Thermodynamics of the Schwinger and Thirring models

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The thermodynamical partition functions for both the Schwinger and Thirring models are evaluated. The imaginary-time formalism of quantum field theory at finite temperature and path-integral methods are used. For the Schwinger model, the partition function displays two features: (i) no physical (transverse) photons exist in $1+1$ dimensions; (ii) the theory also describes just free massive bosons. For the Thirring model, the partition function equals that for free massless fermions. The complete thermodynamical propagators and the energies per unit volume at finite temperature are also given.

I. INTRODUCTION

Quantum field theory at finite temperature has become a useful tool to study plasmas,[1] metals,[2] condensed matter,[2] the variation of physical parameters (mass, magnetic moment) with temperature,[3] or effective potentials at nonzero temperature.[4,5] We can also think of using it to analyze the thermodynamical behavior of relativistic quantum systems.

There are three formulations proposed for a quantum field theory at finite temperature. On the one hand, there is the imaginary-time formalism (ITF) of Bernard[6] and Matsubara,[7] in which energy can take on only discrete values. On the other hand, there are real-time methods: that of Dolan and Jackiw[4] and "thermo field dynamics" (TFD), due to Takahashi and Umezawa,[8] with energy taking on arbitrary values in both of them.

To study thermodynamics, the ITF seems to be more advantageous since it gives an *a priori* definition for the partition function in terms of a functional integration over fields. The corresponding representation for the partition function in the frameworks of the real-time formulations would seem to require further elaboration. We shall limit ourselves to point out that some attempts to define the free energy in TFD have been carried out by Ojima.[9]

In this paper we use the ITF to study the thermodynamics of the Schwinger and Thirring models. We also obtain the finite-temperature complete propagators for both cases. The organization of this paper is as follows. In Sec. II we express the partition function Z for the Schwinger model in terms of a fermion closed-loop functional, which is exactly calculated. In Sec. III we get the electromagnetic (EM) and fermion complete propagators, and, in Sec. IV we explicitly evaluate Z and the thermodynamical energy per unit volume. Finally, in Sec. V, we generalize the work of Secs. II—IV for the Thirring model.

As we have already mentioned we work in Euclidean space-time, for which $x^0=ix^0_M$, $x^1=x^1_M$, $k^0=-ik^0_M$, $k^1=k^1_M$, $\gamma^0=\gamma^0_M$, $\gamma^1=\gamma^1_M$, and $\{\gamma^\mu,\gamma^\nu\}=2\delta^{\mu\nu}$, with M meaning Minkowskian. Notice that x^0 and k^0 are real, and that $kx=k^0x^0+k^1x^1$ and $k^2=(k^0)^2+(k^1)^2$.

II. THE SCHWINGER MODEL: THE FERMION CLOSED-LOOP FUNCTIONAL AT FINITE TEMPERATURE

The Schwinger model is the restriction of quantum electrodynamics with massless fermions to $1+1$ dimensions. The EM field does not have dynamical degrees of freedom and is created by the fermionic charges of the model. Following Bernard,[6] the partition function with external sources in Euclidean space-time is given, using standard notations, by

$$Z[J,\xi,\bar\xi]=NN_F(\beta)\det(-\partial^2)\int_{\substack{\text{periodic } A\\ \text{antiperiodic } \psi,\bar\psi}}\mathscr{D}A\,\mathscr{D}\bar\psi\,\mathscr{D}\psi \times\exp\left[\int_\beta d^2x\left[-\tfrac{1}{4}F^2-\frac{1}{2\alpha}(\partial A)^2-\bar\psi(\partial\!\!\!/+ieA\!\!\!/)\psi+JA+\bar\xi\psi+\bar\psi\xi\right]\right]. \tag{2.1}$$

