General structure of the solutions of the Hamiltonian constraints of gravity

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Abstract

A general framework for the solutions of the constraints of pure gravity is constructed. It provides with well defined mathematical criteria to classify their solutions in four classes. Complete families of solutions are obtained in some cases. A starting point for the systematic study of the solutions of Einstein gravity is suggested.

1 Introduction

This paper is devoted to the search of a general framework for the solutions of the Hamiltonian constraints of gravity. The canonical description of General Relativity is an old problem , which goes on acquiring a renewed and increasing interest as the starting point for the establishment of a quantum theory of gravity [1, 2, 3], which is probably one of the most ambitious programs of modern physics. It is an attempt whose final success seems to be still far away.

The idea to extend the methods of quantum field theories to the case of gravity (perturbative approach) came up against a number of difficulties, the most evident one being the non renormalizability, because of which the number of papers on this topic has diminished in the last years. This is not

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the case with the non perturbative approach which, on the contrary, maintain a considerable appeal. A direct application of the Gauss–Codazzi theorem, allows to split the four dimensional spacetime into a 3+1 space and time (ADM formalism [4, 5, 6, 7], [8, 9]), which provides us with a very useful tool to understand General Relativity in terms of our direct experience of space and time, *i. e.* three dimensional slices evolving with a parametric time variable. The final result is a canonical theory with a Hamiltonian consisting in a linear combination of first class constraints. Then the logical starting point is to characterize the manifold where the dynamics is defined. This is, precisely, the object of this paper. From the work of these pioneers, this problem lives a revival with the introduction of gauge theories of gravity [10, 11, 12, 13], which are on the basis of the most recent proposals by Ashtekar and coworkers. They reach a canonical formalism of gravity, close to standard gauge theories, where the fundamental dynamical variables are given by the canonically conjugate pair (E_i^a, A_{ia}). Here E_i^a are the SO(3) densitized triads, and A_{ia} the corresponding gauge connections. In the following, latin letters a, b, ... of the beginning of the alphabet are assigned to the coordinates defined in the three dimensional slices resulting form a suitable foliation of the space-time, while i, j, ... of the middle of the alphabet are internal SO(3) indices (allways in covariant position) running from 1 to 3, as well as a, b, ...

There have been some attempts to formally solve the constraints [14, 15, 16, **?**], thoroughly discussed in a previous reference [17], in which the close relationship existent between the Ashtekar structure of the constraints and the usual ADM one has been emphasized. Using a suitable choice of the dynamical degrees of freedom, we get, in this work, a general form of the solutions which allows us to classify them in terms of simple and general mathematical conditions. As we will see, these imply restrictions on the three–dimensional metric.

With the purpose to offer a complete and unified version of our approach, we include, in a first section, a brief review of reference [17], adding besides the analysis of some important properties which reveal themselves useful to construct a general framework.

2 Hamiltonian constraints of General Relativity

The Hamiltonian constraints of General relativity, as derived in the literature [20, 21, 22], among others, are the weakly vanishing expressions

$$\nabla_a E_i^a \approx 0\,, \tag{1}$$
$$E_i^b F_{iab} \approx 0\,, \tag{2}$$
$$\epsilon_{ijk} E_j^a E_k^b F_{iab} \approx 0\,, \tag{3}$$

describing $SO(3)$ Gauss law, vector and scalar constraints, respectively. For simplicity, we have chosen here the Barbero–Immirzi parameter $\beta = i$, as long as at the classical level the Einstein's field equations have the same dynamical contents for any value of β. $E_i^a = e e_i^a$ are the densitized triads, $F_{iab} = 2\partial_{[a}A_{|i|b]} + \epsilon_{ijk}A_{ja}A_{kb}$ is the $SO(3)$ field strength tensor, and $\nabla_a E_i^a = \partial_a E_i^a + \epsilon_{ijk}A_{ja}E_k^a$ being the local $S\mathrm{O}(3)$ covariant derivative. Being A_{ia} complex, additional reality conditions are required.

Paying attention to the connections, which are the fundamental dynamical variables, we have considered in the previous paper [17] an approach which offers an alternative writing of the constraints (1)–(3) closer to the ordinary geometrical ADM language. To do that, we redefine the $SO(3)$ connections as

$$A_{ia} = \Gamma_{ia} + i k_{ia}\,,$$

where Γ_{ia} is the part of the connection compatible with the metric and k_{ia} is its intrinsic part, which plays the role of the extrinsic curvature of the three dimensional slices.

