R'^2 + \theta''^2 \Big(\frac{\Lambda}{2} - R' \Big) R''^2 \Big], \end{aligned} \tag{3.1}$$

$$\begin{aligned} R'' - \frac{\Lambda}{2} = \frac{a-b}{8} (2\theta'^2 R'^3 + \theta''^2 R''^3) \\ + \frac{b}{8} \Big[\theta'^2 \Big(\frac{\Lambda}{2} - R'' \Big) R'^2 + \frac{\Lambda}{2} \theta''^2 R''^2 \Big]. \end{aligned} \tag{3.2}$$

If R' and R'' are not constant, the equations above acquire some extra terms involving covariant derivatives of R' and R'', arising from the higher order terms in the action (2.18).

We will exclude from our analysis the cases (i) $a = b = 0$, for it corresponds to no deformations at all, and (ii) $R' = R'' = 0$, for the only solution is then $\Lambda = 0$ and this corresponds to Minkowski spacetime. From Sec. II we know that $\theta'^2 \neq \theta''^2$, so at least one of the two constants θ', θ'' must be nonzero. Since the equations of motion (3.1) and (3.2) remain invariant under the changes

$$(R'', \theta'') \leftrightarrow (R', \theta'), \qquad (a, b) \leftrightarrow -(a, b), \tag{3.3}$$

it is enough to consider $\theta'' \neq 0$. The solutions for $\theta' \neq 0$ are obtained from those for $\theta'' \neq 0$ by making the replacements above. Assuming then $\theta'' \neq 0$, we distinguish two types of solutions:

Type 1: $\theta'' \neq 0$, θ' arbitrary; R', R'' constants given in terms of a, b, θ', θ'',

Type 2: $\theta'' \neq 0$, $\theta' = 0$; R' arbitrary, $R'' = \Lambda = \pm \dfrac{4}{\theta'' \sqrt{2a}}$, $b = a > 0$.

Had we started with $\theta' \neq 0$, instead of $\theta'' \neq 0$, we would have obtained:

Type 3: $\theta' \neq 0$, θ'' arbitrary; R', R'' constants given in terms of a, b, θ', θ'',

Type 4: $\theta' \neq 0$, $\theta'' = 0$; R'' arbitrary, $R' = \Lambda = \pm \dfrac{4}{\theta' \sqrt{-2a}}$, $b = a < 0$.

In the remainder of this section we examine solutions of types 1 and 3. Actually it is enough to look at type 1, for type 3 can be obtained through the substitutions (3.3). The equations of motion form a system of two cubic equations in R' and R'' with coefficients depending on a, b, θ', and θ''. It is convenient to distinguish the three following cases:

Case 1. One of the two-dimensional curvatures R', R'' vanishes. After setting one of them equal to zero, Eqs. (3.1) and (3.2) reduce to two quadratic equations in the nonvanishing curvature that can be easily solved. For example, for $R' = 0$, nontrivial solutions exist only if $\theta'' \neq 0$. In this case, the curvature R'' is

$$R'' = \begin{cases} \frac{32-3b\Lambda^2\theta''^2}{\Lambda\theta''^2(9a-8b)} & \text{if } 9a \neq 8b \\ \frac{3\Lambda}{2} & \text{if } 9a = 8b \end{cases}, \tag{3.4}$$

with the cosmological constant given in terms of θ'' by

$$\left(\frac{\Lambda\theta''}{2}\right)^2 = \begin{cases} \frac{1}{b^3}[36ab - 27a^2 - 8b^2 \pm (9a-8b)\sqrt{a(9a-8b)}] & \text{if } b \neq 0 \\ \frac{32}{27a} & \text{if } b = 0 \end{cases}. \tag{3.5}$$

Recall that by assumption both a and b cannot vanish simultaneously. The right-hand side of Eq. (3.5) must be real and positive. This makes clear that not for all a and b a solution (R'', Λ) exists. Table I collects the allowed ranges for the parameters a and b and the corresponding values for Λ^2. The subscript in $\Lambda^2_\pm$ refers to the $\pm$ sign in front of the square root in (3.5). The solutions for $R'' = 0$ are obtained from those presented here for $R' = 0$ through the replacements (3.3).

Case 2. None of the two-dimensional curvatures vanish, but the cosmological constant does. For $\Lambda = 0$, if $a = b$, the only solution to Eqs. (3.1) and (3.2) is the trivial one $R' = R'' = 0$. We thus take $a \neq b$. Introducing

$$\xi = \frac{R'}{R''}, \qquad k = \frac{\theta'^2}{\theta''^2} \geq 0,$$

the equations of motion can be written as

$$\frac{8\xi}{(\theta'' R'')^2} = -(a-b)(k\xi^3 + 2) + b\xi, \tag{3.6}$$

$$\frac{8}{(\theta'' R'')^2} = (a-b)(2k\xi^3 + 1) - bk\xi^2. \tag{3.7}$$

Eliminating $(\theta'' R'')^2$, one has

$$k\xi^4 + pk\xi^3 + p\xi + 1 = 0, \tag{3.8}$$

where for later convenience we have defined the parameter p as

$$p := \frac{a-2b}{2(a-b)}.$$

Equation (3.8) has degree four in ξ. Its solutions will depend on the parameters p and k. We are only interested in real solutions. For $p = 0$, i.e. for $a = 2b$, all solutions are complex. Hence we take $a \neq b$, $2b$. Given a real solution ξ, Eq. (3.7) provides R'' as a function of a, b, θ, and θ'', and thus a solution $(\xi R'', R'')$ for (R', R''). We must make then sure that Eq. (3.8) has real solutions for ξ. For $k = 0$, the only solution to Eq. (3.8) is

$$\theta' = 0, \qquad R' = -\frac{1}{p}R'', \qquad R'' = \pm\frac{2}{\theta''}\sqrt{\frac{2}{|a-b|}}.$$

For $k > 0$, it is shown in Appendix A that Eq. (3.8) has real solutions except for

$$k_+(p^2) < k < k_-(p^2), \qquad p^2 < 1, \tag{3.9}$$

with $k_\pm(p^2)$ given by

$$k_\pm(p^2) = \frac{1}{27p^4}\{-27p^4 - 2(p^2-4)^3 \pm 2(p^2-4)(p^2+8)\sqrt{(p^2-1)(p^2-4)}\}. \tag{3.10}$$

Hence, for (θ', θ'') with $\theta' = \pm\sqrt{k}\theta''$ and k as in (3.9) there are no real solutions to the field equations. Graphically this is represented in Fig. 1, where only the shaded region is allowed and the angles $\alpha_\pm$ are given by $\tan^2\alpha_\pm = k_\pm$.

Let us illustrate this case with a simple-looking example. For $p = 1$, which corresponds to $a = 0$, Eq. (3.8) has two real solutions, $\xi = -1$ and $\xi = -k^{-1/3}$. The corresponding solutions for R' and R'' are

$$R' = -R'' = \pm\frac{4}{\sqrt{|b\ell^2_{NC}|}}, \qquad \begin{matrix} b < 0 & \text{for } \theta^{\mu\nu} \text{ spacelike} \\ b > 0 & \text{for } \theta^{\mu\nu} \text{ timelike} \end{matrix}$$

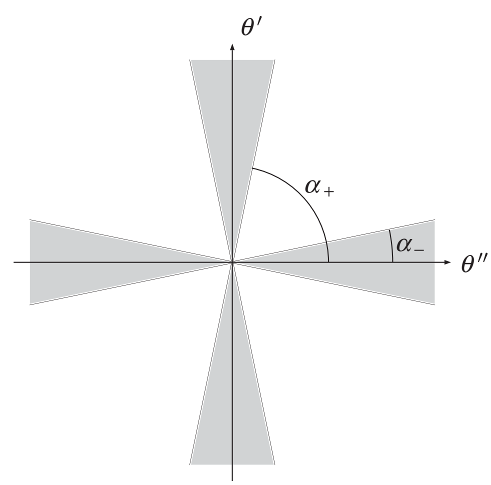

FIG. 1. Allowed region in (θ', θ'')-plane for constant curvatures R' and R''.

TABLE I. Solutions with $R' = 0$, $\theta'' \neq 0$.

$a < b < 0$	$\Lambda^2 = \Lambda^2_-$
$b < 0$, $a \geq 0$	$\Lambda^2 = \Lambda^2_-$, Λ^2_+
$b = 0$, $a > 0$	$\Lambda^2 = \frac{128}{27\lvert a\rvert\theta''^2}$
$0 < 8b \leq 9a$	$\Lambda^2 = \Lambda^2_+$
$0 < 8b < 9a \leq 9b$	$\Lambda^2 = \Lambda^2_-$, Λ^2_+

and

$$\theta'^{2/3}R' = -\theta''^{2/3}R'' = \pm\left[\frac{8}{|b(\theta''^{2/3}-\theta'^{2/3})|}\right]^{1/2},$$
$$b>0 \quad \text{for } \theta^{\mu\nu} \text{ spacelike}$$
$$b<0 \quad \text{for } \theta^{\mu\nu} \text{ timelike}.$$

The gravitational fields solving the equations of motion can be understood as induced by Seiberg-Witten noncommutativity. From this point of view, the equations of motion generate a two-parameter family of NC gravities, with parameters a and b. For instance, the solution above with $R' = -R''$ describes an $\mathrm{AdS}_2 \times S^2$ spacetime, with radii proportional to the NC scale,

$$R^2_{\mathrm{AdS}} = R^2_S = \frac{\sqrt{|b|}|\ell_{\mathrm{NC}}|}{2}.$$

Case 3. None of the curvatures R', R'' vanish, nor the cosmological constant. One may proceed as for $\Lambda = 0$ and derive an equation for ξ. In the general case, that is, leaving aside values of a and b for which simplifications occur, one obtains an equation of degree nine in ξ. This guarantees the existence of at least one real solution. However, being an equation of degree nine, an analytic study as that in Appendix A for $\Lambda = 0$ escapes our abilities. Yet the equations of motion may be used to induce NC gravity, very much as for $\Lambda = 0$. For example, one may be interested in four-dimensional geometries with vanishing curvature, so that $R' + R'' = 0$, but with nonzero cosmological constant. This is achieved e.g. by setting $b = 0$, for which

$$R' = -R'' = -\mathrm{sign}(a)\frac{3|\ell^2_{\mathrm{NC}}|\Lambda}{8(\theta'^2+\theta''^2)} = -\mathrm{sign}(a)\frac{8}{\sqrt{6|a\ell_{\mathrm{NC}}|^2}},$$

with $a > 0$ for spacelike noncommutativity and $a < 0$ for timelike. We note that even though the action is polynomial in θ' and θ'', the solutions for the scalar curvatures are not. This makes sense, since one would naively expect the NC scale to modify lengths, and dimensional analysis makes lengths enter scalar curvatures in a certain way.

It is worth noting that these solutions do not have a smooth $\theta^{\mu\nu} \to 0$ limit. This is expected, since $\theta^{\mu\nu}$ must be spacelike and lightlike deformation bivectors are excluded. The two-parameter family of four geometries solving the field equations found here can be understood as classically induced by the Seiberg-Witten map. Some solutions for induced or emergent NC gravity have been proposed within the context of matrix models [16].

IV. SEIBERG-WITTEN MAPS FOR $SO(1,3)_{\mathrm{loc}}$

Our construction is based on the description [11] of general relativity as gauge theory with gauge algebra $SO(1,3)_{\mathrm{loc}}$. In this section we construct the Seiberg-Witten maps for $SO(1,3)_{\mathrm{loc}}$ without assuming any metric structure. In the next section we extend the maps found here in a way consistent with diffeomorphism invariance by requiring the vierbein postulate.

