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ABOUT A CONJECTURE OF SCHOENBERG ON NESTED PENTAGONS

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RESUMEN.

Presentamos aquí la contestación afirmativa a una conjetura geométrica planteada por I. Scoenberg en 1982, y que había permanecido abierta hasta que en la primavera de 1986 encontré una solución basada en el empleo de sistemas dinámicos lineales en espacios proyectivos. Este trabajo fué presentado como ponencia en la European Conference on Iterations, celebrada en Caldas de Malavella en Septiembre de 1987, y posteriormente ha sido admitida para su publicación en las memorias del citado Congreso.

1. The conjecture.

Let K^1 be a strictly convex pentagon in the affine plane E_2, where by strictly convex we mean convex and with no three colinear vertices Let A_i^1, $i \in Z_5$, be the vertices of K^1. We write $A_i^1 A_j^1$ to represent the straight line joining these two points, and $(A_i^1 A_j^1).(A_k^1 A_m^1)$ to represent the intersection of the two lines.

So, in this way:

$$A_i^2 = (A_{i+1}^1 A_{i+3}^1)(A_{i-1}^1 A_{i-3}^1), \ i \in Z_5$$

we can define the vertices of a strictly convex pentagon $K^2 \subsetneq K^1$ (Fig. 1), and, inductively:

$$A_i^{j+1} = (A_{i+1}^j A_{i+3}^j).(A_{i-1}^j A_{i-3}^j) \ , \ i \in Z_5$$

We get the vertices of a chain of strictly convex pentagons :

$$K^1 \supsetneq K^2 \supsetneq K^3 \supsetneq \dots \supsetneq K^{j+1}$$

we shall prove the following conjecture of Schoenberg: $\bigcap_1^\infty K^1$ is reduced to only one point. This conjecture was formulated by I.J. Schoenberg in "Mathemátical Time Exposures ". Ed. Math. Assoc. of Am. , Wash. 1982.

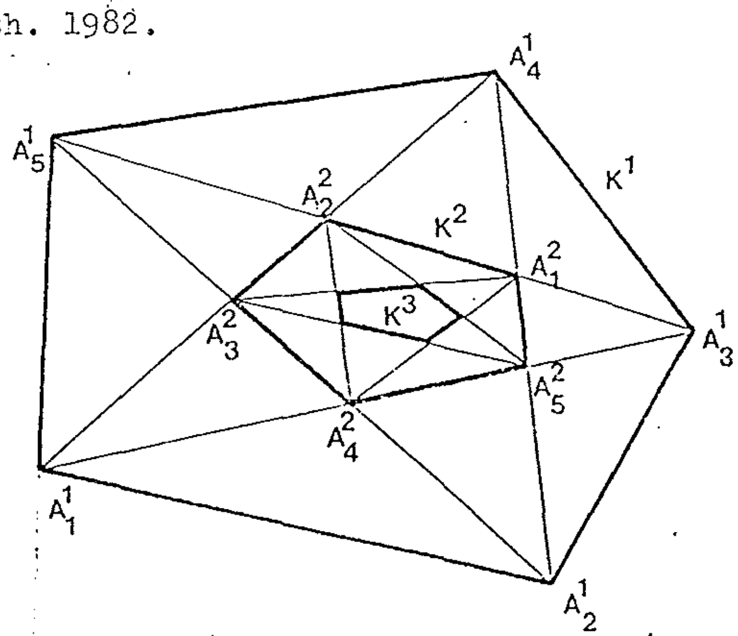

Fig. 1

2. A collineation associated to K^1

We now consider the affine plane E_2 included in the projective plane P_2. Then we have the next lemma:

Lemma 1

There is a collineation $f: P_2 \longrightarrow P_2$ mapping each vertex of K^j, $j \in \mathbb{N}$ onto a different vertex of K^{j+1}.

In order to get f we consider the two sets of four points $\{A_i^1\}$, --- $\{A_i^2\}$, $1 \leqslant i \leqslant 4$ In these sets there are no subsets of three collinear points, and hence there is only a collineation $f: P_2 \longrightarrow P_2$ --- fullfilling the conditions:

$$f(A_i^1) = A_i^2 \qquad 1 \leqslant i \leqslant 4$$

First we shall prove that $f(A_5^1)$ is just A_5^2 .