With these elements and after some algebra (the reader is referred to [17, 19, 23, 24] for the details), one gets, in coordinate language, the three equivalent expressions for the constraints (1)–(3)

$$k_{[ab]} \approx 0, \qquad \text{Gauss law} \tag{4}$$
$$D_a(k_b^a - \delta_b^a \operatorname{Tr} k) \approx 0, \qquad \text{Vector constraint} \tag{5}$$
$$R^{(3)} - \operatorname{Tr}(k^2) + (\operatorname{Tr} k)^2 \approx 0, \qquad \text{Scalar constraint}\,, \tag{6}$$

where $R^{(3)}$ is the scalar curvature defined in the three dimensional space and the covariant derivatives are the ordinary Christoffel ones.

It must be emphasized that (1)–(3) and (4)–(6) are different versions with the same dynamical contents. Three gauges [$SO(3)$, coordinates and reparametrizations] can be arbitrarily chosen to get the final result. The problem simplifies drastically after fixing the reparametrization gauge by imposing the Dirac's maximal slicing gauge [?] $\mathrm{Tr}(k) = 0$ (other choices are possible as we are going to discuss in the last section), so that it reduces to solve

$$D_a k^a_b \approx 0\,, \tag{7}$$
$$R^{(3)} - \mathrm{Tr}(k^2) \approx 0\,, \tag{8}$$

where $k_{ab}$is a traceless symmetric matrix.

It is very easy to verify that the vector constraint (7) transforms covariantly in this gauge with respect to the rescaling of g_{ab} and k^a_b, namely

$$g_{ab} \rightarrow \varphi \tilde{g}_{ab},$$
$$k^a_b \rightarrow \varphi^{-3/2} \tilde{k}^a_b$$

what implies

$$D_a k^a_b = \varphi^{-3/2} \tilde{D}_a \tilde{k}^a_b.$$

Furthermore, the vector constraint always admits an identical solution depending only on the metric tensor. In fact, the Cotton–York tensor C^a_b, defined as

$$C^{ab} = \eta^{acd} D_c \left[R^b_d - \frac{1}{4}\,\delta^b_d R \right], \tag{9}$$

is symmetric, traceless and identically conserved $D_a C^a_b \equiv 0$, as required by the vector constraint (7).

Consequently, we can establish the general form of k^a_b as the sum of two terms

$$k^a_b = \bar{k}^a_b + \alpha C^a_b, \tag{10}$$

where $\bar{k}^a_b$ stands for any solution of (7) and α is an arbitrary constant.

This can be easily understood, as far as one looks primarily for solutions for k^a_b in eq. (7), which are functionals of an arbitrary metric, they must be expressible in terms of the metric tensor and its derivatives up to third order. The reason is that the first derivatives of the invariant $\mathrm{Tr}(k^2)$, implicitly present in (7), leads through the scalar constraint to a derived scalar curvature $R^{(3)}$, which includes up to second order derivatives of the metric.

On the other hand, the cancellation of the Cotton–York tensor being the necessary and sufficient condition for a manifold to be conformally flat, the presence of C^a_b allows us to distinguish, from the very beginning, conformally flat metrics as a very outstanding case, as we will see in the following.

3 The choice of the dynamical variables

A non negligible part of the problems arising in gravity theories is related to the search of a suitable set of dynamical variables. In this section, we consider a parametrization of k^a_b that reveals itself specially useful to deal with the analysis of the possible solutions. For this purpose, we recover the triads formalism to write down k^a_b as

$$k^a_b = e^a_i k_{ij} e_{jb} \,,$$

where k_{ij} is an ordinary traceless, symmetric matrix. In this way the usual $SO(3)$ symmetry of the triads formalism is recovered. Being symmetric, it can be diagonalized with the help of an orthogonal matrix, what enables us to deal with its spectral representation.

$$k_{ij} = u_i \rho_1 u_j + v_i \rho_2 v_j + w_i \rho_3 w_j \,,$$

where ρ_i are the three eigenvalues verifying $\rho_1 + \rho_2 + \rho_3 = 0$ (traceless condition) and u, v and w are the corresponding eigenvectors. In this way, the five degrees of freedom of k^a_b arrange as two scalar eigenvalues plus the three independent parameters associated with the eigenvectors, which are isomorphic to an $SO(3)$ transformation. The last property will be important in what follows. With these assumptions, the final form of k^a_b reads

$$k^a_b = \hat{e}^a_1 \, \rho_1 \, \hat{e}_{1\,b} + \hat{e}^a_2 \, \rho_2 \, \hat{e}_{2\,b} + \hat{e}^a_3 \, \rho_3 \, \hat{e}_{3\,b}, \tag{11}$$

where $\hat{e}^a_1 = e^a_i u_i$, $\hat{e}^a_2 = e^a_i v_i$ and $\hat{e}^a_3 = e^a_i w_i$.