A. BRS characterization of general relativity

We start by reviewing the BRS approach to Kibble's formulation [11] of general relativity as a gauge theory with gauge algebra the local Lorentz algebra $SO(1,3)_{\mathrm{loc}}$. We will use capital Latin letters $A, B, \ldots$ for $SO(1,3)$ indices. The relevant fields are the vierbein $e_\alpha{}^A(x)$ and the spin-connection $\omega_\alpha{}^{AB}(x)$, defined at each spacetime point and regarded as independent.

The inverse vierbein $e_A{}^\alpha(x)$ is defined as

$$e_\alpha{}^A e_A{}^\beta = \delta_\alpha{}^\beta, \qquad e_A{}^\alpha e_\alpha{}^B = \delta_A{}^B. \tag{4.1}$$

Under a Lorentz transformation, the components of $e^{A\alpha}(x) = \eta^{AB}e^\alpha_B(x)$ transform for every α as a vector, which we denote by $F^\alpha(x)$. The components of the inverse vierbein $e_A{}^\alpha(x)$ form then its Hermitian conjugate, which we denote by E^α

$$F^\alpha = \{e^{A\alpha}\}, \qquad E^\alpha = [F^\alpha]^+ = \{e_A{}^\alpha\}.$$

Equations (4.1) define the inverse of $F^\alpha(x)$ as $\mathcal{F}_\alpha(x) = \{e_{\alpha A}(x)\}$, with

$$\mathcal{F}_\alpha F^\beta = \delta_\alpha{}^\beta, \qquad F^\alpha \mathcal{F}_\alpha = 1. \tag{4.2}$$

Taking Hermitian conjugates, the vierbein components form for every α a vector $\mathcal{E}_\alpha(x) = \{e_\alpha{}^A(x)\}$ satisfying

$$E^\alpha \mathcal{E}_\beta = \delta^\alpha{}_\beta, \qquad \mathcal{E}_\alpha E^\alpha = 1. \tag{4.3}$$

The vierbein maps Minkowski's metric η_{AB} onto the object $\mathcal{F}_\alpha(x)\mathcal{E}_\beta(x) = e_\alpha{}^A(x)e_\beta{}^B(x)\eta_{AB}$. Similarly, the inverse vierbein maps η^{AB} onto $E^\alpha(x)F^\beta(x) = e_A{}^\alpha(x)e_B{}^\beta(x)\eta^{AB}$. The transformation properties of F^α and $\mathcal{E}_\alpha$ imply that $E^\alpha F^\beta$ and $\mathcal{F}_\alpha\mathcal{E}_\beta$ are invariant under $SO(1,3)$ transformations. Furthermore, since

$$\mathcal{F}_\alpha \mathcal{E}_\gamma E^\gamma F^\beta = \delta_\alpha{}^\beta, \qquad E^\alpha F^\beta \mathcal{F}_\beta \mathcal{E}_\alpha = 1,$$

$\mathcal{F}_\alpha\mathcal{E}_\beta$ is the inverse of $E^\alpha F^\beta$. All these definitions and transformation properties are local, i.e. hold at every x^α independently of spacetime metric considerations. We emphasize that $\mathcal{F}_\alpha\mathcal{E}_\beta$ is not the spacetime metric $g_{\alpha\beta}(x)$, nor $E^\alpha F^\beta$ is its inverse, since no assumption relating the spacetime metric and Minkowski's metric η_{AB} has as yet been made.

The spin-connection components $\omega_\alpha{}^{AB}(x)$ form the matrix

$$\Omega_\alpha(x) = -\tfrac{1}{2}\omega_\alpha{}^{AB}(x)I_{AB},$$

where we have written I_{AB} for the generators of the vector or adjoint representation of $SO(1,3)$

$$(I_{AB})^C{}_D = \mathrm{i}(\delta_A{}^C\eta_{BD} - \delta_B{}^C\eta_{AD}).$$

The spin connection is the gauge one-form for $SO(1,3)_{\rm loc}$, in terms of which the Lorentz-covariant derivative D_α is defined as

$$D_\alpha = \partial_a - \mathrm{i}[\Omega_\alpha,].$$

The corresponding field strength $\Omega_{\alpha\beta}(x)$ reads

$$\Omega_{\alpha\beta} = \partial_\alpha\Omega_\beta - \partial_\beta\Omega_\alpha - \mathrm{i}[\Omega_\alpha, \Omega_\beta].$$

It is trivial to verify that

$$[D_\alpha, D_\beta]F^\gamma = -\mathrm{i}\Omega_{\alpha\beta}F^\gamma. \qquad (4.4)$$

Furthermore, since $\{F^\alpha\}$ forms a basis, the action of $\Omega_{\alpha\beta}$ on F^α and its Hermitian conjugate E^α can always be written as

$$\mathrm{i}\,\Omega_{\alpha\beta}F^\delta = R_{\alpha\beta\gamma}{}^\delta F^\gamma, \qquad -\mathrm{i}E^\delta\Omega_{\alpha\beta} = E^\gamma R_{\alpha\beta\gamma}{}^\delta, \qquad (4.5)$$

where $R_{\alpha\beta\gamma}{}^\lambda$ are real coefficients. In what follows we will denote by $R_{\alpha\beta}$ the contraction $R_{\alpha\beta} := R_{\alpha\lambda\beta}{}^\lambda$. Note that the coefficients $R_{\alpha\beta}$ are not necessarily symmetric at this stage. One now considers

$$\mathcal{L} = eL, \qquad (4.6)$$

with e and L given by

$$e = [-\det(E^\alpha F^\beta)]^{-1/2} = [-\det(\mathcal{F}_\alpha\mathcal{E}_\beta)]^{1/2} \qquad (4.7)$$

and

$$L = \frac{1}{2\kappa^2}(\mathrm{i}E^\alpha\Omega_{\alpha\beta}F^\beta - \Lambda). \qquad (4.8)$$

It is important to remark that everything so far does not involve any spacetime metric. We recall in this regard that transformation properties under diffeomorphisms of tensor fields do not depend on the existence of a metric. By contrast, partial derivatives of fields do not in general transform covariantly under diffeomorphisms. In the following we assume that partial derivatives of fields exist but do not use their transformation laws. In particular, $\mathcal{L}$ in (4.6) is not assumed to be a scalar under general coordinate transformations, for it involves partial derivatives $\partial_\alpha\Omega_\beta$.

To go from $\mathcal{L}$ and its $SO(1,3)$ local gauge symmetry to general relativity, one imposes the vierbein postulate

$$\nabla_\alpha F^\gamma(x) - \mathrm{i}\Omega_\alpha(x)F^\gamma(x) = 0, \qquad (4.9)$$

where $\nabla_\alpha F^\gamma(x) := \partial_\alpha F^\gamma(x) + \Gamma^\gamma_{\alpha\delta}(x)F^\delta(x)$ denotes the general covariant derivative. As is well known, the solution to Eq. (4.9), together with the torsion-free assumption $\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\beta\alpha}$, gives the spin connection Ω_α in terms of F^α, its inverse, and their Hermitian conjugates. Furthermore, the product $E^\alpha F^\beta$ becomes the inverse spacetime metric $g^{\alpha\beta}$ and $\mathcal{L}$ above the Einstein-Hilbert Lagrangian $\sqrt{-g}(R-\Lambda)/2\kappa^2$, with R the scalar curvature.

The advantage of this approach is that it separates the $SO(1,3)$ gauge symmetry of general relativity from spacetime metric considerations. This is very useful to perform Seiberg-Witten deformations of gravity, for the Seiberg-Witten map provides an algebraic method to construct a deformed symmetry [17]. It is most convenient to describe the $SO(1,3)$ local symmetry in terms of a BRS operator. Let us very briefly recall how this is done. To remind ourselves that no metric assumptions are made, we will use lower case Latin letters $a, b, \ldots$ for indices. Noting that the fields F^a, Ω_a take values in the Lie algebra $SO(1,3)$, we define the BRS operator s through

$$s\Omega_a = D_a\lambda := \partial_a\lambda - \mathrm{i}[\Omega_a, \lambda], \qquad sF^a = \mathrm{i}\lambda F^a, \qquad s\lambda = \mathrm{i}\lambda^2, \qquad (4.10)$$

where $\lambda(x)$ is a ghost field taking values in the vector representation of $SO(1,3)$. It is straightforward to check that $s^2 = 0$ and that

$$s(E^aF^b) = 0, \qquad s\Omega_{ab} = -\mathrm{i}[\Omega_{ab}, \lambda]. \qquad (4.11)$$

Hence $se = 0$. From this, Eqs. (4.10) and

$$(sX)^+ = (-)^{|X|}sX^+, \qquad |X| = \text{ghost number of } X,$$

it then trivially follows that $\mathcal{L}$ is invariant under s. The coefficients $R_{abc}{}^d$ introduced in expansions (4.5) are also BRS invariant,

$$s(R_{abd}{}^c) = 0.$$

Indeed, acting with s on the left-hand side of (4.5) and using Eqs. (4.10) and (4.11), we have $s(\mathrm{i}\Omega_{ab}F^c) = R^c_{abd}(sF^d)$, thus proving BRS invariance of $R_{abd}{}^c$.

B. Construction of Seiberg-Witten maps

Our aim here is to construct a NC extension of the Lagrangian $\mathcal{L}$ using Seiberg-Witten maps. We assume that noncommutativity is characterized by a set of BRS invariant constant parameters $\theta^{ab} = -\theta^{ba}$, $s\theta^{ab} = 0$, with dimensions of $(\text{length})^2$, in terms of which the Moyal-Groenewold $\star$ product reads

$$\star := \exp\left(\frac{\mathrm{i}t}{2}\overleftarrow{\partial}_a\theta^{ab}\vec{\partial}_b\right). \qquad (4.12)$$

The parameter t has been introduced for convenience and takes values on the interval $[0, 1]$. It interpolates between the commutative and the noncommutative cases. Throughout this section θ^{ab} are constant parameters. As a consequence the Moyal-Groenewold product above is associative, a property that plays an essential role in performing the Seiberg-Witten construction to all orders in θ^{ab}.

We recall that, given a set of fields $\{\phi\}$ enjoying a gauge symmetry described by a graded, nilpotent BRS operator s

$$s\phi = \mathcal{P}(\phi), \qquad (4.13)$$

with $\mathcal{P}$ a function of ϕ and their partial derivatives of finite order, the Seiberg-Witten formalism [8,17] yields a NC generalization in terms of fields $\{\hat{\phi}(t)\}$ defined by

$$s\hat{\phi}(t) = \mathcal{P}_\star[\hat{\phi}(t)], \tag{4.14}$$

$$\hat{\phi}|_{t=0} = \phi. \tag{4.15}$$

The functional $\mathcal{P}_\star$ is obtained from $\mathcal{P}$ by replacing the ordinary product with the star product $\star$ of Eq. (4.12). The fields $\{\hat{\phi}(t)\}$ are usually called NC fields, though it is also customary to use for them the name of Seiberg-Witten maps. They are assumed to be power series in θ^{ab} whose coefficients are polynomials in the fields $\{\phi\}$ and their derivatives, hence taking values in the universal enveloping algebra of the gauge Lie algebra under consideration.

A way to solve Eqs. (4.14) and (4.15) is to differentiate the first one with respect to t. This yields

$$s\dot{\hat{\phi}}(t) = \frac{d}{dt}\mathcal{P}_\star[\hat{\phi}(t)]. \tag{4.16}$$

The right-hand side is linear in first derivatives $\dot{\hat{\phi}}(t)$ because of the differentiation rule

$$\frac{d}{dt}(\hat{A}\star\hat{B}) = \dot{\hat{A}}\star\hat{B} + \hat{A}\star\dot{\hat{B}} + \frac{\mathrm{i}}{2}\theta^{ab}\partial_a\hat{A}\star\partial_b\hat{B}. \tag{4.17}$$

Whether or not the system of equations (4.16) can be solved for $\dot{\hat{\phi}}(t)$ must be discussed case by case. Let us assume that a solution $\dot{\hat{\phi}}_0(t)$ exists. At $t = 0$, it yields the first-order contribution in θ^{ab}, while higher derivatives at $t = 0$ provide higher contributions. Taking into account the initial condition (4.15), one may then write

$$\hat{\phi} = \phi + t\phi^{(1)} + t^2\phi^{(2)} + \ldots,$$

where

$$\phi^{(N)} = \frac{1}{N!}\frac{d^{N-1}}{dt^{N-1}}\dot{\hat{\phi}}(t)|_{t=0}$$

is the NC contribution to order N in θ^{ab}. It is clear that, from this point of view, the relevant object to construct the Seiberg-Witten map is $\dot{\hat{\phi}}_0(t)$.