N is a β-independent constant, and $N_F(\beta)$, the dependent one, is

$$\ln N_F(\beta)=-2(\ln\beta)L\sum_{n=-\infty}^{+\infty}\int\frac{dk^1}{2\pi}\,, \tag{2.2}$$

where L is the (infinite) measure of the linear dimension in which the system is contained. For a detailed evaluation of $N_F(\beta)$ see the Appendix. In Eq. (2.1) we have also used the notation

$$\int_\beta d^2x \equiv \int_0^\beta dx^0 \int dx^1 ,$$

$\beta^{-1}=k_BT$, k_B being Boltzmann's constant and T the temperature. On the other hand, $J=J(x)$, $\xi=\xi(x)$, $\bar{\xi}=\bar{\xi}(x)$ are bosonic and fermionic sources $[J(0,x^1)=J(\beta,x^1)$, $\xi(0,x^1)=-\xi(\beta,x^1)$, and $\bar{\xi}(0,x^1)=-\bar{\xi}(\beta,x^1)]$. Then, the thermodynamical partition function is given by

$$Z=Z[0,0,0] . \tag{2.3}$$

Performing the integrations over $\bar{\psi}$, ψ, and A in (2.1) we get[10]

$$Z=[J,\xi,\bar{\xi}]=NZ_0\exp\left[-ie\int_\beta d^2x\frac{\delta}{\delta\xi(x)}\gamma^\nu\frac{\delta}{\delta J^\nu(x)}\frac{\delta}{\delta\bar{\xi}(x)}\right]$$
$$\times\exp\left[\int_\beta d^2x\,d^2y[\tfrac{1}{2}J_\nu(x)D^{\nu\mu}(x-y)J_\mu(y)+\xi(x)S(x-y)\bar{\xi}(y)]\right] , \tag{2.4}$$

where $Z_0=Z_{\rm EM}Z_F$, $Z_{\rm EM}$ and Z_F being, respectively, the partition functions in 1 + 1 dimensions for the free EM field and free massless fermions. Since in 1 + 1 dimensions the free EM field has no dynamical character (there are no physical transverse photons), $Z_{\rm EM}$ is a β-independent quantity which can be absorbed into N. Hence, $Z_0=Z_F$. As shown in the Appendix,

$$\frac{1}{L}\ln Z_F=4\int_0^\infty\frac{dk}{2\pi}\left[\frac{\beta k}{2}+\ln[1+\exp(-\beta k)]\right] . \tag{2.5}$$

The constant $N_F(\beta)$ has been absorbed into Z_F. The $D^{\nu\mu}(x-y)$ and $S(x-y)$ of Eq. (2.4) are the free finite-temperature propagators for the EM and fermion fields:

$$D^{\nu\mu}(x-y)=-\frac{1}{\beta}\sum_{n=-\infty}^{+\infty}\int\frac{dk^1}{2\pi}\exp[-ik(x-y)]\frac{1}{k^2}\left[\delta^{\nu\mu}+(\alpha-1)\frac{k^\nu k^\mu}{k^2}\right],\quad k^0=\frac{2n\pi}{\beta} , \tag{2.6}$$

$$S(x-y)=\frac{1}{\beta}\sum_{n=-\infty}^{+\infty}\int\frac{dk^1}{2\pi}\exp[-ik(x-y)]\frac{1}{\not k},\quad k^0=\frac{(2n+1)\pi}{\beta} . \tag{2.7}$$

Using standard path-integral properties,[11] Eq. (2.4) can be written as

$$Z[J,\xi,\bar{\xi}]=NZ_F\exp\left[\tfrac{1}{2}\int_\beta d^2x\,d^2yJ_\nu(x)D^{\nu\mu}(x-y)J_\mu(y)\right]$$
$$\times\exp\left[-\tfrac{1}{2}\int_\beta d^2x\,d^2y\frac{\delta}{\delta A^\nu(x)}D^{\nu\mu}(x-y)\frac{\delta}{\delta A^\mu(y)}\right]\exp\left[\int_\beta d^2x\,d^2y\bar{\xi}(x)G(x,y\mid eA)\xi(y)\right] , \tag{2.8}$$