We impose now reality conditions on the eigenvalues, which become constrained to the real roots of the cubic canonical equation. This restricts the discriminant to be zero or negative. A very convenient parametrization for both cases is the following

$$\begin{aligned} \rho_1 &= \lambda \cos\omega, \\ \rho_2 &= -\frac{\lambda}{2}(\cos\omega - \sqrt{3}\,\sin\omega) \\ \rho_3 &= -\frac{\lambda}{2}(\cos\omega + \sqrt{3}\,\sin\omega) \end{aligned} \tag{12}$$

where $\sin\omega = 0$ and $\sin\omega \neq 0$ correspond to null and negative discriminant, respectively. Now, we have the essential elements to attempt a classification of the possible solutions, paying mainly attention to the vector constraint. It is worth to emphasize that as a consequence of such a simple parametrization as (12) we have formally solved from the very beginning the scalar constraint. In fact, a straightforward calculation shows that $\mathrm{Tr}(k^2)$, which is independent of ω has the value $\mathrm{Tr}(k^2) = 3\lambda^2/2$. Therefore, taking into account the scalar constraint, we have

$$\frac{3}{2}\,\lambda^2 = R^{(3)}\,. \tag{13}$$

4 Classifying the solutions

Since there are many and very different solutions of the constraints, a way must be found to classify them in a general scheme. For this purpose, we rewrite (11) in a more compact notation as

$$k^a_b = \rho_{ij}\hat{e}^a_i\hat{e}_{jb}\,, \tag{14}$$

where ρ_{ij} is the diagonal matrix of the eigenvalues. The vector constraint becomes then

$$D_a(k^a_b) = D_a(\rho_{ij}\hat{e}^a_i\hat{e}_{jb}) = \hat{e}^a_i\hat{e}_{jb}\partial_a\rho_{ij} + \rho_{ij}\hat{e}^a_i D_a\hat{e}_{jb} + \rho_{ij}\hat{e}_{jb}D_a\hat{e}^a_i = 0\,,$$

which, after multiplication by $\hat{e}^b_k$ gives

$$\hat{e}^a_i\partial_a\rho_{ik} - \rho_{ij}\hat{\gamma}_{ikj} + \rho_{ik}\hat{\gamma}_{jij} = 0\,, \tag{15}$$

where the symbols of anholonomy $\hat{\gamma}_{ijk}$ are given as

$$\hat{\gamma}_{ijk} \equiv -\hat{e}^b_j\hat{e}^a_k D_a\hat{e}_{ib} = \hat{e}_{ib}\hat{e}^a_k D_a\hat{e}^b_j$$

and verify

$$\hat{\gamma}_{ijk} + \hat{\gamma}_{jik} = 0, \qquad \text{and} \qquad \hat{\gamma}_{kjk} = D_a\hat{e}^a_j\,.$$

Note that the covariant derivative are here the Christoffel ones acting on the coordinate indices.

It is convenient for the later discussion to develop eq. (15), which leads to

$$\hat{e}^a_1\partial_a\rho_1 + \hat{\gamma}_{212}(\rho_1 - \rho_2) + \hat{\gamma}_{313}(\rho_1 - \rho_3) = 0\,, \tag{16}$$
$$\hat{e}^a_2\partial_a\rho_2 + \hat{\gamma}_{121}(\rho_2 - \rho_1) + \hat{\gamma}_{323}(\rho_2 - \rho_3) = 0\,, \tag{17}$$
$$\hat{e}^a_3\partial_a\rho_3 + \hat{\gamma}_{131}(\rho_3 - \rho_1) + \hat{\gamma}_{232}(\rho_3 - \rho_2) = 0\,. \tag{18}$$

Notice that the triads $\hat{e}_i^a$ are defined in terms of the matrix elements of k_b^a. This implies that the $S0(3)$ gauge has been fixed in (16)–(18) by imposing $\hat{e}_1^a$, $\hat{e}_2^a$, $\hat{e}_3^a$ to be the eigenvectors of k_b^a with respect to the metric tensor.