Coming to the case we are interested in, the NC fields are now $\hat{\phi} = \hat{\Omega}_m$, $\hat{F}^m$, $\hat{\lambda}$ and take values in the universal enveloping algebra of the vector representation of $SO(1,3)$. The Seiberg-Witten equations (4.14) read

$$s\hat{\Omega}_a = \hat{D}_a\star\hat{\lambda} := \partial_a\hat{\lambda} - \mathrm{i}[\hat{\Omega}_a, \hat{\lambda}]_\star, \qquad s\hat{F}^a = \mathrm{i}\hat{\lambda}\star\hat{F}^a,$$
$$s\hat{\lambda} = \mathrm{i}\hat{\lambda}\star\hat{\lambda}, \tag{4.18}$$

where the notation $[X, Y]_\star = X\star Y - Y\star X$ has been used. It is straightforward to check that s^2 on $\hat{\Omega}_a$, $\hat{F}^a$, and $\hat{\lambda}$ vanishes. Differentiating Eqs. (4.18) with respect to t, we obtain

$$s\dot{\hat{\Omega}}_a = \mathrm{i}[\hat{\lambda}, \dot{\hat{\Omega}}_a]_\star + \hat{D}_a\star\dot{\hat{\lambda}} - \tfrac{1}{2}\theta^{mn}\{\partial_m\hat{\lambda}, \partial_n\hat{\Omega}_a\}_\star, \tag{4.19}$$

$$s\dot{\hat{F}}^a = \mathrm{i}\hat{\lambda}\star\dot{\hat{F}}^a + \mathrm{i}\dot{\hat{\lambda}}\star\hat{F}^a - \tfrac{1}{2}\theta^{mn}\partial_m\hat{\lambda}\star\partial_n\hat{F}^a, \tag{4.20}$$

$$s\dot{\hat{\lambda}} = \mathrm{i}\{\hat{\lambda}, \dot{\hat{\lambda}}\}_\star - \tfrac{1}{2}\theta^{mn}\partial_m\hat{\lambda}\star\partial_n\hat{\lambda}, \tag{4.21}$$

with $\{X, Y\}_\star := X\star Y + Y\star X$. One may check by using Eqs. (4.18) that a particular solution $\dot{\hat{\phi}}_0$ is given by

$$\dot{\hat{\Omega}}_a = \tfrac{1}{2}\theta^{mn}\{\hat{\Omega}_m, -\partial_n\hat{\Omega}_a + \tfrac{1}{2}\hat{D}_a\star\hat{\Omega}_n\}_\star, \tag{4.22}$$

$$\dot{\hat{F}}^a = \frac{1}{2}\theta^{mn}\hat{\Omega}_m\star\Big(-\partial_n\hat{F}^a + \frac{\mathrm{i}}{2}\hat{\Omega}_n\star\hat{F}^a\Big), \tag{4.23}$$

$$\dot{\hat{\lambda}} = -\tfrac{1}{4}\theta^{mn}\{\hat{\Omega}_m, \partial_n\hat{\lambda}\}_\star. \tag{4.24}$$

C. The Seiberg-Witten Lagrangian

The NC extension of the Lagrangian (4.6) is

$$\hat{\mathcal{L}} = \hat{e}\star\hat{L}, \tag{4.25}$$

where $\hat{e}$ is defined by

$$\hat{e}\star\hat{e}\star\det(\hat{E}^a\star\hat{F}^b) = -1 \tag{4.26}$$

and $\hat{L}$ reads

$$\hat{L} = \frac{1}{2\kappa^2}(\mathrm{i}\hat{E}^a\star\hat{\Omega}_{ab}\star\hat{F}^b - \Lambda). \tag{4.27}$$

Here $\hat{E}^a$ is the Hermitian conjugate of $\hat{F}^a$ and $\det(\hat{E}^a\star\hat{F}^b)$ is calculated through

$$\det(\hat{E}^a\star\hat{F}^b) = \frac{1}{4!}\epsilon_{a_1a_2a_3a_4}\epsilon_{b_1b_2b_3b_4}\hat{E}^{a_1}\star\hat{F}^{b_1}\star\hat{E}^{a_2}\star\hat{F}^{b_2}\star\hat{E}^{a_3}\star\hat{F}^{b_3}\star\hat{E}^{a_4}\star\hat{F}^{b_4}.$$

Contributions to $\hat{\mathcal{L}}$ of order N in θ^{ab} are given by

$$\mathcal{L}^{(N)} = \frac{1}{N!}\frac{d^N}{dt^N}\hat{\mathcal{L}}|_{t=0}, \tag{4.28}$$

where derivatives with respect to t are obtained by employing the differentiation rule (4.17). Equation (4.28) involves derivatives with respect to t of $\hat{e}$, $\hat{\Omega}_{ab}$, and $\hat{F}^a$. Those of $\hat{\Omega}_{ab}$ and $\hat{F}^a$ follow straightforwardly from Eqs. (4.22) and (4.23), while those of $\hat{e}$ are computed by differentiating Eq. (4.26) with respect to t as many times as needed. This provides a systematic way to compute the Seiberg-Witten map $\hat{\mathcal{L}}$ to any order in θ^{ab}. The algebra may be, and in fact is, long but the method is straightforward. In this paper we consider first and second-order corrections in θ^{ab}.

To ease the writing we introduce the notation

$$h_{ab} := \mathcal{F}_a\mathcal{E}_b,$$

with $\mathcal{F}_a$ and $\mathcal{E}_a$ as in Eqs. (4.2) and (4.3). For the order-one correction we obtain, after some algebra,

$$\mathcal{L}^{(1)} = e^{(1)}L + eL^{(1)} + \frac{i}{2}\theta^{mn}(\partial_m e)\partial_n L, \tag{4.29}$$

where L is as in (4.8), and $e^{(1)}$ and $L^{(1)}$ are given by

$$e^{(1)} = -\frac{i}{4}e\theta^{mn}h_{ab}(D_m E^a)D_n F^b \tag{4.30}$$

and

$$L^{(1)} = \frac{i}{4\kappa^2}\theta^{mn}\Big\{(D_m E^a)\Omega_{ab}D_n F^b + iE^a\Big(\{\Omega_{ma},\Omega_{nb}\} - \frac{1}{2}\{\Omega_{ab},\Omega_{mn}\}\Big)F^b + \partial_m[(D_n E^a)\Omega_{ab}F^b + \text{H.c.}]\Big\}. \tag{4.31}$$

The second-order contribution $\mathcal{L}^{(2)}$ reads

$$\mathcal{L}^{(2)} = e^{(2)}L + e^{(1)}L^{(1)} + eL^{(2)} + \frac{i}{2}\theta^{mn}[(\partial_m e^{(1)})\partial_n L + (\partial_m e)\partial_n L^{(1)}] - \frac{1}{8}\theta^{mn}\theta^{rs}(\partial_m\partial_r e)\partial_n\partial_s L. \tag{4.32}$$

The quantities in this equation have the following expressions: $e^{(2)}$ is the order-two contribution to $\hat{e}$ and has the form

$$e^{(2)} = e\theta^{mn}\theta^{rs}\{\tfrac{1}{16}h_{ab}[(D_m D_r E^a)D_n D_s F^b - 2i(D_m E^a)\Omega_{ns}D_r F^b] - \tfrac{1}{32}(h_{ab}h_{cd} + 2h_{ad}h_{bc})(D_m E^a)(D_n F^b)(D_r E^c)D_s F^d + \tfrac{1}{32}(h_{ab}h_{cd} - h_{ad}h_{bc})\partial_m\partial_r(E^a F^b)\partial_n\partial_s(E^c F^d) - \tfrac{1}{12}\partial_m\partial_r(E^a F^b) \times [(\partial_n h_{ab} - h_{ab}\partial_n \ln e)\partial_s \ln e + \tfrac{1}{4}(2h_{bc}\partial_n h_{ad} - h_{ab}\partial_n h_{cd})\partial_s(E^c F^d)] + \tfrac{1}{16}(\partial_m\partial_r \ln e)[2(\partial_n \ln e)\partial_s \ln e - 3(\partial_n\partial_s \ln e)]\}. \tag{4.33}$$

$L^{(2)}$ is the second-order contribution in $\hat{L}$ and can be written as the sum

$$L^{(2)} = L^{(2)}_{\text{v}} + L^{(2)}_{\text{s}} \tag{4.34}$$

of two terms given by

$$L^{(2)}_{\text{v}} = \frac{1}{16\kappa^2}\theta^{mn}\theta^{rs}\Big\{-i(D_m D_r E^a)\Omega_{ab}D_n D_s F^b + [D_m(E^a\Omega_{ra})\Omega_{sb}D_n F^b + D_m(E^a\Omega_{sb})\Omega_{ra}D_n F^b + \Big[-\frac{1}{2}D_m(E^a\Omega_{rs})\Omega_{ab}D_n F^b + \frac{1}{2}D_m(E^a\Omega_{ab})(\Omega_{rs}D_n F^b - 4\Omega_{ns}D_r F^b) + \text{H.c.}\Big] + (D_m E^a)\{\Omega_{ab},\Omega_{ns}\}D_r F^b + \frac{1}{2}(D_m E^a)(\{\Omega_{ra},\Omega_{sb}\} - 2\{\Omega_{ab},\Omega_{rs}\})D_n F^b + iE^a\Big[2\Omega_{na}\Omega_{mr}\Omega_{sb} - \frac{1}{2}\Omega_{ma}\Omega_{rs}\Omega_{nb} - (a\leftrightarrow b)\Big]F^b + iE^a\Big[\frac{1}{2}\Omega_{mn}\Omega_{ab}\Omega_{rs} + \{\{\Omega_{na},\Omega_{sb}\},\Omega_{mr}\} - \{\{\Omega_{ra},\Omega_{sb}\},\Omega_{mn}\}\Big]F^b\Big\} \tag{4.35}$$

and

$$L^{(2)}_{\text{s}} = \frac{1}{16\kappa^2}\theta^{mn}\theta^{rs}\partial_m\Big\{i(D_{(n}D_{s)}E^a)\Omega_{ab}D_r F^b + \frac{i}{2}\partial_r[2(D_n E^a)\Omega_{ab}D_s F^b - (D_{(n}D_{s)}E^a)\Omega_{ab}F^b] - (D_r E^a)\Omega_{ns}\Omega_{ab}F^b - D_n(E^a\Omega_{ab})\Omega_{rs}F^b - \frac{1}{2}(D_n E^a)(2\{\Omega_{ra},\Omega_{sb}\} - \{\Omega_{ab},\Omega_{rs}\})F^b - \frac{1}{2}[D_n(E^a\Omega_{ra})\Omega_{sb} + D_n(E^a\Omega_{sb})\Omega_{ra}]F^b + \text{H.c.}\Big\}. \tag{4.36}$$

In general, the contribution $\mathcal{L}^{(N)}$ of order N will be a polynomial of degree $N+1$ in Ω_{ab}, with covariant derivatives acting on F^c and $\Omega_{ab}F^c$, and on their Hermitian conjugates E^a and $E^c\Omega_{ab}$.