In order to do that, let $r_2r_3r_4r_5$ be the pencil of rays centered in A_1^1 , where $r_i = A_1^1A_i^1$ (Fig 2). We consider as well the pencil $s_5s_4s_3s_2$, centered in A_1^2 , where $s_i = A_1^2A_i^2$. These two pencils intersect in the - range of points $A_2^1A_4^2A_3^2A_5^1$ which lies on the line $A_2^1A_5^1$ and therefore they are projective pencils which we shall indicate by , :---------

$$s_5s_4s_3s_2 \;\overline{\wedge}\; r_2r_3r_4r_5$$

Obviously $r_2r_3r_4r_5 \,\overline{\wedge}\, r_5r_4r_3r_2$ and hence $s_5s_4s_3s_2 \,\overline{\wedge}\, r_5r_4r_3r_2$. Now - we ask for the image of the pencil $r_5r_4r_3r_2$ under f . If i = 2,3,4, then $f(r_i) = f(A_1^1A_i^1) = A_1^2A_i^2 = s_i$. So, f maps the pencil $r_5r_4r_3r_2$ on

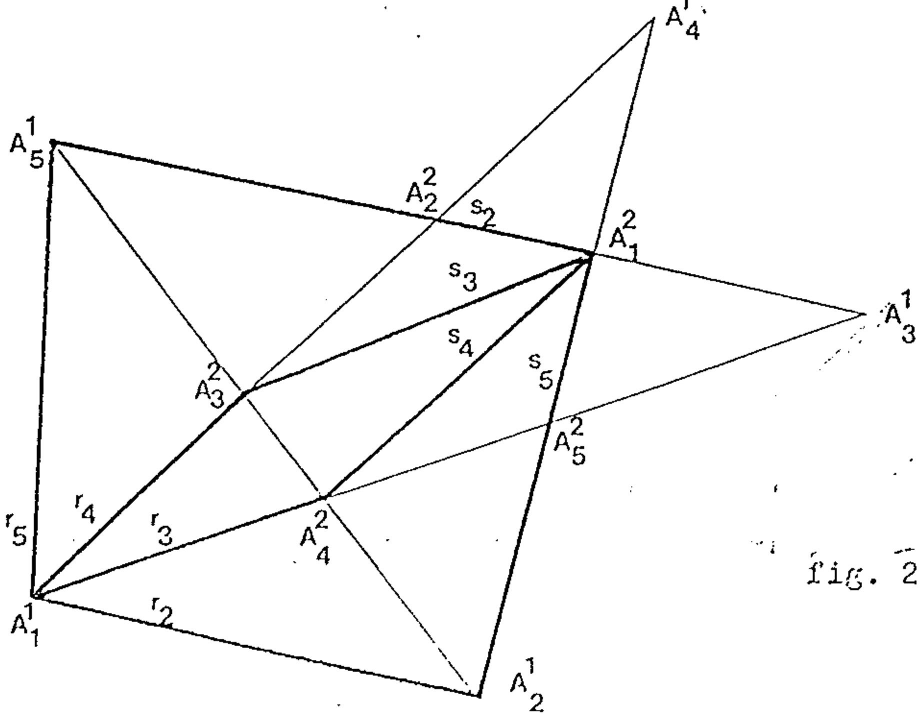

fig. 2

onto the pencil $f(r_5)s_4s_3s_2$, and, f being a colineation, we can write $r_5r_4r_3r_2 \barwedge f(r_5)s_4s_3s_2$, and hence $s_5s_4s_3s_2 \barwedge f(r_5)s_4s_3s_2$. This shows that $s_5=f(r_5)$. In a similar way, by replacing in our reasoning the pencils $r_2r_3r_4r_5$ and $s_5s_4s_3s_2$ by the pencils $m_3m_2m_1m_5$, where $m_i = A^1_4A^1_i$ and $t_5t_1t_2t_3$, where $t_i = A^2_4A^2_i$ we get $f(m_5) = t_5$. Therefore $f(A_5) = f(r_5 \cap m_5) = s_5 \cap t_5 = A^2_5$, and the lemma is proved when $j = 1$.

Now we can proceed by induction. We assume the lemma to be true for $1 \leqslant j < n$. So :

$$f(A^n_i) = f((A^{n-1}_{i+1}A^{n-1}_{i+3}).(A^{n-1}_{i-1}A^{n-1}_{i-3})) =$$
$$(A^n_{i+1}A^n_{i+3}).(A^n_{i-1}A^n_{i-3}) = A^{n+1}_i \quad , \; i \in Z_5$$

and the lemma is proved.