The value of the dynamical variable ω in the parametrization (12) provides us with a well defined mathematical criterion to classify all the real solutions. As will be seen, there are four different classes.

- **Class A.** It corresponds to $\sin\omega = 0$. This is precisely the case where the discriminant of the secular equation cancels. Starting from (16)-(18) a bit of algebra leads to the following system.

$$\hat{e}_1^a \partial_a \lambda + \frac{3}{2}\lambda(\hat{\gamma}_{212} + \hat{\gamma}_{313}) = 0\,, \tag{19}$$

$$\hat{e}_2^a \partial_a \lambda + 3\lambda\hat{\gamma}_{121} = 0\,, \tag{20}$$

$$\hat{e}_3^a \partial_a \lambda + 3\lambda\hat{\gamma}_{131} = 0\,. \tag{21}$$

  We recall that $\sin\omega = 0$ is not the only value leading to the last equations, in fact taking $\sin\omega = \pm\sqrt{3}/2$ one recovers (19)–(21) simply by redefining the eingenvalues. So that a general definition of this class can be formulated by imposing $(\mathrm{Tr}\, k^3)^2 = (\mathrm{Tr}\, k^2)^3/6\,$.

- **Class B.** The second class corresponds to $\cos\omega = 0$. In this case k_b^a is a singular matrix (*i. e.* $\det(k) = 0$), verifying the equations

$$\lambda(\hat{\gamma}_{212} - \hat{\gamma}_{313}) \;=\; 0\,, \tag{22}$$

$$\hat{e}_2^a \partial_a \lambda + \lambda(\hat{\gamma}_{121} + 2\hat{\gamma}_{323}) \;=\; 0\,, \tag{23}$$

$$\hat{e}_3^a \partial_a \lambda + \lambda(\hat{\gamma}_{131} + 2\hat{\gamma}_{232}) \;=\; 0\,. \tag{24}$$

  As in the previous case one remains in the same class when $\cos\omega = \pm\sqrt{3}/2\,$, leading to the general definition of this class by means the condition $\mathrm{Tr}\, k^3 = 0$.

- **Class C.** The third class is the general case, in which the characteristic equations can be written as

$$\hat{e}_1^a \partial_a \mu + \frac{\mu}{2}(3-\delta)\hat{\gamma}_{212} + \frac{\mu}{2}(3+\delta)\hat{\gamma}_{313} \;=\; 0\,, \tag{25}$$

$$\hat{e}_2^a \partial_a [\mu(1-\delta)] + \mu(3-\delta)\hat{\gamma}_{121} - 2\mu\delta\hat{\gamma}_{323} \;=\; 0\,, \tag{26}$$

$$\hat{e}_3^a \partial_a [\mu(1+\delta)] + \mu(3+\delta)\hat{\gamma}_{131} + 2\mu\delta\hat{\gamma}_{232} \;=\; 0\,, \tag{27}$$

  with $\delta = \sqrt{3}\,\tan\omega$ and $\mu = \lambda\cos\omega$.

- **Class D.** Finally a fourth class occurs when we take directly $\bar{k}^a_b = 0$ in (11). Therefore, all the solutions in this class depend only on the metric tensor. In this case, the distinction between spaces that are conformally flat and those which are not acquires a specially relevant role, which will be analyzed in the following.

5 The choice of the coordinates

Once characterized the different classes of solutions, we can use 3-diff invariance to choose coordinates. According to a Gauss theorem, this can be done by fixing the values of the elements of the metric tensor to render it diagonal. Although other choices are clearly possible, some of the known relevant solutions can be written in diagonal form, which on the other hand greatly simplifies the calculations. We start, therefore, by writing the metric tensor in the form

$$g_{ab} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} . \tag{28}$$

We introduce now a "natural" triad associated with this form and constructed with the three vectors $e_{1a} = (\sqrt{a}, 0, 0)$, $e_{2a} = (0, \sqrt{b}, 0)$ and $e_{3a} = (0, 0, \sqrt{c})$, which are an orthonormal basis with respect to g_{ab}. Any other basis can differ from this one only by the application of an orthogonal matrix O_{ij} , so that we can write

$$\hat{e}_{ia} = O_{ij}\, e_{ja} \,, \tag{29}$$

which expresses the relation between the eigenvectors of k^a_b and our "natural" triad e_{ia}. Equation (29) allows thus to parametrize all the arbitrariness of k^a_b simply in terms of a rotation matrix. Inserting (29) in (19)-(27), it is easy to find the final form of the vector constraint in each class. In this way, once a metric is adopted, the presence of O_{ij} describes all the generality of the theory.