There are three important observations concerning the action of partial derivatives ∂_m and covariant derivatives D_m in these expressions. The first one concerns ∂_m. From Eqs. (4.2) and (4.3) it is clear that

$$\partial_m h_{ab} = -h_{ac}\partial_m(E^c F^d)h_{db}. \tag{4.37}$$

This and the definition (4.7) of e as a linear combination of products $E^a F^b$ implies that ∂_m and $\partial_m\partial_n$ on e and h_{ab} are given in terms of $\partial_m(E^a F^b)$ and $\partial_m\partial_n(E^a F^b)$. Noting now that

$$\partial_m(E^a F^b) = (D_m E^a)F^b + E^a D_m F^b, \quad (4.38)$$

$$\partial_m\partial_n(E^a F^b) = (D_m D_n E^a)F^b + (D_n E^a)D_m F^b + (D_m E^a)D_n F^b + E^a(D_m D_n F^b), \quad (4.39)$$

we have that ∂_m and $\partial_m\partial_n$ acting on e and h_{ab} are linear combinations of $D_m E^a$, $(D_m E^a)D_n F^b$, $D_m D_n F^a$, and their Hermitian conjugates.

The second observation concerns the action of products $D_m D_n$ of two covariant derivatives on E^a and F^b. Noting Eqs. (4.4) and (4.5), every product of two covariant derivatives D_m acting on F^a can then be written as

$$D_m D_n F^a = -\tfrac{1}{2}R_{mnb}{}^a F^b + \tfrac{1}{2}\{D_m, D_n\}F^a. \quad (4.40)$$

We also note here that contributions with D_m acting on $\Omega_{ab}F^c = -\mathrm{i}R_{abd}{}^c F^d$, or its Hermitian conjugate $E^c\Omega_{ab}$, will yield a term $(\partial_m R^c_{abd})F^d$ with partial derivatives and a term $R^c_{abd}D_m F^d$.

The third observation concerns terms with products of covariant derivatives acting on the field strength, $D_m \cdots D_n \Omega_{ab}$. In this case, integration by parts is performed as many times as necessary until no covariant derivatives acting on Ω_{ab} is left and they all act on F^c and/or E^c. This procedure has already been used to obtain Eqs. (4.35) and (4.36).

V. DIFFEOMORPHISM-INVARIANT SEIBERG-WITTEN LAGRANGIAN

The Seiberg-Witten Lagrangian $\hat{\mathcal{L}}$ constructed in the previous section is a power series in θ^{ab}. This construction has been performed in the universal enveloping algebra of (the vector representation of) $SO(1,3)$ and is metric independent. Note that, precisely because of this metric independence, covariance under diffeomorphisms only holds for the vierbein and spin connection and not for their derivatives. The problem we face now is to extend $\hat{\mathcal{L}}$ to a generally invariant expression without losing BRS invariance.

A. The vierbein postulate

To relate the underlying spacetime metric to the spin connection Ω_a and the vierbein F^a, we proceed as in general relativity. We take a point x^a to be the origin of a locally inertial frame, impose the vierbein postulate

$$[\bar{\nabla}_a - \mathrm{i}\Omega_a(\bar{x})]F^b(\bar{x}) = 0 \quad (5.1)$$

in a neighborhood $\bar{x}^a$ of x^a, and demand a torsion-free geometry

$$\Gamma^a_{cb}(\bar{x}) = \Gamma^a_{bc}(\bar{x}). \quad (5.2)$$

Note that $\bar{x}^a - x^a$ are normal coordinates. The covariant derivative $\bar{\nabla}_a$ at $\bar{x}^a$ in (5.1) is defined as usual,

$$\bar{\nabla}_a F^b(\bar{x}) := \bar{\partial}_a F^b(\bar{x}) + \Gamma^b_{ac}(\bar{x})F^c(\bar{x}),$$

and involves the Christoffel symbols $\Gamma^a_{bc}(\bar{x})$, which depend on the metric $g_{ab}(\bar{x})$. Since $\hat{\mathcal{L}}$ is written in terms of fields and their derivatives at x^a, it is convenient to write the vierbein postulate and the torsion-free condition in terms of fields and derivatives ∇_a at x^a. In Appendix B, it is shown that conditions (5.1) and (5.2) are equivalent to the infinite set of conditions

$$(\nabla_{a_1} - \mathrm{i}\Omega_{a_1})\cdots(\nabla_{a_n} - \mathrm{i}\Omega_{a_n})F^c(x) = 0, \quad (5.3)$$

$$\Gamma^b_{cd}(x) = 0 \qquad \partial_{a_1}\cdots\partial_{a_n}\Gamma^b_{[cd]}(x) = 0, \quad (5.4)$$

with $n = 1, 2, \ldots$ The condition $\Gamma^b_{cd}(x) = 0$ reminds us that x^a is the origin of a locally inertial frame. It is convenient to recall that in such a frame the derivatives of the Christoffel symbols do not vanish, so that conditions (5.4) are not trivial.

The treatment of Eqs. (5.1) and (5.2), equivalently (5.3) and (5.4), is the same as in general relativity. By solving them, the spin connection $\Omega_a(\bar{x})$ and the Christoffel symbols $\Gamma^a_{bc}(\bar{x})$ are uniquely determined in terms of the (inverse) vierbein $F^a(\bar{x})$ and its partial derivatives. Once the Christoffel symbols are known, it follows trivially that $E^a(\bar{x})F^b(\bar{x})$ is covariantly constant and becomes the inverse metric $g^{ab}(\bar{x})$. Furthermore, from Eqs. (5.3), (5.4), and (5.5) it follows that

$$[\nabla_a - \mathrm{i}\Omega_a, \nabla_b - \mathrm{i}\Omega_b]F^c = 0 \Rightarrow [\nabla_a, \nabla_b]F^c = R_{abd}{}^c F^d,$$

so that the coefficients $R_{abd}{}^c$ become the components of the Riemann tensor.

Having used Eqs. (5.1) and (5.2), equivalently (5.3) and (5.4), to relate the Lorentz algebra connection Ω_a and the vierbein F^a with the underlying metric, we must make sure that (5.1) and (5.2) are compatible with the Lorentz BRS symmetry described by s. To establish the latter, one must show (i) that the spin connection $\Omega_a(\bar{x})$, which is no longer an independent field, transforms under s as in Eq. (4.10), and (ii) that the Christoffel symbols $\Gamma^a_{bc}(\bar{x})$ are BRS invariant. Both statements are proved in Appendix B.

Although conceptually the situation is similar to general relativity, there is a very important technical difference. The Seiberg-Witten Lagrangian $\hat{\mathcal{L}}$ is far more complicated than the Einstein-Hilbert Lagrangian of general relativity, for it contains products of arbitrary numbers of Lorentz-covariant derivatives $D_a = \partial_a - \mathrm{i}\Omega_a$ acting on the Lorentz field strength Ω_{ab} and the vierbein F^a that must be taken care of. To obtain a diffeomorphism-invariant extension of the Seiberg-Witten construct $\hat{\mathcal{L}}$, we now proceed in two steps:

Step 1. The antisymmetric part of every product of more than two covariant derivatives D_a is extracted and all partial derivatives ∂_a are replaced with covariant derivatives ∇_a, so that D_m is replaced with $\nabla_m - \mathrm{i}\Omega_m$. In doing so, Christoffel symbols in ∇_a are provided by the solution to the vierbein postulate (5.3). A comment concerning the antisymmetrization of products $D_{m_1}\cdots D_{m_n}F^a$ with two or more covariant derivatives is due here. Consider the Lagrangian (4.6), from which the Einstein-Hilbert action is recovered, and view Ω_{ab} as $D_aD_b - D_bD_a$. Replacing $\partial_m \to \nabla_m$ and blindly using the vierbein postulate (5.3) with $n=2$ leads to $\mathcal{L}=0$. In other words, although not explicitly spelled out, antisymmetrization is built in the $SO(1,3)$ gauge description of general relativity.

Step 2. In step 1, $\hat{\mathcal{L}}$ has been extended to a power series in the parameters θ^{mn} with coefficients generally covariant at x^a and in a neighborhood $\bar{x}^a$ of it. To achieve a generally covariant $\hat{\mathcal{L}}_{\rm cov}$, a prescription to deal with θ^{mn} is necessary. We identify θ^{mn} with the components at x^a of a bivector. Since this bivector has constant components at the origin of a locally inertial frame $\{\bar{x}^a - x^a\}$, it must be covariantly constant, i.e. its components $\bar{\theta}^{mn}$ at $\bar{x}^a$ must satisfy $\bar{\nabla}_r\bar{\theta}^{mn}(\bar{x})=0$. This yields a NC Lagrangian $\hat{\mathcal{L}}$ that is generally covariant in a normal coordinate patch. Transition to the whole four-dimensional manifold is performed in the standard way and Greek indices may be restored. In particular the components $\theta^{\mu\nu}$ of the covariantly constant bivector satisfy $\nabla_\rho\theta^{\mu\nu}=0$.

Following these two steps, the generally invariant extensions of $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$ will be computed in the next subsection. As a general word of caution, it is convenient to put first the various terms of the Seiberg-Witten construct $\hat{\mathcal{L}}$ in a *manifestly* gauge invariant form and then apply steps 1 and 2.

We close this subsection by noting that once steps 1 and 2 have been performed, the original Seiberg-Witten construction with the Moyal-Groenewold star product (4.12) *cannot be recovered.* Indeed, generally covariant derivatives do not commute even at the origin of a locally inertial frame. Clearly, the deformation parameters in our approach do not form a Poisson tensor.

B. The deformed classical action: explicit expressions up to order θ^2

We first look at $\mathcal{L}^{(1)}$ in (4.29). From Eq. (4.30) it follows that $e^{(1)}$ is linear in D_nF^a and its Hermitian conjugate D_nE^a. Step 1 above, i.e. the replacement $D_a \to \nabla_a - \mathrm{i}\Omega_a$ and the vierbein postulate (5.3), then yields that the generally covariant extension of $e^{(1)}$ vanishes identically. Similar arguments show that the generally covariant extension of $\partial_m e$ is also identically zero. We are thus left with the term $eL^{(1)}$ in (4.29) as the only source of generally covariant contributions to order one in θ^{mn}, the only piece in $L^{(1)}$ that may give a nontrivial contribution being

$$-\frac{1}{4\kappa^2}\theta^{mn}E^a\left(\{\Omega_{ma},\Omega_{nb}\}-\frac{1}{2}\{\Omega_{ab},\Omega_{mn}\}\right)F^b. \tag{5.5}$$

Using now Eqs. (4.5) and recalling from Appendix B that E^aF^b becomes the inverse metric g^{ab} after solving the vierbein postulate (5.3), it is straightforward to see that (5.5) is identically zero. We thus conclude that there is no diffeomorphism-invariant first-order deformation of the Einstein-Hilbert action,

$$\mathcal{L}^{(1)}_{\rm cov}=0,$$

in accordance with the general arguments in Sec. II.

Let us next compute the generally invariant extension of the second-order contribution $\mathcal{L}^{(2)}$. Inspection of Eq. (4.32) for $\mathcal{L}^{(2)}$ and the arguments used at first order imply that the only nonzero second-order contributions will arise from terms in $e^{(2)}L$ and $eL^{(2)}$ without factors D_mF^a and D_mE^a. From Eqs. (4.33) and (4.34) for $e^{(2)}$ and $L^{(2)}$ it follows that there are only three different types of such terms:

(i) Terms with products of two or more Lorentz-covariant derivatives D_m acting on F^a. They are treated as follows. Consider e.g. the first term of $L^{(2)}_{\rm v}$ in (4.35). It gives to $\mathcal{L}^{(2)}$ a contribution

$$-\frac{\mathrm{i}e}{16\kappa^2}\theta^{mn}\theta^{rs}(D_mD_rE^a)\Omega_{ab}(D_nD_sF^b). \tag{5.6}$$

According to step 1, we use Eq. (4.40) to extract the antisymmetric part of the products D_mD_r and D_nD_s, replace $\partial_n \to \nabla_n$, and impose the vierbein postulate (5.3). In accordance with step 2, we take θ^{mn} as the components at x^a of a generally covariant bivector $\theta^{\mu\nu}$. This gives

$$-\frac{\sqrt{-g}}{64\kappa^2}\theta^{\mu\nu}\theta^{\rho\sigma}R_{\mu\rho\gamma}{}^{\alpha}R_{\nu\sigma\delta}{}^{\beta}R_{\alpha\beta}{}^{\gamma\delta} \tag{5.7}$$

for the diffeomorphism-invariant extension of (5.6).