with $A^\nu(x)$ the external field

$$A^\nu(x)=-i\int_\beta d^2y\,D^{\nu\mu}(x-y)J_\mu(y) ,$$

$G(x,y\mid eA)$ is the solution of the differential equation

$$(\not\partial+ie\not A)G(x,y\mid eA)=-i\delta^{(2)}(x-y) , \tag{2.9}$$

and $L[A]$ is the already regularized fermion closed-loop functional

$$L[A]=i\,{\rm tr}\int_0^e de'\int_\beta d^2x\,\not A(x)\lim_{x\to y}G(x,y\mid e'A)\exp\left[ie'\int_y^x dz_\nu A^\nu(z)\right] , \tag{2.10}$$

where the limit must be taken symmetrically.

For Eq. (2.9) we make the ansatz

$$G(x,y\mid eA)=\exp\{ie[\phi(x)-\phi(y)]\}S(x-y) . \tag{2.11}$$

The $\phi(x)$ is the solution of $-\partial^2\phi(x)=\not\partial\not A(x)$, which is given by

$$\phi(x)=\int_\beta d^2x\,\Delta(x-y)\not\partial_y\not A(y) \tag{2.12}$$

and

$$\Delta(x-y)=\frac{1}{\beta}\sum_{n=-\infty}^{+\infty}\int\frac{dk^1}{2\pi}\exp[-ik(x-y)]\frac{1}{k^2},\quad k^0=\frac{2n\pi}{\beta} . \tag{2.13}$$

Notice the key fact that k^0 in (2.13) is a boson frequency (instead of a fermion one): this guarantees that (2.11) satisfies the required antiperiodic boundary conditions. From Eqs. (2.10) and (2.11) it follows that in order to evaluate exactly $L[A]$ we need to know how $S(x-y)$ behaves as $x-y$ approaches zero. This can be achieved by using the well-known fermion frequency series[12] for any function $f(k^0)$ which is analytic in a neighborhood of the imaginary k^0 axis

$$\frac{2\pi i}{\beta}\sum_{n=-\infty}^{+\infty} f\left[\frac{(2n+1)\pi i}{\beta}\right]=\int_{-i\infty}^{+i\infty} dk^0 f(k^0)-\int_{-i\infty+\epsilon}^{+i\infty+\epsilon} dk^0\frac{f(k^0)}{\exp(\beta k^0)+1}-\int_{-i\infty-\epsilon}^{+i\infty-\epsilon} dk^0\frac{f(k^0)}{\exp(-\beta k^0)+1} \tag{2.14}$$

in Eq. (2.7) and taking the limit $x\to y$. In doing so it turns out that the only divergent part comes from the first term on the right-hand side of Eq. (2.14) and is given by

$$\lim_{x\to y} S(x-y)=\lim_{x\to y}\frac{(-i)}{2\pi}\frac{(x-y)_\nu\gamma^\nu}{(x-y)^2}\ . \tag{2.15}$$

Introducing Eqs. (2.11)—(2.13) and (2.15) into (2.10) we get

$$L[A]=\tfrac{1}{2}\int_\beta d^2x\, d^2y A_\nu(x)\Pi^{\nu\mu}(x-y)A_\mu(y)\ , \tag{2.16}$$

$$\Pi^{\nu\mu}(x-y)=\frac{e^2}{\pi}\frac{1}{\beta}\sum_{n=-\infty}^{+\infty}\int\frac{dk^1}{2\pi}\exp[-ik(x-y)]\left[\delta^{\nu\mu}-\frac{k^\nu k^\mu}{k^2}\right],\ k^0=\frac{2n\pi}{\beta}\ , \tag{2.17}$$

for the fermion closed-loop functional.