Moreover, the straight lines on which the sides of K^j lay map over the lines on which the sides of K^{j+1} lay. We shall prove that the sides of K^j map onto the sides of K^{j+1} and the inner points of K^j map also on to the inner points of K^{j+1}. As a matter of fact, when we think about the inner points of K^j, we think about the inner with respect the usual topology of $E_2 \subset P_2$. We are identifying E_2 with the points of P_2 which are not ideal points. So $E_2 = P_2 - E_1$, where E_1 is the line at infinity. Let $E_1' = f^{-1}(E_1)$, $F_2 = P_2 - E_1'$, and $g = f/F_2$. Then g maps F_2 on to E_2, because if $x \in F_2$, $x \notin f^{-1}(E_1)$ and $f(x) \notin E_1$, and therefore $f(x) \in P_2 - E_1 = E_2$. Now, by selecting a suitable coordinate system, we can see that $g: F_2 \longrightarrow E_2$ is an injective continuous mapping. In this coordinate system the points of P_2 have homogeneous coordinates (x_0, x_1, x_2) in such a way that the ideal points, that is, the E_1 line, have equation $x_0 = 0$. Then the homogeneous equations of f are:

$$\lambda x_i' = a_{i0}x_0 + a_{i1}x_1 + a_{i2}x_2 \qquad i = 0,1,2$$

where the matrix $A = (a_{ij})$, $0 \leqslant i,j \leqslant 2$ is nonsingular

In this coordinate system, all the points in E_2 have well defined affine coordinates:

$$\xi_i = x_i / x_0$$

The affine equations of g in F_2 are:

$$\xi_i' = \frac{a_{i0} + a_{i1}\xi_1 + a_{i2}\xi_2}{a_{00} + a_{01}\xi_1 + a_{02}\xi_2} \qquad i = 1,2$$

These equations show that g is an injective mapping in F_2, and continuous with respect to the ususal topology of F_2 as a subespace of E_2 and P_2. So when P,Q are points in E_2 and the segment $\overline{PQ} \subset F_2$, $g(\overline{PQ})$ is the segment $\overline{g(P)g(Q)}$. But if we assume that some point $X \in E_1'$ is an inner point of $\overline{PQ}$, then $g(\overline{PQ})$ is the set $g(PQ)-\overline{g(P)g(Q)}$ (where $g(PQ)$ is the line onto which maps the line PQ), and reciprocally, the set $PQ-\overline{PQ}$ maps onto the segment $\overline{g(P)g(Q)}$.

Let $X \in E_1'$ and let X belong to some side of K^1, say $X \in \overline{A_2^1 A_2^1}$. It is impossible $X = A_1^1$ or $X = A_2^1$ because if $X \in E_1'$, $f(X) \in E_1$, and we know that the points $f(A_1^1) = A_1^2$ and $f(A_2^1) = A_2^2$ can not be in E_1. Hence X must be an inner point of $A_1^1 A_2^1$ and therefore $g(\overline{A_1^1 A_2^1}) = A_1^2 A_2^2 - \overline{A_1^2 A_2^2}$ From this we conclude that some point $Y \in \overline{A_1^1 A_2^1}$ maps onto A_5^1 (see fig 3) and this is not possible, because the only point mapping under g onto A_5^1 is $Z = (A_1^1 A_2^1).(A_3^1 A_4^1)$, and $Z \notin \overline{A_1^1 A_2^1}$. So we have proved that there is not any point of E_1' in the border of K^1. Moreover $K^1 \subset F_2$, because if $K^1 \cap E_1' \neq \emptyset$, K^1 being a convex set, there would be points of E_1' on the border of K^1.