As a first simple approach, let us consider the case $O_{ij} = \delta_{ij}$. A bit of algebra suffices then to find the expression of the vector constraint in the four classes. In the case of the class A, it adopts the very simple integrable form

$$\begin{aligned}
\partial_1 \log\left[\lambda (bc)^{3/4}\right] &= 0\,, && (30)\\
\partial_2 \log\left[\lambda a^{3/2}\right] &= 0\,, && (31)\\
\partial_3 \log\left[\lambda a^{3/2}\right] &= 0\,. && (32)
\end{aligned}$$

As is seen, this implies some restrictions on the fundamental structure of the matrix elements of the metric tensor, $\lambda a^{3/2}$ being a function of only the first coordinate and $\lambda (bc)^{3/4} = f(x_2, x_3)$, f being an arbitrary function of two variables.

The equations in class B can be expressed also in directly integrable form, namely

$$\begin{aligned}
\partial_1 \log\left[\sqrt{c/b}\right] &= 0\,, && (33)\\
\partial_2 \log\left[\lambda\, c\sqrt{a}\right] &= 0\,, && (34)\\
\partial_3 \log\left[\lambda\, b\sqrt{a}\right] &= 0\,, && (35)
\end{aligned}$$

so that the quotient c/b is independent of the first coordinate while $\lambda\, c\sqrt{a}$ and $\lambda\, b\sqrt{a}$ do not depend on the second and third coordinates, respectively. It must be stressed that that, in both cases A and B, these conditions are a condensed manner to express families of solutions. Notice, besides, that since $\lambda = \sqrt{2/3}\, R^{(3)}$ in Dirac's gauge, they involve only the elements of the metric tensor.

The study of the third class C leads us to the more complex system

$$\begin{aligned}
\partial_1 \log\left[\mu (bc)^{3/4}\right] - \delta\, \partial_1 \log\left[(b/c)^{1/4}\right] &= 0, && (36)\\
\partial_2 \log\left[\mu (1-\delta) a^{3/2}\right] + \frac{\delta}{1-\delta}\, \partial_2 \log\left[a/c\right] &= 0, && (37)\\
\partial_3 \log\left[\mu (1+\delta) a^{3/2}\right] - \frac{\delta}{1+\delta}\, \partial_3 \log\left[a/b\right] &= 0. && (38)
\end{aligned}$$

It is clear that this system is, by no means, so easy to solve as are the previous ones. Nevertheless, assuming in general that $b \neq c$, δ can be obtained from the first equation so that, after substitution in the other two, two conditions can be deduced which involve the elements of the metric tensor and their first derivatives and define families of solutions. We recall that this results correspond to the most simple choice $O_{ij} = \delta_{ij}$, so that this procedure puts in evidence the richness of solutions of the problem.

6 The search for the solutions

In order to find the solutions, we must handle a problem parametrized in terms of eight degrees of freedom. Two of them are the eigenvalues of k^a_b although, as emphasized before, one is formally found by the condition $R^{(3)} = 3\lambda^2/2$. After fixing the gauge, we have three more which correspond to the independent elements of the metric tensor. Finally, the remaining three are parametrized by the defining elements of an orthogonal matrix, the Eulerian angles for instance. We will obtain in such a way the four degrees of freedom of pure gravity by solving the scalar and the vector constraints.

From the mathematical point of view, different possibilities are open. Nevertheless, it seems convenient to remain close to our experience of the world, so that the best approach is probably to use geometry as a primary input. We will consider, therefore, in this section several interesting cases in order to test our treatment in geometrical language.

No doubt, isotropic spaces are obvious candidates for that. Moreover, as will be seen later, they allow us to propose an interesting slight modification of our approach. An isotropic space is described by a diagonal metric tensor that can be written in spherical coordinates as

$$g_{11} = 1/f(r)\,, \quad g_{22} = r^2\,, \quad g_{33} = r^2 \sin^2\theta\,,$$

$f(r)$ being a function depending only on the radial coordinate r.