III. THE SCHWINGER MODEL: THE COMPLETE THERMODYNAMICAL PROPAGATORS IN ITF

Taking Eqs. (2.16) and (2.17) into Eq. (2.8) and using path-integral techniques, we obtain

$$Z[J,0,0]=NZ_F\exp\left[\tfrac{1}{2}\,\mathrm{tr}\ln(I+D\Pi)^{-1}\right]\exp\left[\tfrac{1}{2}\int_\beta d^2x\, d^2y\, J_\nu(x)\,\mathbb{D}^{\nu\mu}(x-y)J_\mu(y)\right], \tag{3.1}$$

$I+D\Pi$ is a 2×2 matrix which in momentum space is represented by

$$\frac{1}{k^2}\left[\delta^{\nu\mu}\left[k^2+\frac{e^2}{\pi}\right]-\frac{e^2}{\pi}\frac{k^\nu k^\mu}{k^2}\right].$$

On the other hand, $\mathbb{D}^{\nu\mu}(x-y)$ is the complete propagator for the EM field [symbolically $\mathbb{D}=(D^{-1}-\Pi)^{-1}$]. One has

$$\mathbb{D}^{\nu\mu}(x-y)=-\frac{1}{\beta}\sum_{n=-\infty}^{+\infty}\int\frac{dk^1}{2\pi}\exp[-ik(x-y)]\left[\frac{1}{k^2+e^2/\pi}\left[\delta^{\nu\mu}-\frac{k^\nu k^\mu}{k^2}\right]+\alpha\frac{k^\nu k^\mu}{(k^2)^2}\right]. \tag{3.2}$$

Recall that the zero component of the momentum in (3.2) is discretized as for a boson particle, $k^0=2n\pi/\beta$. Notice that the gauge-independent part has acquired a mass $e^2/\sqrt{\pi}$ and the longitudinal part is the same as for the free propagator [see (2.6)].

The complete fermion propagator is obtained from Eqs. (2.8), (2.16), and (2.17) by logarithmic differentiation with respect to the external currents $\bar{\xi}(x)$ and $\xi(y)$:

$$\mathbb{S}(x-y)=\exp\left[\frac{1}{2}\frac{1}{\beta}\sum_{n=-\infty}^{+\infty}\int dk^1\{1-\exp[-ik(x-y)]\}\left[\frac{1}{k^2}-\frac{1}{k^2+\frac{e^2}{\pi}}+\frac{\alpha e^2/\pi}{(k^2)^2}\right]\right]S(x-y),\ k^0=\frac{2n\pi}{\beta}\ . \tag{3.3}$$

Notice that the structures of $\mathbb{D}^{\nu\mu}(x-y)$ and $\mathbb{S}(x-y)$ are the same as for zero temperature.[13] These results can also be obtained using TFD.[14]

IV. THE SCHWINGER MODEL: THE THERMODYNAMICAL PARTITION FUNCTION

The thermodynamical partition function, given by Eq. (2.3), is the result of letting $J=0$ in Eq. (3.1):

$$Z=NZ_F\exp\left[\tfrac{1}{2}L\sum_{n=-\infty}^{+\infty}\int\frac{dk^1}{2\pi}\ln\left[\frac{k^2}{k^2+\frac{e^2}{\pi}}\right]\right],\ k^0=\frac{2n\pi}{\beta}\ , \tag{4.1}$$

where L is the length of the linear dimension where the system is contained. By recalling Eq. (2.5) and Cauchy's residue theorem (compare with Bernard[6]), we get

$$\frac{1}{L}\ln Z = 4\int_0^\infty \frac{dk}{2\pi}\left[\frac{\beta k}{2}+\ln[1+\exp(-\beta k)]\right]+2\int_0^\infty \frac{dk}{2\pi}\left[\frac{\beta}{2}(k-w_k)+\ln\left[\frac{1-\exp(-\beta k)}{1-\exp(-\beta w_k)}\right]\right]+\text{const}\ , \tag{4.2}$$

where

$$w_k=\left[k^2+\frac{e^2}{\pi}\right]^{1/2} . \tag{4.3}$$

According to Eqs. (2.8), (2.11)–(2.13), (2.16), and (2.17) the Schwinger model is equivalent to a massless fermion together with a nondynamical massive EM field with mass $e/\sqrt{\pi}$. Therefore, we could have conjectured that its partition function would be that of the nondynamical EM field times the one of the free massless fermions. The actual EM partition function is not the one for a massive free boson field: rather, it equals the latter divided by that for a massless one. In other words, for any degree of freedom of the boson with mass $e/\sqrt{\pi}$, one subtracts a related quantity for the corresponding massless boson in the partition function. All of this is just what Eq. (4.2) accomplishes. Notice that Z is gauge independent and that when the interaction is removed (i.e., $e=0$) it reduces to the partition function for free massless fermions, as expected.