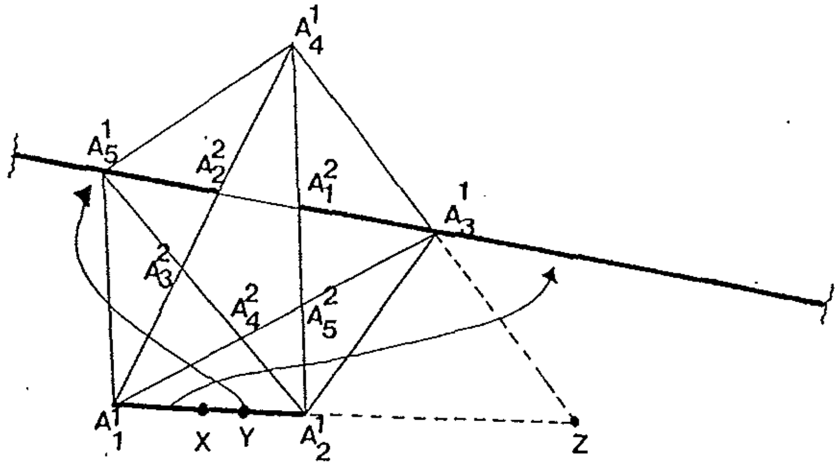

fig 3

Now it is clear that if $A, B \in K^n$, $n \in N$, then $g(\overline{AB})$ is a segment with endpoints $g(A)$ and $g(B)$. Therefore each side of K^n maps onto the respective side of K^{n+1} and $g(Fr(K^n)) = Fr(K^{n+1})$. The set $g(K^n)$ is a convex set with the same border than K^{n+1}. Hence $g(K^n) = K^{n+1}$, $n \in N$.

From these results we can get easily the next lemma:

Lemma 2

The collineation f associated to K^1 has the following properties:

a) It has a fixed point $P \in \bigcap_1^{\infty} K^n$

b) P is the only fixed point under f on K^1

c) P is on each straight line invariant under f and intersecting K^1.

The existence of P is a result from Brouwer's theorem applied to the restriction g of f to K^1, which is, like we have proved above, a continuous mapping on K^1 such that $g(K^1) = f(K^1) = K^2 \subset K^1$. So g has a fixed point $P \in K^1$ and $f(P) = g(P) = P$. If we assume $P \in K^n$, $f(P) = P \in f(K^n) = K^{n+1}$, and by induction we conclude $P \in \bigcap_1^{\infty} K^n$.

Let be now Q a point on the border of K^1, like in fig. 4 , and $\overline{PQ}$ be the segment joining Q and the fixed point P. If we assume $Q \in \overline{A_1^1 A_2^1}$, then $Q' = f(Q) \in \overline{A_1^2 A_2^2}$ and $g(\overline{PQ}) = \overline{PQ}'$, and from $PQ \cap PQ' = P$ we conclude that the only fixed point on $\overline{PQ}$ is P. If Q changes round the border, the segments PQ cover K^1, and thereby P is the only fixed point under f on K^1.

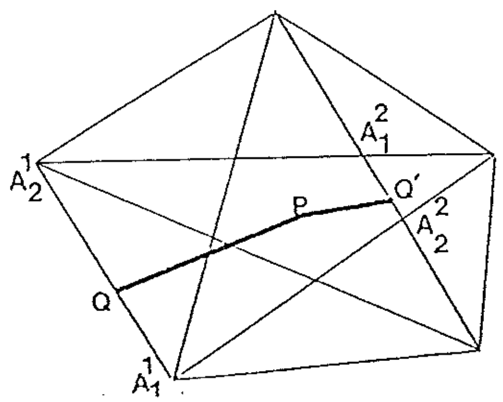

fig. 4

Let us ~~suppose~~ assume r to be a straight line such that $f(r) = r$, and ----- $r \cap K^1 \neq \emptyset$. Then $f(r \cap K^1) = r \cap f(K^1) = r \cap K^2 \subset r \cap K^1$. From Brouwer's theorem we get the existence on $r \cap K^1$ of a fixed point under f. This point must be P by b) property of f, and c) is proved.

From now on, we shall always write f for the collineation associated to K^1 and P for its only fixed point on K^1.

3. Classification of f.

First we shall exclude a possibility in which the proof of the conjecture would be trivial. The next example proves that, in general, f is not contractive. In fact $d(A_1^1 A_2^1) < d(A_1^2 A_2^2) = d(f(A_1^1), f(A_2^1))$