It is easy to see that the scalar curvature adopts the form

$$R^{(3)} = \frac{2}{r^2}\,\partial_r\{r[f(r)-1]\}\,. \tag{39}$$

Thanks to our parametrization, we can thus write immediately the scalar constraint as

$$\frac{3}{2}\,\lambda^2 = \frac{2}{r^2}\,\partial_r\{r[f(r)-1]\}\,, \tag{40}$$

which expresses a differential relation between f and λ. At the same time, this defines λ as a function depending only on r, a property that highly simplifies the calculation, as will be seen. As in last section, we may start by choosing the matrix $O_{ij} = \delta_{ij}$, obtaining in this way simple relations enabling us to construct solutions of Einstein's equations. For isotropic spaces, a simple calculation shows that the vector constraint can be written as the system

$$
\begin{aligned}
\partial_r(r^3\lambda\cos\omega) &= 0\,, \quad (a)\\
\lambda\left[\partial_\theta\cos\omega - \frac{\sqrt{3}}{\sin^2\theta}\,\partial_\theta(\sin\omega\sin^2\theta)\right] &= 0\,, \quad (b)\\
\lambda\,\partial_\varphi(\cos\omega+\sqrt{3}\sin\omega) &= 0\,, \quad (c)
\end{aligned}
\tag{41}
$$

In the case of $\sin\omega = 0$ (class A), (41a) reduces to

$$
\partial(r^3\lambda) = 0\,, \quad \text{so that} \quad \lambda = 2k_0/r^3.
$$

From this and the scalar constraint, it follows

$$
\partial_r\left\{r[f(r)-1]+\frac{k_0^2}{r}\right\} = 0\,,
$$

what leads to the solution

$$
f = 1 - \frac{2m}{r} - \frac{k_0^2}{r^4}\,.
$$

It is straightforward to check that the conditions (30)–(32) are clearly satisfied.

The class D occurs when $\lambda = 0$. Since the Cotton–York tensor C^a_b cancels for isotropic metrics, only the scalar constraint remains, and it is

$$
R^{(3)} = \frac{2}{r^2}\,\partial_r\{r[f(r)-1]\} = 0,
$$

what gives the Schwarzshild solution

$$
f = 1 - \frac{2m}{r}.
$$

This function describes a static situation since $k^a_b = 0$ in this case.

In the class C, both $\sin\omega$ and $\cos\omega$ are different from zero, so that we write (41a) and (41b) in the form

$$
\partial_r\,\omega = \cot\omega\,\partial_r\,\log(r^3\lambda), \qquad \partial_\theta\,\omega = \frac{-2\sqrt{3}\cot\theta}{1+\sqrt{3}\cot\omega}.
$$

The integrability condition $\partial_\theta\partial_r\omega = \partial_r\partial_\theta\omega = 0$ leads to $\tan\omega = -2\sqrt{3}$, from which some algebra shows that it appears, curiously, an obstruction to the integrability as the condition $\cos\theta = 0$ on the coordinates. The same happens in class B.

This obstruction $\cos\theta = 0$ in classes B and C requires a comment. One can verify that this is not a property related with isotropy, but rather a consequence of the choice of the matrix O_{ij} as a Kronecker δ_{ij}, what puts in evidence the relevance of the role of the matrix O_{ij} in the construction of the different solutions. In fact, there are surely families of solutions corresponding to other choices of the matrix. To understand this, one can take O_{ij} as a rotation with Eulerian angles $(\bar\phi,\, \bar\theta,\, \bar\psi)$ such that $\cos\bar\theta = 0$, $\sin\bar\psi = \cos\bar\psi = 1/\sqrt{2}$ and arbitrary value of the azimuthal angle, i. e,

$$O = \begin{pmatrix} \sin\bar\phi & -\cos\bar\phi & 0 \\ \cos\bar\phi/\sqrt{2} & \sin\bar\phi/\sqrt{2} & 1/\sqrt{2} \\ -\cos\bar\phi/\sqrt{2} & -\sin\bar\phi/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}. \tag{42}$$

The result, not detailed here for simplicity, is that the Eulerian angle $\bar\phi$ is independent on the azimuthal coordinate φ, what does not pose any restriction on the coordinates.

Isotropic spaces have a property suitable to be used in a slightly different approach, due to the fact that the Ricci tensor becomes itself diagonal in these spaces. This suggests to parametrize the solution of the vector constraint in the alternative form

$$k^a_b = \eta^{acd} D_c M_{db},$$

where the tensor M_{db} must be symmetric to account for the property of k^a_b of being traceless. Moreover, it is easy to show that M_{db} and the Ricci tensor R_{ab} commute, otherwise k^a_b would not be divergenceless. Finally, the symmetry of k^{ab} requires that

$$D_a[M^a_b - \delta^a_b \,\mathrm{Tr}(M)] = 0,$$

as is easy to see, just cancelling its antisymmetric part. From this, it is very easy to obtain again the arbitrariness related to the presence of the Cotton–York tensor. In fact, if we take simply

$$M^a_b - \delta^a_b \mathrm{Tr}(M) = G^a_b,$$

G^a_b being the Einstein's tensor, it turns out that

$$M^a_b = R^a_b - \frac{1}{4}\,\delta^a_b R$$

reproduces an identical solution. In this way one can work with a symmetric matrix M_{ab} that can be simultaneously diagonalized with the Ricci tensor, a property very useful in some cases.