The Helmholtz free energy is $F=-\beta^{-1}\ln Z$. The expectation value over the thermodynamical ensemble of the energy per unit volume is given by $E/L=-\partial \ln Z/\partial\beta$. We make use of Eq. (4.2), ignore the zero-point energy of the vacuum, perform the resulting integrations and cancel out terms using

$$2\int_0^\infty \frac{dk}{2\pi}\frac{k}{\exp(\beta k)+1}=\int_0^\infty \frac{dk}{2\pi}\frac{k}{\exp(\beta k)-1}=\frac{\pi}{12\beta^2} . \tag{4.4}$$

Then, we obtain

$$\frac{E}{L}=\frac{1}{\pi}\int_0^\infty dk\frac{w_k}{\exp(\beta w_k)-1} . \tag{4.5}$$

If we consider the mass $e/\sqrt{\pi}$ as an energy scale, we can define high and low temperatures as those satisfying $\beta e/\sqrt{\pi}\ll 1$ and $\beta e/\sqrt{\pi}\gg 1$, respectively. By making the change $k^2+(e^2/\pi)\to k^2$, using

$$\frac{1}{\exp(x)-1}=\sum_{n=1}^{\infty}\exp(-nx),\quad x>0\ ,$$

the representation for second-class modified Bessel functions $K_r(z)$,

$$\int_0^\infty (x^2-u^2)^{r-1}\exp(-zx)dx =\frac{1}{\sqrt{\pi}}\left[\frac{2u}{z}\right]^{r-1/2}\Gamma(z)K_{r-1/2}(z)$$
$$(u>0,\ \mathrm{Re}\,r>0,\ \mathrm{Re}\,z>0)\ ,$$

and their behavior for small and large values of z (Ref. 15), we find

$$\frac{E}{L}=\frac{e^2}{4\pi^2}\left[\frac{2\pi^3}{3\beta^2e^2}+3(\gamma-\ln 4\pi)-\tfrac{1}{2}+3\ln\frac{\beta e}{\sqrt{\pi}}-\frac{5\zeta(3)}{16}\frac{\beta^2e^2}{\pi}+O\left[\frac{\beta^4e^4}{\pi^2}\right]\right]$$

for high temperatures [γ being the Euler constant and $\zeta(z)$ being Riemann's ζ function] and

$$\frac{E}{L}=\left[\frac{\pi\sqrt{\pi}}{2\beta e}\right]^{1/2}\exp(-\beta e/\sqrt{\pi})\times e^2\left[\frac{7}{8\sqrt{\pi}\beta e}+\frac{1}{\pi}+\frac{3}{8\beta^2e^2}+\cdots\right]+O(\exp(-2\beta e/\sqrt{\pi}))$$

for low temperatures.

The result (4.5) indicates that, up to the additive zero-point terms, (i) the energy per unit volume is just the one for free scalar bosons of mass $e/\sqrt{\pi}$ [since the contributions from free massless bosons and fermions have disappeared due to (4.4)], and (ii) $\ln Z$ is also the logarithim of the partition function for only those free scalar massive bosons [just by integrating (4.5) over β]. That is, in our analysis of the thermodynamics, we recover an alternative (and, as a matter of fact, known[16]) feature of the Schwinger model: it describes just free massive bosons (as the fermions disappear in this complementary description).