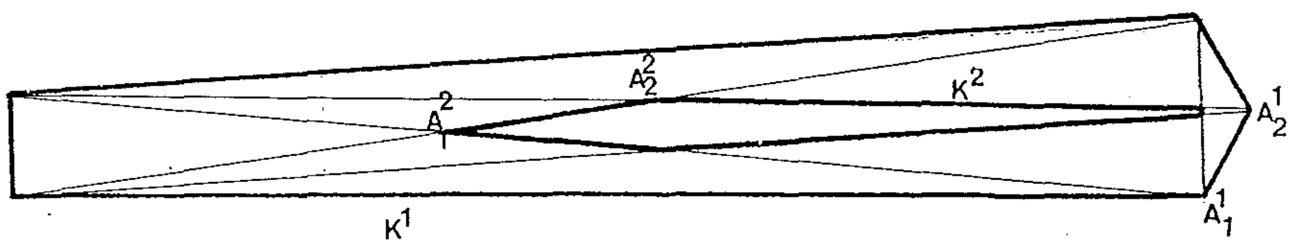

fig 5

So, in order to get the proof of the conjecture, we must study the iterations of f. We can distinguish two possibilities:

a) There is a straight line such that $f(r) = r$ and $P \notin r$.

The analytic affine equations of f in a reference system in which -- the equation of f is $x_0 = 0$ are the equations of an ~~affinity~~ affine mapping. Let S_n be the surface of K^n in this reference system. All the points of -- K^1 have finite affine coordinates, because according to lemma 2-c) we know that $r \cap K^1 \neq \emptyset$ is not possible when $P \notin r$ and $f(r) = r$, such as we are assuming. Hence S_n is a finite positive real number, and --- there is an affinity modulus t such that $S_{n+1} = t.S_n$. Moreover, $K_2 \subsetneq K_1$ implies $t < 1$, and therefore $\lim_{n \to \infty} S_n = 0$. From this we conclude that $\bigcap_1^\infty K^n$ is a convex set of zero surface, that is, either a point or a segment. We proceed to prove the impossibility of this last hypothesis. Let $\bigcap_1^\infty K^n = \overline{BC}$. Then $f(\overline{BC}) = f(\bigcap_1^\infty K^n) = \bigcap_1^\infty K^{n+1} = \overline{BC}$. The border - of the segment BC maps onto itself (by the continuity and injectivity of $g = f/K^1$), in such a way that we have either $f(C)=B$ and $f(B)= C$ or $f(C) = C$ and $f(B) = B$. This second possibility is excluded by lemma 2 - b), and the first possibility means that the restriction of f to the line s=BC is an involution, and if $Q \in F_r(K^1) \cap s$, $f^2(Q) = Q \in F_r(K^3)$ which is contradictory with $F_r(K^1) \cap F_r(K^3) = \emptyset$. This shows that --- $\bigcap_1^\infty K^n$ can not be a segment, and it reduces to a single point, and therefore the conjecture is proved in this case a).

The other possibility is :

b) P belongs to every straight line such that $f(r)=r$.

In this case we get the proof of the conjecture via the following lemma:

<u>Lemma 3</u>

If P belongs to every invariant straight line under f, for every --- $X \in K^1$, $\lim_{n \to \infty} f^n(X) = P$

In order to prove this lemma, we must consider every Jordan canonical form possible for f, and then analyze its iterations. As it is well known, the fixed point associated to a simple root of the characteristical equation of f can not belong to the invariant straight line associated to the same root, and so we know that P is associated to a double or triple root of the characteristical equation of f. Two are the possibilities that we must analyze:

b_1) The canonical forms of the maps f and f^n are respectively:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & n & \binom{n}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$$

These canonical forms mean that f is a mapping with a triple root in its caracteristical equation which we can always assume to be 1 because, f being a collineation, we could multiply it by any no zero real number if it were necessary, without changing f.

The case that we are analyzing is the only one in which f has a triple root, because, as we can easily see, the other two possible canonical forms of f with triple roots must have an infinity of fixed points on K^1 in contradiction with lemma 2 - b). These two cases are the identity and an elation with an invariant straight line of fixed points to which P must belong.

Let f be the mapping with the above written canonical form. P(1,0,0) is its only fixed point, and it is the same point P that we have got in lemma 2. Let $X(x_0,x_1,x_2)$ be a general point on P_2, and $f^n(X) = X(n)(x_0(n),x_1(n),x_2(n))$. We have:

$$\begin{aligned} \lambda x_0(n) &= x_0 + x_1 n + x_2\binom{n}{2} \\ \lambda x_1(n) &= x_1 + x_2 n \\ \lambda x_2(n) &= x_2 \end{aligned}$$