To end this section, we include a brief example of the inverse problem, starting from a well known solution and reconstructing from it the different elements of the method. The example is a stationary four dimensional metric with axial symmetry and Papapetrou's structure, given as

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & 0 & 0 & g_{03} \\ 0 & g_{11} = a & 0 & 0 \\ 0 & 0 & g_{22} = b & 0 \\ g_{03} & 0 & 0 & g_{33} = c \end{pmatrix},$$

the components of which are functions depending only on r and θ. The extrinsic curvature is in this case

$$k_{ab} = \frac{1}{\sqrt{-g_{00}}} \Gamma^0_{ab} = \begin{pmatrix} 0 & 0 & \alpha_1 \\ 0 & 0 & \alpha_2 \\ \alpha_1 & \alpha_2 & 0 \end{pmatrix}.$$

Solving now the eigenvalues equation with respect to the three-dimensional restriction of the metric

$$k_{ab} \hat{e}^b_i = \rho_i g_{ab} \hat{e}^b_i \,,$$

we obtain the eigenvalues

$$\rho_1 = 0, \qquad \rho_2 = -\rho_3 = \frac{\rho}{\sqrt{abc}} = \frac{\sqrt{b\alpha_1^2 + a\alpha_2^2}}{\sqrt{abc}}\,.$$

The corresponding eigenvectors can be easily deduced. They are

$$\begin{aligned} \hat{e}^a_1 &= \left(\frac{\alpha_2}{\rho}, -\frac{\alpha_1}{\rho}, 0 \right), \\ \hat{e}^a_2 &= \left(\sqrt{\frac{b}{2a}}\,\frac{\alpha_1}{\rho}, \sqrt{\frac{a}{2b}}\,\frac{\alpha_2}{\rho}, \frac{1}{\sqrt{2c}} \right), \quad \hat{e}^a_3 = \left(-\sqrt{\frac{b}{2a}}\,\frac{\alpha_1}{\rho}, -\sqrt{\frac{a}{2b}}\,\frac{\alpha_2}{\rho}, \frac{1}{\sqrt{2c}} \right). \end{aligned}$$

The matrix O_{ij} relating them with the "natural" triad $\tilde{e}^a_1 = (1/\sqrt{a}, 0, 0)$,

$\tilde{e}_2^a = (0, 1/\sqrt{b}, 0)$, $\tilde{e}_3^a = (0, 0, 1/\sqrt{c})$, is

$$O = \begin{pmatrix} \frac{\sqrt{a}\,\alpha_2}{\rho} & -\frac{\sqrt{b}\,\alpha_1}{\rho} & 0 \\ \sqrt{\frac{b}{2}}\,\frac{\alpha_1}{\rho} & \sqrt{\frac{a}{2}}\,\frac{\alpha_2}{\rho} & \frac{1}{\sqrt{2}} \\ -\sqrt{\frac{b}{2}}\,\frac{\alpha_1}{\rho} & -\sqrt{\frac{a}{2}}\,\frac{\alpha_2}{\rho} & \frac{1}{\sqrt{2}} \end{pmatrix},$$

in which one recognizes eq. (42) if $\sin\bar{\phi} = \sqrt{a}\,\alpha_2/\rho$ and $\cos\bar{\phi} = \sqrt{b}\,\alpha_1/\rho$. It is therefore a typical B-class solution.

7 Gauge dependence and concluding remarks

The main result of this paper is the establishment of a general framework for the study of the solutions of Einstein's equations in the case of pure gravity, which gives a classification based on simple and well defined mathematical grounds. For the sake of simplicity, we have presented our results in Dirac's "maximal slicing gauge", in which the problem becomes particularly simple. Nevertheless, as is well known, the symmetries of the gravitational Hamiltonian are non internal. This implies that the different gauges maybe more or less adapted to the global geometrical properties, essentially open or closed spaces. As a consequence, a number of gauge conditions have been proposed (ADM, Dirac or York gauges for instance) [2] and [18].