V. THE THIRRING MODEL

With the Thirring model we proceed analogously as for the Schwinger one. Its partition function, in Euclidean space-time, is given by

$$Z=NN_F(\beta)\int_{\text{antiperiodic}}\mathscr{D}\bar\psi\,\mathscr{D}\psi\exp\left[\int_\beta d^2x[-\bar\psi\partial\psi-\tfrac{1}{2}g^2(\bar\psi\gamma^\nu\psi)(\bar\psi\gamma_\nu\psi)]\right] ,$$

with N a β-independent constant, $N_F(\beta)$ being given by (2.2), and g being a dimensionless coupling constant.

Z can be cast as

$$Z=Z[0,0]\ ,$$

where $Z[\xi,\bar\xi]$ is of the form

$$Z[\xi,\bar{\xi}]=NZ_F\exp\left[-\tfrac{1}{2}g^2\int_\beta d^2x\left[\frac{\delta}{\delta\xi(x)}\gamma^\nu\frac{\delta}{\delta\bar{\xi}(x)}\right]\left[\frac{\delta}{\delta\xi(x)}\gamma_\nu\frac{\delta}{\delta\bar{\xi}(x)}\right]\right]\exp\left[\int_\beta d^2x\,d^2y\,\bar{\xi}(x)S(x-y)\xi(y)\right],$$

Z_F being the partition function for free massless fermions [see Eq. (2.5)] and $S(x-y)$ being the free propagator [see Eq. (2.7)]. We can also write

$$Z[\xi,\bar{\xi}]=NZ_F\exp\left[-g\int_\beta d^2x\frac{\delta}{\delta\xi(x)}\gamma^\nu\frac{\delta}{\delta J^\nu(x)}\frac{\delta}{\delta\bar{\xi}(x)}\right]\exp\left[-\tfrac{1}{2}\int_\beta d^2x\,J^2(x)\right]\exp\left[\int_\beta d^2x\,d^2y\,\bar{\xi}(x)S(x-y)\xi(y)\right]\Bigg|_{J=0}. \tag{5.1}$$

As the current $\bar{\psi}\gamma^\nu\psi$ is conserved, we can include a coupling of it to an arbitrary longitudinal source, and, instead of Eq. (5.1), we write

$$Z[\xi,\bar{\xi}]=NZ_F\exp\left[-g\int_\beta d^2x\frac{\delta}{\delta\xi(x)}\gamma^\nu\frac{\delta}{\delta J^\nu(x)}\frac{\delta}{\delta\bar{\xi}(x)}\right]$$

$$\times\exp\left[\tfrac{1}{2}\int_\beta d^2x\,d^2y\,J_\nu(x)K^{\nu\mu}(x-y)J_\mu(y)\right]\exp\left[\int_\beta d^2x\,d^2y\bar{\xi}(x)S(x-y)\xi(y)\right]\Bigg|_{J=0} \tag{5.2}$$

with

$$K^{\nu\mu}(x-y)=\frac{1}{\beta}\sum_{n=-\infty}^{+\infty}\int\frac{dk^1}{2\pi}\exp[-ik(x-y)]\left[f(k^2)\frac{k^\nu k^\mu}{k^2}-\delta^{\nu\mu}\right],\quad k^0=\frac{2n\pi}{\beta},$$

where $f(k^2)$ is an arbitrary dimensionless function of k^2.

Now, proceeding with Eq. (5.2) in the same way as with Eq. (2.4) we get for the fermion complete propagator

$$\mathbb{S}(x-y)=\exp\left[-\frac{g^2}{2\pi}\frac{1}{\beta}\sum_{n=-\infty}^{+\infty}\int dk^1\{1-\exp[-ik(x-y)]\}\frac{1}{k^2}\left[f(k^2)-\frac{g^2/\pi}{1+(g^2/\pi)}\right]\right]S(x-y),\quad k^0=\frac{2n\pi}{\beta}.$$

There are three particularly interesting choices of $f(k^2)$. For $f(k^2)=0$ we have the propagator for the original Thirring model, that is, without any coupling of the current $\bar{\psi}\gamma^\nu\psi$ to longitudinal sources. The second one is

$$f(k^2)=\frac{g^2/\pi}{1+(g^2/\pi)},$$

that corresponds to remove the interaction. Finally, the choice

$$f(k^2)=\frac{g^2/\pi}{1+(g^2/\pi)}-\frac{e^2}{g^2}\left[\frac{1}{k^2+(e^2/\pi)}+\frac{\alpha}{k^2}\right],$$

$$k^0=\frac{2\pi n}{\beta},$$

with e a constant of dimension $(\text{length})^{-1}$ and α a dimensionless arbitrary parameter, reproduces the solution (3.2) for the fermion propagator of the Schwinger model.