But $x_0 = x_1 = x_2 = 0$ does not hold for any point on P_2, and this shows that the polynomial $x_0(n)$ is not the zero polynomial. So, for some n_0, $n > n_0 \Longrightarrow x_0(n) \neq 0$ and the affine coordinates of $X(n)$ are well - defined : They are $\xi_i(n) = x_i(n)/x_0(n)$, $i = 1,2$. In these rational funtions, fixed the point X, the denominator polynomial has greater degree than the numerator , with the only exceptional case in which $x_1 = x_2 = 0$, and in this case X is the fixed point P and the lemma 3 is trivial. But in all other cases, $\lim_{n \to \infty} X(n) = (0,0)$ in affine coordinates, and (0,0) is the point P(1,0,0) in homogeneous coordinates q.e.d. It is remarkable that f, having an only attractive fixed point which is P, nevertheless is not a contractive map.

b_2) The canonical forms of f and f^n are respectively :

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \rho \end{bmatrix} \qquad \begin{bmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \rho^n \end{bmatrix}$$

We can assume $\rho \neq 1$, because if $\rho = 1$, the characteristical equation of f would have 1 as a triple root, and we are just coming to analyze that case.

So, 1 is a double root of the characteristical equation of f. The fixed points of f are $P(1,0,0) \in K^1$ (because this point is associated to the double root 1) and $Q(0,0,1) \in K^1$ (because lemma 2 - b)).

Let $X \in P_2$ and $f^n(X) = X(n)$ with the same notation as in b_1). We have:

$$\begin{aligned} \lambda x_0(n) &= x_0 + x_1 n \\ \lambda x_1(n) &= x_1 \\ \lambda x_2(n) &= x_2 \rho^n \end{aligned}$$

Excluding the case $X = Q$, the polynomial $x_0(n)$ is not the zero polynomial, and it is not zero if n exceeds some n_0 . For these values of n the affine coordinates of $X(n)$ are well defined. We are going -

to prove that $|\rho| < 1$. The line $r \equiv x_1 = 0$ which joins the points P and Q is invariant under f. Its affine equation is $\xi_1 = 0$, and the restriction of f^n to r has equation $\xi_2(n) = \rho^n x_2/x_0 = \rho^n \xi_2$. --- Let $X \in r \cap K^1$ and $|\rho| > 1$. Then, the sequence $\xi_2(n)$ would be not ---- bounded. But this is not possible because the point $X(n) \in r \cap K^1$, - and it has affine coordinates $(0, \xi_2(n))$, and we can always find a - bound for the affine coordinates of the points of $r \cap K^1$, since the only ideal point on r, for which is $x_0 = 0$, is $Q(0,0,1) \notin K^1$ and --- Q is not adherent to $r \cap K^1$. This contradiction shows $|\rho| \leqslant 1$. Now, - $\rho = 1$ being excluded by hypothesis, we must only prove $\rho \neq -1$. -- Let us assume $\rho = -1$. We have for the restriction of f to $r \cap K^1$ -- the equation $\xi_2(n) = (-1)^n \xi_2(n)$, which is an involution, and --- this is impossible as we know according to the argument presented in the case a). And, as the last possibility, remains only $|\rho| < 1$

Let $|\rho| < 1$ and $X \in P_2$, $X \neq Q$. Then $\lim_{n \to \infty} \xi_1(n) = 0$ and $\lim_{n \to \infty} \xi_2(n) = 0$. Hence $\lim_{n \to \infty} X(n) = (0,0)$ in affine coordinates and (0,0) is the point - P(1,0,0) in homogeneous coordinates. But every $X \in K^1$ is $X \neq Q$, and - the lemma 3 is proved in the case b_2): The iterations of the points of P_2 converge to the attractive fixed point P, except those of the repulsive fixed point Q.

With lemma 3 we can easily get the proof of the conjecture. Let $B_P(\delta)$ be a circle with center on P and radius δ. By taking a suitable n, $f^n(A_i^1) \in B_P(\delta)$, $i \in Z_5$, and therefore $K^n \subset B_P(\delta)$. Hence $\bigcap_1^\infty K^j \subset K^n \subset B_P(\delta)$, and from this we can conclude $\bigcap_1^\infty K^j = P$.

4. Graphical construction of P

If we know any invariant straight line under f or any fixed point, - we can get P with straight edge and compass. In order to this, we - shall get all the fixed points of the mapping f, and then select -- the only one belonging to K^1. To get the fixed points of f is, in -- general, a third degree problem. But if we know a fixed point or invariant straight line, then the problem is of second degree and can be solved with ruler and compass.

