

Line spread function formulation proposed by W. H. Steel: a revision

A. Manzanares, M. L. Calvo, M. Chevalier, and V. Lakshminarayanan

A revised formulation of the image light distribution of an incoherent line source proposed by Steel [Rev. Opt. **31**, 334–340 (1952)] is presented. Analytical and numerical results based on this new representation are given. We explicitly show that a major error in Steel's final expression generates singularities, thereby preventing convenient numerical computation. © 1997 Optical Society of America

1. Introduction

Steel¹ formulated the light distribution of an incoherent line source in the image plane in terms of three separate expressions: (1) direct calculation (in terms of the Struve function of first order), (2) the Fourier transform (integral representation), and (3) a series of Bessel functions.

In the context of image-forming systems, the image of a line source is usually called the line spread function (LSF), similar to the image of a point source called the point spread function (PSF). According to these well-known criteria for PSF and LSF, the formalism given by Steel for representing the incoherent image intensity distribution of a line behaves as the LSF of a diffraction-limited system with a circular pupil of unit radius in conditions of incoherent illumination.

Based on the classical theory of image-forming systems limited by diffraction, an invariant linear system can be characterized by its impulse response: the response of the system to a point source (PSF) or by its line response, the response of the system to a line source (LSF).² The LSF is easier to measure and sometimes is preferred for characterizing the system response instead of the PSF. Moreover the

LSF characterization gives an alternative procedure in which the edge spread function (ESF) or edge trace of the system is directly derived. When the research of Marchand is used,^{3,4} the reciprocity between the LSF and modulation transfer function (MTF) of the system establishes the characterization of the ESF as a Fourier transform of the LSF. This gives a systematic procedure that could be applied to edge-imaging quality assessment and its influence on optical instrumentation.⁵ This is the main focus of interest of this note. Defining the LSF of the system, one calculates the output image of the system directly by convolving the LSF with the input image. This is the main procedure that we have considered in our study.

To obtain an explicit expression for the convolution operation, one may search for a convenient analytical formulation for the LSF. This goal has a double purpose: (1) obtaining a simple and manageable formulation and (2) assuring proper mathematical behavior (avoiding divergences) and a fast speed of convergence of the solution. As a first attempt we introduced Steel's LSF expression in our calculations.¹ We have found that the LSF in terms of a series of the Bessel function of the order of 1 given by Steel presents odd symmetry, and it is negative for some values of the argument. This fact contradicts the mathematical behavior of the LSF that is defined as a positive function. As for the mathematical representation, Steel employed the so-called standard notation of Watson,⁶ which coincides with the one that we used in our study.

According to Steel's formulation, the LSF is directly obtained by assuming that the system being considered is diffraction limited. The function $\text{circ}(r)$ represents the circular pupil of the system (with a unit radius). We consider such a system to

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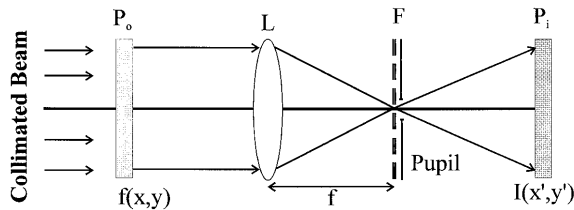


Fig. 1. Simplified diffraction-limited system with a telecentric effective stop. Incoherent conditions of illumination are assumed.

be illuminated by an incoherent and collimated light beam.

We assume that the optical system is composed of an arbitrarily defined object plane, where the input function (object), a lens and its associated Fourier transform plane F (where the spectrum of the object is defined), and the image plane, which is located far from plane F in the Fraunhofer approximation (see Fig. 1) are located. For simplicity the pupil of the system is located close to the Fourier transform plane (a negligible distance to it). Note that this simplification implies that we consider a telecentric effective stop.⁷

2. Procedure

On the plane P_o is located the object intensity distribution or input function, $f(x, y)$. On plane F the Fourier transform of the input function, $\text{FT}[f(x, y)]$, is defined. It is affected by the optical transfer function (OTF) of the system; then

$$F(\alpha, \beta) = \text{FT}[f(x, y)]\text{OTF}(\alpha, \beta), \quad (1)$$

where FT is the Fourier transform operation and

$$\text{OTF}(\alpha, \beta) = \text{circ}(\rho/2)**\text{circ}(\rho/2). \quad (2)$$

The ** denotes autocorrelation and $\rho = (\alpha^2 + \beta^2)^{1/2}$ is the radial coordinate.

In plane P_i , one has the image intensity distribution or output function. By applying the inverse Fourier transform in Eq(1), we have

$$\begin{aligned} I(x', y') &= \text{FT}^{-1}\{\text{FT}[f(x, y)]\text{OTF}(\alpha, \beta)\} \\ &= f(x', y')*\text{FT}^{-1}[\text{circ}(\rho/2)**\text{circ}(\rho/2)], \end{aligned} \quad (3)$$

where * denotes convolution and $(x', y') \equiv (-x, -y)$ are the spatial coordinates at the image plane. Note that we assume a unit magnification.