When comparing the propagator $\mathbb{S}(x-y)$ with that of the zero-temperature case[11] we observe that all β dependences are included in the discretization of k^0. A real-time expression for $\mathbb{S}(x-y)$ in TFD can be found in Ref. 14.

For the partition function we get

$$Z=N'Z_F,\quad N'=N\exp\left[-\ln\left[1+\frac{g^2}{\pi}\right]L\sum_{n=-\infty}^{+\infty}\int\frac{dk^1}{2\pi}\right].$$

Notice that N' is a β-independent constant, which amounts to conclude that the partition function for the Thirring model is that of free massless fermions, save a constant which depends on the coupling parameter g but not on temperature. Therefore, the thermodynamical energy per unit volume is the same in both cases. Notice also that Z is independent of $f(k^2)$.

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APPENDIX

The β-dependent divergent constant $N_F(\beta)$ is determined from the free partition function Z_F. In our case,

$$Z_F=NN_F(\beta)\int_{\text{antiperiodic}}\mathscr{D}\bar{\psi}\,\mathscr{D}\psi\exp\left[-\int_\beta d^2x\,\bar{\psi}\partial\psi\right],$$

which implies

$$\ln Z_F = \ln N_F(\beta) + \sum_{n=-\infty}^{+\infty} L \int \frac{dk^1}{2\pi} \ln\left[\frac{(2n+1)^2\pi^2}{\beta^2} + (k^1)^2\right] + \text{const} ,$$

L being the length of the linear dimension in which the system is contained. By using Cauchy's residue theorem, the last equation can be cast as

$$\ln Z_F = \ln N_F(\beta) + 4L \int_0^\infty \frac{dk}{2\pi}\left[\frac{\beta k}{2} + \ln[1+\exp(-\beta k)]\right] - 2(\ln\beta)\sum_{n=-\infty}^{+\infty}\int \frac{dk^1}{2\pi} + \text{const} .$$

Identifying

$$\ln N_F(\beta) = 2(\ln\beta)\sum_{n=-\infty}^{+\infty}\int\frac{dk^1}{2\pi}$$

and ignoring the β-independent constant we get the Fermi-Dirac distribution for massless electrons:

$$\frac{1}{L}\ln Z_F = 4\int_0^\infty \frac{dk}{2\pi}\left[\frac{\beta k}{2} + \ln[1+\exp(-\beta k)]\right] .$$

In the thermodynamical limit $L \to \infty$ while $L^{-1}\ln Z$ approaches a finite limit, save the first term on the right-hand side, which corresponds to the zero-point energy.

Notice that for each (k^0, k^1), $N_F(\beta)$ has dimension (length)^2. This agrees with the fact that $\int \mathscr{D}\bar{\psi}\mathscr{D}\psi$ has dimension (length)$^{-2}$.

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$$\det(-\partial^2)\int_{\text{periodic}} \mathscr{D}A \exp\left[\tfrac{1}{2}\int_\beta d^2x\, A_\nu(x) D^{-1\nu\mu}(x) A_\mu(x)\right]$$

which has been absorbed into the normalization constant N of Eq. (2.3). Actually, the analogue of the representation (2.3) for 1 + 3 dimensions has been obtained a long time ago by E. S. Fradkin, Dok. Akad. Nauk. SSSR **125**, 66 (1959) [Sov. Phys. Dokl. **4**, 327 (1959)]; Nucl. Phys. **12**, 465 (1959).

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