(ii) Terms with one covariant derivative D_m acting on $\Omega_{ab}F^c$ (or on its Hermitian conjugate $E^c\Omega_{ab}$). Consider, for example, the last term in $L^{(2)}_{\rm s}$, namely

$$-\frac{e}{32\kappa^2}\theta^{mn}\theta^{rs}\partial_m[D_n(E^a\Omega_{ra})\Omega_{sb} + D_n(E^a\Omega_{sb})\Omega_{ra}]F^b + \mathrm{H.c.} \tag{5.8}$$

Using expansions (4.5), recalling that D_m acts on $R_{abc}{}^d$ only through ∂_m, replacing $\partial_m \to \nabla_m$, imposing the vierbein postulate (5.3) with $n=1$, and noting that after solving the vierbein postulate E^dF^e becomes the inverse metric g^{de}, one obtains

$$-\frac{\sqrt{-g}}{16\kappa^2}\theta^{\mu\nu}\theta^{\rho\sigma}\nabla_\mu[(\nabla_\nu R_{\rho\delta})R_\sigma{}^\delta + (\nabla_\nu R_{\sigma\delta\gamma\beta})R_\rho{}^{\beta\gamma\delta}] \tag{5.9}$$

for the generally covariant extension of (5.8). Here $R_{\alpha\beta}:=R_{\alpha\gamma\beta}{}^{\gamma}$ has been used. Contribution (5.9) is diffeomorphism invariant under the assumption $\nabla_\rho\theta^{\mu\nu}=0$.

(iii) Terms only involving products of Ω_{ab}, E^a, and F^a, and no covariant derivatives D_m. In this case, all that needs to be used are the expansions (4.5).

Proceeding in this way with all the terms in $e^{(2)}L$ and $eL^{(2)}$, we obtain after some algebra the following generally covariant extension for $\mathcal{L}^{(2)}$:

$$\mathcal{L}^{(2)}_{\rm cov} = \frac{\sqrt{-g}}{16\kappa^2}\Big[\mathcal{R}_1 - 2\mathcal{R}_2 - \mathcal{R}_3 + \frac{1}{2}\mathcal{R}_4 - \frac{1}{8}\mathcal{R}_5 + 2\mathcal{R}_6 + \frac{1}{4}\mathcal{R}_7 - \frac{1}{2}\mathcal{R}_8 - \mathcal{R}_9 + \mathcal{R}_{10} + \frac{1}{8}(R-\Lambda)\mathcal{Q}_1 + \frac{1}{2}\mathcal{B}_1 - \mathcal{B}_2\Big], \tag{5.10}$$

the invariants $\mathcal{R}_1, \ldots, \mathcal{R}_{10}$ and $\mathcal{Q}_1$, $\mathcal{B}_1$, $\mathcal{B}_2$ being three

$$\begin{aligned}
\mathcal{R}_1 &= \theta^{\mu\nu}\theta^{\rho\sigma}R_{\mu\rho}{}^{\alpha\beta}R_{\nu\alpha\gamma\delta}R_{\sigma\beta}{}^{\gamma\delta},\\
\mathcal{R}_6 &= \theta^{\mu\nu}\theta^{\rho\sigma}R_{\nu\alpha}R_{\sigma\beta}R_{\mu\rho}{}^{\alpha\beta},\\
\mathcal{R}_2 &= \theta^{\mu\nu}\theta^{\rho\sigma}R_{\mu\rho}{}^{\alpha\beta}R_{\nu\gamma\alpha}{}^{\delta}R_{\sigma\delta\beta}{}^{\gamma},\\
\mathcal{R}_7 &= \theta_{\mu\nu}\theta^{\rho\sigma}R_{\mu\nu}{}^{\alpha\beta}R_{\rho\sigma}{}^{\gamma\delta}R_{\alpha\beta\gamma\delta},\\
\mathcal{R}_3 &= \theta_{\mu\nu}\theta^{\rho\sigma}R_{\mu\nu}{}^{\alpha\beta}R_{\rho\alpha\gamma\delta}R_{\sigma\beta}{}^{\gamma\delta},\\
\mathcal{R}_8 &= \theta^{\mu\nu}\theta^{\rho\sigma}R_{\rho\alpha}R_{\sigma\beta}R_{\mu\nu}{}^{\alpha\beta},\\
\mathcal{R}_4 &= \theta^{\mu\nu}\theta^{\rho\sigma}R_{\mu\nu}{}^{\alpha\beta}R_{\rho\gamma\alpha}{}^{\delta}R_{\sigma\delta\beta}{}^{\gamma},\\
\mathcal{R}_9 &= \theta^{\mu\nu}\theta^{\rho\sigma}g^{\gamma\delta}R_{\nu\gamma}R_{\sigma\delta\alpha\beta}R_{\mu\rho}{}^{\alpha\beta},\\
\mathcal{R}_5 &= \theta^{\mu\nu}\theta^{\rho\sigma}R_{\mu\rho}{}^{\alpha\beta}R_{\nu\sigma}{}^{\gamma\delta}R_{\alpha\beta\gamma\delta},\\
\mathcal{R}_{10} &= \theta^{\mu\nu}\theta^{\rho\sigma}g^{\gamma\delta}R_{\rho\gamma}R_{\sigma\delta\alpha\beta}R_{\mu\nu}{}^{\alpha\beta},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{Q}_1 &= \theta^{\mu\nu}\theta^{\rho\sigma}R_{\mu\rho\alpha\beta}R_{\nu\sigma}{}^{\alpha\beta},\\
\mathcal{B}_1 &= \theta^{\mu\nu}\theta^{\rho\sigma}\nabla_\mu(R_{\rho\beta\gamma\delta}\nabla_\nu R_\sigma{}^{\beta\gamma\delta}),\\
\mathcal{B}_2 &= \theta^{\mu\nu}\theta^{\rho\sigma}\nabla_\mu(R_{\rho\beta}\nabla_\nu R_\sigma{}^{\beta}).
\end{aligned}$$

It was already mentioned in Sec. II that the only spacetime metrics satisfying $\nabla_\mu\theta^{\nu\rho}=0$ are either of pp-wave type or $(2+2)$ decomposable. There it was discussed that all invariants of order two in $\theta^{\mu\nu}$ are identically zero for pp-wave metrics. As a consistency check one may verify that $\mathcal{R}_i$, $\mathcal{Q}_1$, and $\mathcal{B}_i$ vanish for such metrics. Using the notation of Sec. II for $(2+2)$-decomposable metrics, it is straightforward to check that

$$4\mathcal{R}_1 = -8\mathcal{R}_2 = 2\mathcal{R}_3 = -4\mathcal{R}_4 = 2\mathcal{R}_5 = 8\mathcal{R}_6 = \mathcal{R}_7 = 4\mathcal{R}_8 = -4\mathcal{R}_9 = -2\mathcal{R}_{10} = -16I_1, \tag{5.11}$$

$$\mathcal{Q}_1 = -4I_5, \tag{5.12}$$

$$\mathcal{B}_1 = \mathcal{B}_2 = 0, \tag{5.13}$$

with I_1 and I_4 as given in Eqs. (2.2) and (2.13). Taking into account that $R = 2(R' + R'')$ and the expression (2.13) for I_5, the second-order Lagrangian becomes

$$\mathcal{L}^{(2)}_{\rm cov,2+2} = \frac{1}{16\kappa^2}\sqrt{-h'h''}\Big[I_1 - \Big(R' + R'' - \frac{\Lambda}{2}\Big)I_5\Big]. \tag{5.14}$$

This corresponds to taking $a = b = 1/2$ in the second-order terms of the action (2.18).

VI. MORE GENERAL SEIBERG-WITTEN LAGRANGIANS

The expression (5.10) for the second-order contribution $\mathcal{L}^{(2)}_{\rm cov}$ has been found using the Seiberg-Witten maps (4.22) and (4.23) for $\hat{\Omega}_a(t)$ and $\hat{F}^a(t)$. These are not, however, the most general solutions to the Seiberg-Witten equations (4.18), as the following argument shows. Assume that $\{\hat{\Omega}_a(t), \hat{F}^a(t)\}$ is a solution to the Seiberg-Witten equations (4.18), with $\hat{\lambda}(t)$ as in (4.24). It is then clear that $\{\hat{\Omega}_a(t) + \delta\hat{\Omega}_a(t), \hat{F}^a(t) + \delta\hat{F}^a(t)\}$ is also a solution provided $\delta\hat{\Omega}_a(t)$ and $\delta\hat{F}^a(t)$ satisfy [18]

$$\begin{aligned}
s\delta\hat{\Omega}_a(t) &= {\rm i}[\hat{\lambda}(t), \delta\hat{\Omega}_a(t)]_\star, \qquad \delta\hat{\Omega}_a|_{t=0} = 0,\\
s\delta\hat{F}^a(t) &= {\rm i}\hat{\lambda}(t)\star\delta\hat{F}^a(t), \qquad \delta\hat{F}^a|_{t=0} = 0.
\end{aligned} \tag{6.1}$$

To find more general Seiberg-Witten solutions, these two equations must be solved. In what follows we do it. Recall that we are interested in solutions which are formal power series in θ^{mn} whose coefficients depend polynomially in the fields Ω_a, F^a and their derivatives.

Equations (6.1) are homogeneous in $\delta\hat{\Omega}_a$ and $\delta\hat{F}^a$, and do not contain contributions of order zero in θ^{mn}. Their solutions will then be power series in θ^{mn}

$$\begin{aligned}
\delta_N\hat{\Omega}_a &= t^N(\delta_N\Omega_a)^{(N)} + t^{N+1}(\delta_N\Omega_a)^{(N+1)} + \ldots,\\
\delta_N\hat{F}^a &= t^N(\delta_N F^a)^{(N)} + t^{N+1}(\delta_N F^a)^{(N+1)} + \ldots
\end{aligned}$$

starting at any order $N \geq 1$ and satisfying

$$s(\delta_N\hat{\Omega}_a) = {\rm i}[\hat{\lambda}, \delta_N\hat{\Omega}_a]_\star, \qquad s(\delta_N\hat{F}^a) = {\rm i}\hat{\lambda}\star\delta_N\hat{F}^a. \tag{6.2}$$

The most general solution for $\delta\hat{\phi}(t)$ will be

$$\delta\hat{\phi}(t) = \sum_{N=1}^{\infty}\delta_N\hat{\phi}(t), \qquad \phi = \Omega_a, F^a.$$

The form of $\delta_N\hat{\Omega}_a$ and $\delta_N\hat{F}^a$ can be determined as follows. Their lowest-order contributions $(\delta_N\Omega_a)^{(N)}$ and $(\delta_N F^a)^{(N)}$ satisfy

$$\begin{aligned}
s(\delta_N\Omega_a)^{(N)} &= {\rm i}[\lambda, (\delta_N\Omega_a)^{(N)}],\\
s(\delta_N F^a)^{(N)} &= {\rm i}\lambda(\delta_N F^a)^{(N)}
\end{aligned} \tag{6.3}$$

for $N = 1, 2, \ldots$ Using dimensional analysis, BRS covariance, and Eqs. (4.10), it is easy to solve these equations.