We present an example of this kind, where the points $B = (A_2^1 A_3^1).(A_4^1 A_5^1)$, A_1^1 and A_1^2 are collinear. Then the line r on which these three points lay is invariant, because $f(r) = f(BA_1^1) = f(B)A_1^2 = A_1^1 A_1^2 = r$. We can get the fixed points of f/r with a well known construction (see Rf. 2), and P is the one of them which belongs to K_1^1. It is remarkable -- that we can apply the Desargues theorem to the triangles $A_3^1 A_2^1 A_5^2$ and $A_4^1 A_5^1 A_2^2$ getting the concurrency of the lines $A_3^1 A_4^1$, $A_2^1 A_5^1$ and $A_5^2 A_2^2$ on the point Q (fig 6), which is a fixed point, because f(Q) = ------ $= f((A_3^1 A_4^1).(A_2^1 A_5^1)) = (A_3^2 A_4^2).(A_2^2 A_5^2) = Q$. Reciprocally from the concurrency of the lines $A_3^1 A_4^1$, $A_2^1 A_5^1$ and $A_5^2 A_2^2$ on Q, we can conclude that Q is a fixed point and the line $r = A_1^1 A_1^2$ is an invariant line.

Other characteristical example is that in which K^1 is the regular pentagon. In that case the line at infinity is pointwise fixed, and so f is an affine mapping like in the case a). P is clearly the center of the regular pentagon.

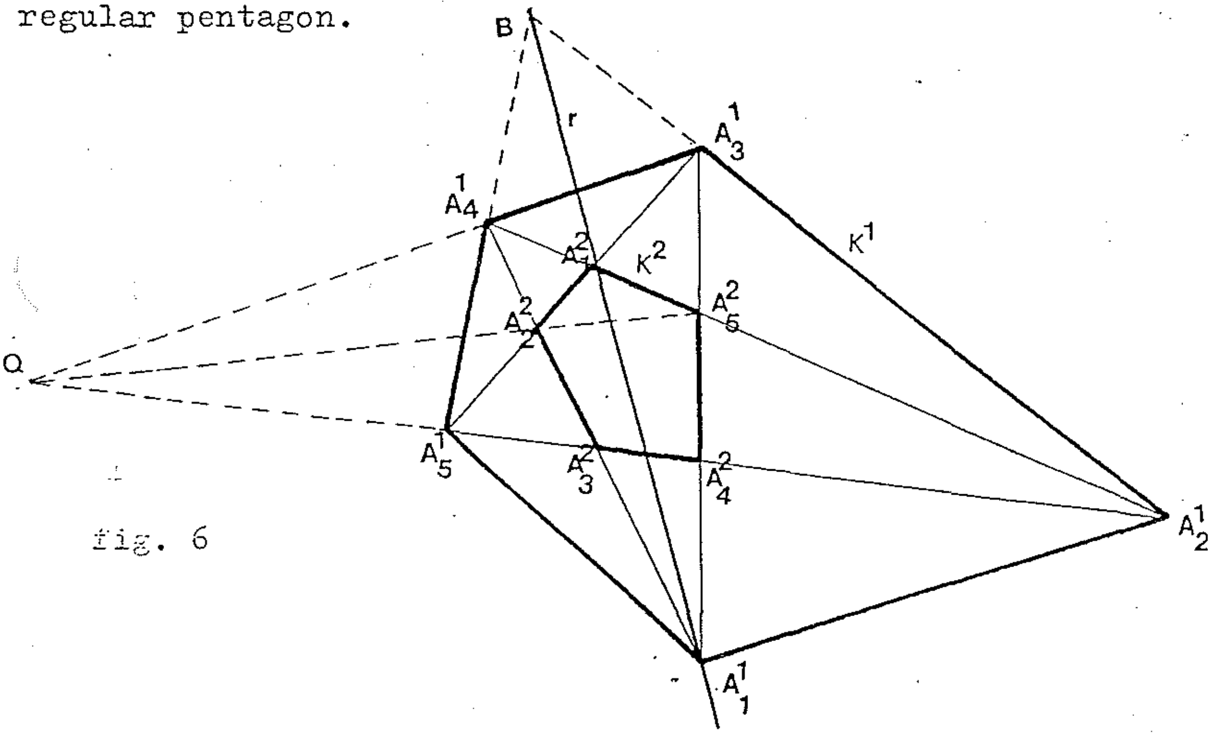

fig. 6

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