It is not the aim of this paper to discuss the deep dynamical meaning of a non internal symmetry, we limit ourselves here to look for a parametrization of the problem, in terms of triads language, suitable for the study, from a purely mathematical point of view, of the characterization of the possible solutions.

At this purpose, we consider the constraints (4), (5) and (6) as a system of equations providing us the natural way to obtain the extrinsic curvature, using as input a given triad or metric tensor. This is the reason to adopt the basis in which K^a_b is diagonal, in such a way that the problem reduces to the calculation of the three extrinsic curvature eigenvalues.

If we maintain $\mathrm{Tr}K = \Delta$ different from zero, the parametrization (12) reads

$$\begin{aligned}
\rho_1 &= \lambda\cos\omega + \frac{1}{3}\,\Delta = \hat\rho_1 + \frac{1}{3}\,\Delta,\\
\rho_2 &= -\frac{\lambda}{2}\,(\cos\omega - \sqrt{3}\sin\omega) + \frac{1}{3}\,\Delta = \hat\rho_2 + \frac{1}{3}\,\Delta,\\
\rho_3 &= -\frac{\lambda}{2}\,(\cos\omega + \sqrt{3}\sin\omega) + \frac{1}{3}\,\Delta = \hat\rho_3 + \frac{1}{3}\Delta.
\end{aligned} \tag{43}$$

A brief calculation shows that the general form of the constraints (5) and (6) becomes

$$\begin{aligned}
\hat e_1^a\partial_a\hat\rho_1 + \hat\gamma_{212}(\hat\rho_1 - \hat\rho_2) + \hat\gamma_{313}(\hat\rho_1 - \hat\rho_3) &= \frac{2}{3}\,\hat e_1^a\,\partial_a\Delta,\\
\hat e_2^a\partial_a\hat\rho_2 + \hat\gamma_{121}(\hat\rho_2 - \hat\rho_1) + \hat\gamma_{323}(\hat\rho_2 - \hat\rho_3) &= \frac{2}{3}\,\hat e_2^a\,\partial_a\Delta,\\
\hat e_3^a\partial_a\hat\rho_3 + \hat\gamma_{131}(\hat\rho_3 - \hat\rho_1) + \hat\gamma_{232}(\hat\rho_3 - \hat\rho_2) &= \frac{2}{3}\,\hat e_3^a\,\partial_a\Delta,
\end{aligned} \tag{44}$$

and

$$R^{(3)} - \frac{3}{2}\,\lambda^2 + \frac{2}{3}\,\Delta^2 = 0. \tag{45}$$

We notice that, in this parametrization, the scalar constraint is independent on ω, thus it can be used to express for instance λ as a function of the three dimensional scalar curvature and the trace Δ. In this way, the problem reduces to discuss the vector constraint (44) as defined by a set of partial differential equations depending on two unknown functions, Δ and ω. Therefore, to solve them an additional condition (gauge) in Δ, ω or both is needed. We emphasize that, the eigenvalues being scalar quantities, a condition of this kind is invariant with respect to general coordinate transformations in the three-dimensional space.

The structure of the vector constraint strongly suggests to fix the gauge giving a condition on the trace Δ. In this manner, the vector constraint becomes an ordinary differential equation for ω. As a matter of fact, the most common choices occurs precisely when Δ is taken as a constant function, here of course "constant" means a function independent on the three dimensional space coordinates, a true constant for instance or a function depending on the time variable (Dirac or York proposals [25]). In both cases the second term in eq. (44) vanishes, so that we recover eqs. (16)-(18) and therefore our classification holds. It must be emphasized that, the scalar constraint being independent on ω, it is "formally" solved, once the value of Δ is fixed as long as λ is algebraically given in terms of Δ and the three dimensional scalar curvatures.

Therefore, it remains only the vector constraint as the natural criterion to classify the different solutions of the problem, a classification that, in our scheme, is simply given in terms of the possible roots of the cubic secular equation. A criterion on the other hand that remains valid in any possible gauge. In fact, it must be emphsized that eq. (44) can alternatively be used to directly characterize the different classes. Not only that but, due to the fact that A and B cases correspond to fixed values of ω, the problem is formally solved without any additional condition. However, Dirac's gauge is an useful frame to illustrate the richness of solutions as long as it describes, in a natural way, the relevant role of the Cotton-York tensor in the search for three dimensional geometrical configurations.

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