See below for explicit examples. The solutions will be functions $\omega_a^{(N)}$ and $f^{a(N)}$

$$(\delta_N\Omega_a)^{(N)} = \omega_a^{(N)}[\Omega_m, F^m],$$
$$(\delta_N F^a)^{(N)} = f^{a(N)}[\Omega_m, F^m]$$

of Ω_m, F^m and their Lorentz-covariant derivatives. Let us multiply $\omega_a^{(N)}$ and $f^{a(N)}$ with t^N and replace in them the ordinary product with the $\star$ product, the spin connection Ω_m with the full $\hat{\Omega}_a + \delta\hat{\Omega}_a$, and vierbein F^n with $\hat{F}^a + \delta\hat{F}^a$. This results in power series

$$t^N\omega_a^{(N)}[\cdot \to \star; \hat{\Omega}_m + \delta\hat{\Omega}_m, \hat{F}^m + \delta\hat{F}^m],$$
$$t^N f^{a(N)}[\cdot \to \star; \hat{\Omega}_m + \delta\hat{\Omega}_m, \hat{F}^m + \delta\hat{F}^m],$$

starting at order N which solve Eqs. (6.2). Hence

$$\delta_N\hat{\Omega}_a = t^N\omega_a^{(N)}\Big[\cdot \to \star; \hat{\Omega}_m + \sum_{N=1}^{\infty}\delta_N\hat{\Omega}_m, \hat{F}^m + \sum_{N=1}^{\infty}\delta_N\hat{F}^m\Big], \tag{6.4}$$

$$\delta_N\hat{F}^a = t^N f^{a(N)}\Big[\cdot \to \star; \hat{\Omega}_m + \sum_{N=1}^{\infty}\delta_N\hat{\Omega}_m, \hat{F}^m + \sum_{N=1}^{\infty}\delta_N\hat{F}^m\Big] \tag{6.5}$$

provide through iteration explicit solutions to (6.2).

Our interest in this paper is contributions up to second order in θ^{mn}. It is then enough to consider

$$\delta\hat{\phi} = t(\delta_1\phi)^{(1)} + t^2(\delta_1\phi)^{(2)} + t^2(\delta_2\phi)^{(2)} + \ldots \tag{6.6}$$

for $\phi = \Omega_a$ and F^a. First-order contributions $(\delta_1\phi)^{(1)}$ and second-order contributions $(\delta_2\phi)^{(2)}$ are obtained by solving Eqs. (6.3) for $N = 1, 2$. In turn, contributions $(\delta_1\phi)^{(2)}$ are computed by iterating Eqs. (6.4) and (6.5) once for $N = 1$ and by retaining terms quadratic in t. All we need is thus the solutions to (6.3) for $N = 1, 2$. It is straightforward to see that the most general solution to (6.3) for $N = 1$ is

$$(\delta_1\Omega_a)^{(1)} = \omega_a^{(1)} = \frac{c}{2}\theta^{mn}D_a\Omega_{mn},$$
$$(\delta_1 F^a)^{(1)} = f^{a(1)} = \frac{\mathrm{i}}{2}(p\theta^{mn}\Omega_{mn}F^a + q\theta^{am}\Omega_{mb}F^b + r\theta^{am}\{D_m, D_b\}F^b),$$

with c, p, q, and r arbitrary real coefficients. Equations (6.4) and (6.5) for $N = 1$ then read

$$\delta_1\hat{\Omega}_a = \frac{t}{2}c\theta^{mn}\hat{D}_a^{\hat{\Omega}+\delta\hat{\Omega}} \star (\hat{\Omega}_{mn} + \delta\hat{\Omega}_{mn}), \tag{6.7}$$

$$\delta_1\hat{F}^a = \frac{\mathrm{i}t}{2}[p\theta^{mn}(\hat{\Omega}_{mn} + \delta\hat{\Omega}_{mn}) \star (\hat{F}^a + \delta\hat{F}^a) + q\theta^{am}(\hat{\Omega}_{mb} + \delta\hat{\Omega}_{mb}) \star (\hat{F}^b + \delta\hat{F}^b) + r\theta^{am}\{\hat{D}_m^{\hat{\Omega}+\delta\hat{\Omega}}, \hat{D}_b^{\hat{\Omega}+\delta\hat{\Omega}}\}_\star \star (\hat{F}^b + \delta\hat{F}^b)] \tag{6.8}$$

with $\hat{D}_a^{\hat{\Omega}+\delta\hat{\Omega}}$ the Lorentz $\star$-covariant derivative $\hat{D}_a^{\hat{\Omega}+\delta\hat{\Omega}} = \partial_a - \mathrm{i}[\hat{\Omega}_a + \delta\hat{\Omega}_a,]_\star$ and

$$\delta\hat{\Omega}_{mn} = \hat{D}_m \star \delta\hat{\Omega}_n - \hat{D}_n \star \delta\hat{\Omega}_m - \mathrm{i}[\delta\hat{\Omega}_m, \delta\hat{\Omega}_n]_\star.$$

From here it follows that

$$(\delta_1\Omega_a)^{(2)} = \frac{1}{2}\frac{d^2}{dt^2}\Big\{\frac{t}{2}c\theta^{mn}[\hat{D}_a \star \hat{\Omega}_{mn} + tD_a(\delta_1\Omega_{mn})^{(1)} - \mathrm{i}t[(\delta_1\Omega_a)^{(1)}, \Omega_{mn}]]\Big\}\Big|_{t=0}$$

and

$$(\delta_1 F^a)^{(2)} = \frac{1}{2}\frac{d^2}{dt^2}\Big\{\frac{\mathrm{i}t}{2}p\theta^{mn}[\hat{\Omega}_{mn} \star \hat{F}^a + t\Omega_{mn}(\delta_1F^a)^{(1)} + t(\delta_1\Omega_{mn})^{(1)}F^a] + \frac{\mathrm{i}t}{2}q\theta^{am}[\hat{\Omega}_{mb} \star \hat{F}^b + t\Omega_{mb}(\delta_1F^b)^{(1)} + t(\delta_1\Omega_{mb})^{(1)}F^b] + \frac{\mathrm{i}t}{2}r\theta^{am}[\{\hat{D}_m, \hat{D}_b\}_\star \star \hat{F}^b + t\{D_m, D_b\}(\delta_1F^b)^{(1)} - \mathrm{i}t\{D_m, (\delta_1\Omega_b)^{(1)}\}F^b - \mathrm{i}t\{D_b, (\delta_1\Omega_m)^{(1)}\}F^b]\Big\}\Big|_{t=0}.$$

Let us now turn to Eqs. (6.3) for $N = 2$. Using dimensional analysis and BRS covariance, it follows that the solution for $(\delta_2\Omega_a)^{(2)}$ is an arbitrary linear combination

$$(\delta_2\Omega_a)^{(2)} = \sum_{i=1}^{6}c_i(\delta_2\Omega_a)_i^{(2)}$$

with real coefficients c_i of the linearly independent solutions

$$(\delta_2\Omega_a)^{(2)}_1 = \frac{i}{4}\theta^{mn}\theta^{rs}[D_a\Omega_{mn}, \Omega_{rs}],$$
$$(\delta_2\Omega_a)^{(2)}_2 = \frac{i}{4}\theta^{mn}\theta^{rs}[D_a\Omega_{mr}, \Omega_{ns}],$$
$$(\delta_2\Omega_a)^{(2)}_3 = \frac{i}{4}\theta^{mn}\theta^{rs}D_r[\Omega_{mn}, \Omega_{sa}],$$
$$(\delta_2\Omega_a)^{(2)}_4 = \frac{1}{4}\theta^{mn}\theta^{rs}D_a(\hat{\Omega}_{mn}\hat{\Omega}_{rs})$$
$$(\delta_2\Omega_a)^{(2)}_5 = \frac{1}{4}\theta^{mn}\theta^{rs}D_a(\Omega_{mr}\Omega_{ns})$$
$$(\delta_2\Omega_a)^{(2)}_6 = \frac{1}{4}\theta^{mn}\theta^{rs}D_r\{\Omega_{mn}, \Omega_{sa}\}.$$

Similarly, the solution for $(\delta_2 F^a)^{(2)}$ is a linear combination

$$(\delta_2 F^a)^{(2)} = \sum_{i=1}^{3} p_i(\delta_2 F^a)^{(2)}_i + \sum_{j=1}^{11} q_j(\delta_2 F^a)^{(2)}_{j+3} \qquad (6.9)$$

of linearly independent solutions

$$(\delta_2 F^a)^{(2)}_1 = \frac{1}{4}\theta^{mn}\theta^{rs}\Omega_{mn}\Omega_{rs}F^a,$$
$$(\delta_2 F^a)^{(2)}_2 = \frac{1}{4}\theta^{mn}\theta^{rs}\Omega_{mr}\Omega_{ns}F^a,$$
$$(\delta_2 F^a)^{(2)}_3 = \frac{i}{4}\theta^{mn}\theta^{rs}(D_m\Omega_{rs})(D_n F^a),$$
$$(\delta_2 F^a)^{(2)}_4 = \frac{1}{4}\theta^{am}\theta^{rs}\Omega_{rs}\Omega_{mb}F^b,$$
$$(\delta_2 F^a)^{(2)}_5 = \frac{1}{4}\theta^{am}\theta^{rs}\Omega_{ms}\Omega_{rb}F^b,$$
$$(\delta_2 F^a)^{(2)}_6 = \frac{1}{4}\theta^{am}\theta^{rs}\Omega_{mb}\Omega_{rs}F^b,$$
$$(\delta_2 F^a)^{(2)}_7 = \frac{1}{4}\theta^{am}\theta^{rs}\Omega_{rb}\Omega_{ms}F^b,$$
$$(\delta_2 F^a)^{(2)}_8 = \frac{i}{4}\theta^{am}\theta^{rs}(D_m\Omega_{rs})(D_b F^b),$$
$$(\delta_2 F^a)^{(2)}_9 = \frac{i}{4}\theta^{am}\theta^{rs}(D_b\Omega_{rs})(D_m F^b),$$
$$(\delta_2 F^a)^{(2)}_{10} = \frac{i}{4}\theta^{am}\theta^{rs}(D_b\Omega_{ms})(D_r F^b),$$
$$(\delta_2 F^a)^{(2)}_{11} = \frac{i}{4}\theta^{am}\theta^{rs}(D_m\Omega_{sb})(D_r F^b),$$
$$(\delta_2 F^a)^{(2)}_{12} = \frac{1}{4}\theta^{am}\theta^{rs}(\{D_m, D_b\}\Omega_{rs})F^b,$$
$$(\delta_2 F^a)^{(2)}_{13} = \frac{1}{4}\theta^{am}\theta^{rs}(\{D_m, D_r\}\Omega_{sb})F^b,$$
$$(\delta_2 F^a)^{(2)}_{14} = \frac{1}{4}\theta^{am}\theta^{rs}(\{D_s, D_b\}\Omega_{mr})F^b.$$

This is not a complete list of all independent solutions for $(\delta_2 F^a)^{(2)}$. For example, together with $(\delta_2 F^a)^{(2)}_4$, one also has the solution $\theta^{am}\theta^{rs}\Omega_{rs}\{D_m D_b\}F^b$. This, however, does not contribute to $\mathcal{L}^{(2)}$ since, according to step 1 in Subsection VA, symmetrized products of more than one covariant derivative D_a acting on F^a vanish. In the list above we have omitted solutions with symmetrized products of Lorentz-covariant derivatives acting on F^b. By contrast, terms with symmetrized products $\{D_m, D_n\}$ acting on Ω_{ab} may give a nonvanishing contribution, since integration by parts to move the covariant derivatives on F^c will pick, upon antisymmetrization in (m, n), a nonvanishing contribution. We finally note that we have taken the coefficients r, p_j, q_k to be real, to avoid complexifications of the local $SO(1, 3)$ symmetry into a $U(1, 3)$ symmetry and the difficulties that such complexifications, in terms of unwanted ghost states, introduce [7,19].

Since our interest here is corrections in θ up to order two, it is enough to consider δ_1 and δ_2. Writing for the fields

$$\hat{\Omega}'_a = \hat{\Omega}_a + t(\delta_1\Omega_a)^{(1)} + t^2(\delta_1\Omega_a)^{(2)} + t^2(\delta_2\Omega_a)^{(2)} + \ldots,$$
$$\hat{F}'^a = \hat{F}^a + t(\delta_1 F^a)^{(1)} + t^2(\delta_1 F^a)^{(2)} + t^2(\delta_2 F^a)^{(2)} + \ldots \qquad (6.10)$$

we go over the construction in Secs. IV and V. After quite a bit of work we obtain that there are no diffeomorphism-invariant first-order corrections to the Einstein-Hilbert Lagrangian, in agreement with the general arguments of Sec. II. See Appendix C for intermediate results. For second-order corrections we obtain that, while for pp-wave metrics, second-order contributions, for (2 + 2) metrics there is a nonvanishing contribution, given by

$$\mathcal{L}'^{(2)}_{\rm cov} = \frac{\sqrt{-h'h''}}{8\kappa^2}\Big[aI_1 - b\Big(R' + R'' - \frac{\Lambda}{2}\Big)I_5\Big], \qquad (6.11)$$

where the coefficients a and b are given by Eqs. (C2) and (C3) in terms of the coefficients c, p, q, r and c_i, p_j, q_k entering $(\delta_1\phi)^{(1)}$ and $(\delta_2\phi)^{(2)}$. Since c, p, q, r and c_i, p_j, q_k are themselves arbitrary, the coefficients a and b are arbitrary. Putting together the Einstein-Hilbert action and its second-order deformation (6.11), we reproduce the action written in Eq. (2.18).

VII. CONCLUSION AND OUTLOOK

The Seiberg-Witten map can be viewed as a method to extend a local gauge symmetry to a larger symmetry living in the universal enveloping algebra of the original Lie algebra. The method is not explicit, in the sense that it provides equations that must be solved for every gauge algebra. The solutions are power series in constant antisymmetric parameters $\theta^{\mu\nu}$ whose coefficients depend polynomially on the fields involved and their derivatives. In the past, the solutions have been found to low orders in θ for the gauge groups of particle physics, which in turn has led to anomaly-free [20] NC extensions of particle models [21].

In this paper we have solved the analogous problem for general relativity's symmetry group, namely, the group of local Lorentz transformations. This has resulted in a model for NC gravity whose classical action is a power series in a covariantly constant bivector $\theta^{\mu\nu}(x)$. First and second-order contributions to the classical action have been explicitly computed.

The condition $\nabla_\mu \theta^{\rho\sigma} = 0$ restricts four-dimensional geometries to pp-wave metrics and direct sums of two two-dimensional metrics. For pp-wave metrics, $\theta^{\mu\nu}$ is null and first and second-order corrections to general relativity's classical action vanish. In turn, $(2+2)$-decomposable metrics correspond to either spacelike/timelike $\theta^{\mu\nu}$. For them first-order corrections vanish but second-order corrections do not. The curvatures of the two two-dimensional metrics depend on $\theta^{\mu\nu}$ and two arbitrary parameters a and b, their $\theta^{\mu\nu} \to 0$ limit being not smooth since $\theta^{\mu\nu}$ cannot be zero.

One of the motivations behind today's interest in NC gravity is studying whether noncommutativity may act as a source for gravity. From this point of view, the classical action obtained in this paper provides a field theory model with $\theta^{\mu\nu}$ a "gravity source." Furthermore, the family of $(2+2)$ geometries and their gravitational fields found here as can be understood as classically induced by noncommutativity through the Seiberg-Witten map. Some solutions for induced, or emergent, NC gravity have been proposed within the context of matrix models [16].

In this paper we have not coupled gravity to matter. Matter couplings introduced in the classical action produce contributions of order one in $\theta^{\mu\nu}$. The simplest case is that of a $U(1)$ gauge field. One may keep gravity undeformed and only construct the Seiberg-Witten map for the $U(1)$ field. Yet, by going to the generally covariant extension of the Seiberg-Witten construction along the lines explained here, the condition $\nabla_\rho \theta^{\mu\nu} = 0$ comes in and one is again limited to pp-wave metrics and $(2+2)$-decomposable metrics. In this case [22], first-order contributions in the classical action provide NC deformations of pp-wave metrics.

We want to finish with a few words about associativity of the Moyal-Groenewold product. It is precisely the fact that the deformation parameters θ^{mn} are constant in the Seiberg-Witten construction that ensures associativity. Turning the construction point into the origin of a locally inertial frame already destroys associativity. It is convenient to recall at this point that covariant constancy of $\theta^{\mu\nu}$ does not ensure associativity [14]. Furthermore, in the simple case of functions on four-dimensional Euclidean space, the Moyal-Groenewold product for bivectors of rank four only is associative for constant $\theta^{\mu\nu}$ [15].

It remains an open problem to extend the construction presented here to metrics which are not of pp-type or $2+2$. For example, by considering nonconstant deformation parameters; or by solving the Seiberg-Witten equations for other star products, e.g. Kontsevich's, for which the deformation parameters form a Poisson tensor. However, we are not aware of a systematic way to compute Seiberg-Witten maps at higher order with such deformation parameters.

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APPENDIX A: SOLUTIONS TO EQUATION (3.8)

Being (3.8) a quartic equation, its solutions can be determined analytically. They can be cast in the form

$$\xi_i = y_i - \frac{p}{4}, \qquad i = 1, 2, 3, 4,$$

with y_i given by

$$\begin{aligned} y_1 &= \tfrac{1}{2}(\sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3}),\\ y_3 &= \tfrac{1}{2}(-\sqrt{z_1} + \sqrt{z_2} - \sqrt{z_3}),\\ y_2 &= \tfrac{1}{2}(\sqrt{z_1} - \sqrt{z_2} - \sqrt{z_3}),\\ y_4 &= \tfrac{1}{2}(-\sqrt{z_1} - \sqrt{z_2} + \sqrt{z_3}) \end{aligned}$$

in terms of the solutions z_1, z_2, and z_3 of the resolvent cubic associated to Eq. (3.8). These, in turn, have the following explicit expressions:

$$\begin{aligned} z_1 &= \frac{p^2}{4} + s_1 + s_2,\\ z_{\frac{2}{3}} &= \frac{p^2}{4} - \frac{1}{2}[(s_1 + s_2) \mp \mathrm{i}\sqrt{3}(s_1 - s_2)], \end{aligned}$$

where s_1 and s_2 are the cubic roots

$$s_{\frac{1}{2}} = \left[\frac{p^2}{2k^2}(k+1) \pm \frac{p^2}{2k^2}\sqrt{\Delta(k, p^2)}\right]^{1/3},$$

$\Delta(k, p^2)$ is the discriminant

$$\begin{aligned} \Delta(k, p^2) &= (k+1)^2 + \frac{4(p^2-4)^3 k}{27p^4}\\ &= [k - k_+(p^2)][k - k_-(p^2)], \end{aligned}$$

and $k_\pm(p^2)$ are as in (3.10). The real or complex nature of z_1, z_2, z_3, hence of the solutions ξ_i, depend on the sign of Δ. To study Δ, we note that

$$\begin{aligned} &k_- \geq k_+ > 0 \quad \text{for } p^2 \leq 1,\\ &k_- = k_+^* \quad \text{for } 1 < p^2 < 4,\\ &k_- \leq k_+ < 0 \quad \text{for } 4 \leq p^2. \end{aligned}$$

Attending to the sign of Δ, the (p^2, k) domain can be divided into the following subdomains:

$$D_> = \{p^2 \le 1, k > 0\} \cup \{p^2 < 1, k > k_-\} \cup \{p^2 < 1, k < k_+\},$$
$$D_0 = \{p^2 < 1, k = k_-\} \cup \{p^2 < 1, k = k_+\},$$
$$D_< = \{p^2 < 1, k_+ < k < k_-\}.$$

We now have

(1) In $D_>$, the discriminant Δ is positive, z_1 is real and positive, and z_2, z_3 are complex conjugate of each other. This implies that there are two real and two complex solutions ξ_i.

(2) In D_0, Δ vanishes and z_1 and $z_2 = z_3$ are real and positive. Hence there are four real solutions ξ_i, with at most three of them distinct.

(3) Finally, in $D_<$, Δ is negative and one can easily deduce that

$$k < \frac{36}{k}\left(\frac{k+1}{4-p^2}\right)^2 < \frac{16(4-p^2)}{3p^4}. \tag{A1}$$

We are only interested in positive z_2 and z_3, since otherwise the four y_i are complex and there are no real solutions for ξ. From Vieta's relation we obtain

$$z_1z_2 + z_2z_3 + z_3z_1 = \frac{3p^4}{16} - \frac{4-p^2}{k} > 0,$$

in contradiction with (A1). We conclude that in $D_<$ there are no real solutions for ξ, thus proving the restriction (3.9)
.

APPENDIX B: THE VIERBEIN POSTULATE AND BRS LORENTZ SYMMETRY

We collect here some technical issues concerning Subsection VA. First we prove the equivalence between Eqs. (5.1), (5.2), (5.3), and (5.4). Going from (5.1) and (5.2) to (5.3) and (5.4) is straightforward. Indeed, condition $\Gamma^b_{cd}(x) = 0$ in (5.4) holds trivially, for x^a is the origin of a locally inertial frame. Furthermore, since (5.1) and (5.2) hold for arbitrary $\bar{x}^a$ close to x^a, one may act on them with products of $\bar{\partial}_a$ and $\bar{\nabla}_a - \mathrm{i}\bar{\Omega}_a$ and then set $\bar{x}^a = x^a$. This leads to (5.3) and (5.4). Let us now prove that (5.3) and (5.4) imply (5.1) and (5.2). Expanding $(\bar{\nabla}_a - \mathrm{i}\bar{\Omega}_a)\bar{F}^c$ in power series of $\bar{x}^a - x^a$, we have

$$(\bar{\nabla}_b - \mathrm{i}\bar{\Omega}_b)\bar{F}^c = \sum_{k=1}^{\infty}\frac{1}{k!}(\bar{x}-x)^{a_1}\cdots(\bar{x}-x)^{a_k} \times \partial_{a_1}\cdots\partial_{a_k}(\nabla_b - \mathrm{i}\Omega_b)F^c(x). \tag{B1}$$

For the first term in the sum, it follows from Eq. (5.3) that

$$\partial_{a_k}(\nabla_b - \mathrm{i}\Omega_b)F^c(x) = \mathrm{i}\Omega_{a_k}(x)(\nabla_b - \mathrm{i}\Omega_b)F^c(x) = 0. \tag{B2}$$

Acting with ∂_{a_k-1} on (B2), we obtain for the second term in the sum (B1)

$$\begin{aligned}\partial_{a_{k-1}}\partial_{a_k}(\nabla_b - \mathrm{i}\Omega_b)F^c(x) &= \mathrm{i}[\partial_{a_{k-1}}\Omega_k][(\nabla_b - \mathrm{i}\Omega_b)F^c(x)] \\ &\quad + \Omega_{a_k}(x)[\partial_{a_{k-1}}(\nabla_b - \mathrm{i}\Omega_b)F^c(x)].\end{aligned}$$

The first contribution on the right-hand side vanishes because of (5.3), the second one because of (B2). Repeating the argument we arrive at

$$\partial_{a_1}\partial_{a_2}\cdots\partial_{a_k}(\nabla_b - \mathrm{i}\Omega_b)F^c(x) = 0,$$

which implies (5.1) upon substitution in (B1). Similarly, expanding $\Gamma^b_{cd}(\bar{x})$ about x^a and using Eq. (5.4), we have

$$\begin{aligned}\Gamma^b_{cd}(\bar{x}) &= \sum_{k=0}^{\infty}\frac{1}{k!}(\bar{x}-x)^{a_1}\cdots(\bar{x}-x)^{a_k}\partial_{a_1}\cdots\partial_{a_k}\Gamma^b_{cd}(x) \\ &= \Gamma^b_{dc}(\bar{x}),\end{aligned}$$

which is the torsion-free condition (5.2).

To illustrate this equivalence, let us show that the solutions to (5.3) and (5.4) can be retrieved from those to (5.1) and (5.2) by taking derivatives $\bar{\partial}_{a_1}\cdots\bar{\partial}_{a_n}$ and setting $\bar{x}^a = x^a$. Indeed, the solutions to (5.1) and (5.2) at any point $\bar{x}^a$ of the locally inertial frame with origin x^a is known to be

$$\begin{aligned}\bar{\Omega}_a &= \frac{\mathrm{i}}{2}\{\bar{g}_{ab}[\bar{F}^c(\bar{\partial}_c\bar{E}^b) - (\bar{\partial}_c\bar{F}^b)\bar{E}^c] + \bar{\mathcal{E}}_b\bar{\partial}_a\bar{E}^b \\ &\quad + (\bar{\partial}_b\bar{\mathcal{E}}_a)\bar{E}^b - (\bar{\partial}_a\bar{F}^b)\bar{\mathcal{F}}_b - \bar{F}^b\bar{\partial}_b\bar{\mathcal{F}}_a\}\end{aligned} \tag{B3}$$

for the spin connection and

$$\bar{\Gamma}^b_{cd} = -\frac{1}{2}[\bar{g}^{ba}(\bar{\partial}_a\bar{g}_{cd}) + \bar{g}_{ac}(\bar{\partial}_d\bar{g}^{ba}) + \bar{g}_{ad}(\bar{\partial}_c\bar{g}^{ba})] \tag{B4}$$

for the Christoffel symbols, with $\bar{g}_{ab} = \bar{\mathcal{F}}_a\bar{\mathcal{E}}_b = \bar{\mathcal{F}}_b\bar{\mathcal{E}}_a$ the metric and $\bar{g}^{ab} = \bar{E}^a\bar{F}^b = \bar{E}^b\bar{F}^a$ the inverse metric. By taking $\bar{x}^a = x^a$ in (B3) and by using that at the origin of a locally inertial frame

$$\begin{aligned}\partial_c g^{ab} = 0 &\Rightarrow (\partial_c E^a)F^b + E^a(\partial_c F^b) = 0 \\ &\Rightarrow \mathcal{E}_a(\partial_c F^a) + (\partial_c F^b)\mathcal{F}_b = 0, \\ \partial_c g_{ab} = 0 &\Rightarrow (\partial_c\mathcal{F}_a)\mathcal{E}_b + \mathcal{F}_a(\partial_c\mathcal{E}_b) = 0 \\ &\Rightarrow g_{ab}(\partial_c E^b) - \partial_c\mathcal{F}_a = 0,\end{aligned}$$

we obtain

$$\Omega_a = -\mathrm{i}(\partial_a F^b)\mathcal{F}_b.$$

This is precisely the solution to Eq. (5.4) for $n = 1$.

Next we show the consistency of the BRS operator on which the Seiberg-Witten construction for the Lagrangian is based with the vierbein postulate (5.1) and the torsion-free condition (5.2). This amounts to proving that $s\bar{\Gamma}^b_{cd} = 0$ and that $\bar{\Omega}_a$ in (B3) transforms as in (4.10). From Eqs. (4.2) and (4.3) and the transformation law $s\bar{F}^a = \mathrm{i}\bar{\lambda}\bar{F}^a$, it follows that

$$s\bar{E}^a = -\mathrm{i}\bar{E}_a\bar{\lambda}, \qquad s\bar{\mathcal{E}}_a = \mathrm{i}\bar{\lambda}\bar{\mathcal{E}}_a, \qquad s\bar{\mathcal{F}}_a = -\mathrm{i}\bar{\mathcal{F}}_a\bar{\lambda}. \tag{B5}$$

Equations (B5), $\bar{g}_{ab} = \bar{\mathcal{F}}_a\bar{\mathcal{E}}_b$ and $\bar{g}^{ab} = \bar{E}^a\bar{F}^b$, imply $s\bar{g}_{ab} = s(\partial_c\bar{g}_{ab}) = 0$ and $s\bar{g}^{ab} = s(\bar{\partial}_c\bar{g}^{ab}) = 0$. Upon substitution in (B4) one has $s\bar{\Gamma}^b_{cd} = 0$. Analogously, acting with s on Eq. (B3), employing (B5), and simplifying with the help of (4.2) and (4.3), we obtain after some simple algebra that $s\bar{\Omega}_a = \bar{\partial}_a\bar{\lambda} - \mathrm{i}[\bar{\Omega}_a, \bar{\lambda}]$, in agreement with (4.10).

APPENDIX C: DERIVATION OF EQ. (6.11)

Equation (C1) is obtained by using the construction explained in Secs. IV and V to the Seiberg-Witten maps $\hat{\Omega}'_a$ and $\hat{F}'^a$ in Eqs. (6.10). In this appendix we present some partial results of this computation. Equation (4.28) gives for the first-order contribution

$$\mathcal{L}'^{(1)} = \mathcal{L}^{(1)} + \delta\mathcal{L}^{(1)},$$

where $\mathcal{L}^{(1)}$ is as in Eq. (4.29) and $\delta\mathcal{L}^{(1)}$ reads

$$\delta\mathcal{L}^{(1)} = \frac{e}{2\kappa^2}\theta^{mn}[(c-p)E^aR_{ab}R^b_{mnc}F^c + qR_{mn}(R_{ab}E^aF^b - \Lambda)].$$

The coefficients c, p, and q are those in $\delta_1\hat{\Omega}_a$ and $\delta_1\hat{F}^a$. From Sec. V we know that $\mathcal{L}^{(1)}_{\rm cov} = 0$. For the contribution $\delta\mathcal{L}^{(1)}$, steps 1 and 2 in Subsection VA yield $\delta\mathcal{L}^{(1)}_{\rm cov} = 0$. Hence $\mathcal{L}'^{(1)}_{\rm cov} = 0$.

The second-order contribution can also be written as

$$\tilde{\mathcal{L}}^{(2)} = \mathcal{L}^{(2)} + \delta\mathcal{L}^{(2)},$$

where $\mathcal{L}^{(2)}$ is as in Eq. (4.32) and $\delta\mathcal{L}^{(2)}$ has a very complicated expression. Here we only display its result after going through the "covariantization procedure" of steps 1 and 2 in Sec. V:

$$\begin{aligned}\delta\mathcal{L}^{(2)}_{\rm cov} = \frac{\sqrt{-g}}{16\kappa^2}[&m_6\mathcal{R}_6 + m_7\mathcal{R}_7 + m_8\mathcal{R}_8 + m_9\mathcal{R}_9 \\ &+ m_{10}\mathcal{R}_{10} + m_{11}\mathcal{R}_{11} \\ &+ (R-\Lambda)(-n_1\mathcal{Q}_1 + n_2\mathcal{Q}_2 + n_3\mathcal{Q}_3 - q^2\mathcal{Q}_4) \\ &+ 2(2c_1 + c^2)(4\mathcal{B}_3 - \mathcal{B}_4) + 4c_2(4\mathcal{B}_5 - \mathcal{B}_6) \\ &+ 2c_3(2\mathcal{B}_7 + \mathcal{B}_8)],\end{aligned} \tag{C1}$$

The coefficients $m_6, \cdots, m_{11}$ and n_1, n_2, n_3 are given in terms of those in $\delta_1\hat{\phi}$ and $\delta_2\hat{\phi}$ by

$$\begin{aligned} m_6 &= -2r - 2q^2 + 4q_2 - q_7, \\ m_7 &= (c-p)^2, \\ m_8 &= 4pq - 4q_1 + (q_5 - q_6), \\ m_9 &= -r + 2q_4 + \tfrac{1}{2}q_8, \\ m_{10} &= 2q_3, \\ m_{11} &= 4(pc - p^2 - c^2 + p_1 - p_3 + c_1) + 2(p_2 + c_2) \end{aligned}$$

and

$$\begin{aligned} n_1 &= \frac{r}{2} + 2p_2 + q_4, \\ n_2 &= p^2 - pq - 2p_1 + q_3, \\ n_3 &= \frac{r}{2} + q^2 + 2q_1 - q_2. \end{aligned}$$

With respect to $\mathcal{L}^{(2)}_{\rm cov}$, new invariants occur in $\delta\mathcal{L}^{(2)}_{\rm cov}$, namely

$$\mathcal{R}_{11} = \theta^{\mu\nu}\theta^{\rho\sigma}g^{\gamma\delta}R_{\alpha\beta}R_{\mu\nu\gamma}{}^{\alpha}R_{\rho\sigma\delta}{}^{\beta}$$

and

$$\begin{aligned} \mathcal{Q}_2 &= \theta^{\mu\nu}\theta^{\rho\sigma}R_{\mu\nu\alpha\beta}R_{\rho\sigma}{}^{\alpha\beta}, \\ \mathcal{Q}_3 &= \theta^{\mu\nu}\theta^{\rho\sigma}R_{\mu\nu\rho}{}^{\alpha}R_{\sigma\alpha}, \\ \mathcal{Q}_4 &= \theta^{\mu\nu}\theta^{\rho\sigma}R_{\mu\rho}R_{\nu\sigma}, \\ \mathcal{B}_3 &= \theta^{\mu\nu}\theta^{\rho\sigma}\nabla_\alpha(R_{\mu\nu}{}^{\alpha\beta}\nabla_\rho R_{\sigma\beta}), \\ \mathcal{B}_4 &= \theta^{\mu\nu}\theta^{\rho\sigma}\nabla_\alpha\nabla_\beta(R_{\mu\nu\gamma}{}^{\alpha}R_{\rho\sigma}{}^{\gamma\beta}), \\ \mathcal{B}_5 &= \theta^{\mu\nu}\theta^{\rho\sigma}\nabla_\alpha(R_{\mu\rho}{}^{\alpha\beta}\nabla_\nu R_{\sigma\beta}), \\ \mathcal{B}_6 &= \theta^{\mu\nu}\theta^{\rho\sigma}\nabla_\alpha\nabla_\beta(R_{\mu\rho\gamma}{}^{\alpha}R_{\nu\sigma}{}^{\gamma\beta}), \\ \mathcal{B}_7 &= \theta^{\mu\nu}\theta^{\rho\sigma}\nabla_\alpha\nabla_\mu(R_{\rho\sigma}{}^{\alpha\gamma}R_{\nu\gamma}), \\ \mathcal{B}_8 &= \theta^{\mu\nu}\theta^{\rho\sigma}\nabla_\alpha\nabla_\mu(R_{\rho\sigma}{}^{\gamma\delta}R_{\gamma\delta\nu}{}^{\alpha}). \end{aligned}$$

We only have to compute these invariants for pp-wave metrics and (2 + 2)-decomposable metrics. For pp-wave metrics, they vanish identically, in agreement with the discussion of Sec. II. For (2 + 2)-decomposable metrics, they become

$$\begin{aligned} &\mathcal{R}_{11} = -8I_1, \qquad \mathcal{Q}_2 = 2\mathcal{Q}_3 = 4\mathcal{Q}_4 = -8I_5, \\ &4\mathcal{B}_3 = \mathcal{B}_4 = 8\mathcal{B}_5 = 2\mathcal{B}_6 = 2\mathcal{B}_7 = -\mathcal{B}_8 = -4J_1, \end{aligned}$$

with I_1, I_5 ,and J_1 as in Eqs. (2.2), (2.6), and (2.13). Substituting in (C1) and summing the contribution (5.14) from $\mathcal{L}^{(2)}$, we reproduce Eq. (2.18), with the coefficients a and b given by

$$a = \tfrac{1}{2} - \tfrac{1}{2}m_6 - 4m_7 - m_8 + m_9 + 2m_{10} - 2m_{11}, \tag{C2}$$

$$b = \tfrac{1}{2} - 4n_1 + 8n_2 + 4n_3 - q^2. \tag{C3}$$

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