By using the properties of the Fourier transform related to the convolution, we have

$$\begin{aligned} \text{FT}^{-1}[\text{circ}(\rho/2)**\text{circ}(\rho/2)] &= [\text{FT}^{-1}[\text{circ}(\rho/2)]]^2 \\ &= \left[\frac{J_1(r)}{r} \right]^2, \end{aligned} \quad (4)$$

where $r = [(x')^2 + (y')^2]^{1/2}$ and J_1 is the Bessel function of first order. Therefore

$$\begin{aligned} I(x', y') &= f(x', y') * \left[\frac{J_1(r)}{r} \right]^2 \\ &= \int \int_{-\infty}^{+\infty} f(x' - u, y' - v) \\ &\quad \times \frac{[J_1(u^2 + v^2)^{1/2}]^2}{u^2 + v^2} du dv. \end{aligned} \quad (5)$$

The LSF is defined by the convolution integral given in Eq. (5) provided that the object intensity distribution $f(x, y)$ at the input of the system depends on a single variable, i.e., x . In this situation

$$I(x') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x' - u) \left[\frac{J_1(u^2 + v^2)^{1/2}}{(u^2 + v^2)^{1/2}} \right]^2 du dv \quad (6)$$

where (u, v) are dummy variables.

$$\begin{aligned} I(x') &= \int_{-\infty}^{\infty} f(x' - u) \left\{ \int_{-\infty}^{\infty} \left[\frac{J_1(u^2 + v^2)^{1/2}}{(u^2 + v^2)^{1/2}} \right]^2 dv \right\} du \\ &= \int_{-\infty}^{\infty} f(x' - u)\text{LSF}(u) du. \end{aligned} \quad (7)$$

According to Eq. (7) we arrive at a well-known result expressing the output image intensity distribution as a convolution of the one-dimensional input object intensity distribution with the normalized LSF of the system, given in terms of the Struve function^{1,8} of first order $H_1(u)$:

$$\text{LSF}(u) = \int_{-\infty}^{\infty} \left[\frac{J_1(u^2 + v^2)^{1/2}}{(u^2 + v^2)^{1/2}} \right]^2 dv = \frac{3\pi}{8} \frac{H_1(2u)}{u^2}. \quad (8)$$

3. Series Expansion

By substituting the Struve function in terms of its series expansion,⁸ one obtains

$$\text{LSF}(u) = \frac{3\pi}{8} \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n}}{\Gamma(n + 3/2)\Gamma(n + 5/2)}. \quad (9)$$

Figure 2 shows the numerical behavior of Eq. (9) for $0 \leq n \leq 30$. The convergence of the series is assured. Nevertheless the speed of convergence of the series appears to be slow, even if the number of terms of the series increases. (For $n > 30$ no improvement has been found.) The time required to compute Eq. (9) with 31 terms is 27.95 s, when a common mathematical software (Mathematica 2.2.1 on a Pentium computer operating at 120 MHz) is used. It is interesting therefore to look for an alternative series expansion for which the convergence becomes faster. Steel proposed the use of the integral representation of $H_1(u)$ as well as the Fourier expansion of the ar-

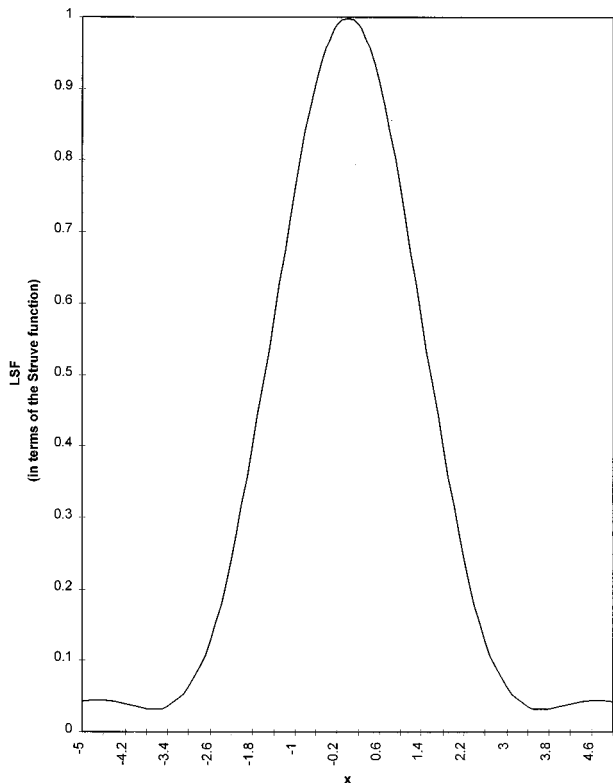


Fig. 2. LSF representation in terms of a polynomial series of the argument [see Eq. (9)]. The x represent the spatial coordinate. The ordinate axis represents the line image radiant flux density (in arbitrary units).

gument,¹ which depends on a series expansion of Bessel functions.

We have reproduced this calculation and found an analogous result:

$$\text{LSF}(u) = \frac{3}{4u^2} \left[1 - J_0(2u) + 2 \sum_{n=1}^{\infty} \frac{J_{2n}(2u)}{4n^2 - 1} \right], \quad (10)$$

where J_0 is the Bessel function of zero order and J_{2n} is the Bessel function with $n > 0$ and even order. Steel proposed a simplification of Eq. (10) where the mathematical procedure was not specified, leading to a new expression that exhibits a divergent behavior:

$$\text{LSF}(u) = \frac{3}{2u^2} \left[u - 2 \sum_{n=1}^{\infty} \frac{J_1(4nu)}{4n^2 - 1} \right]. \quad (11)$$

This is an odd function. Figure 3 displays this result for $1 \leq n \leq 100$. The figure also shows negative values for the LSF and a loss of symmetry. There is also a lack of lobes and the center exhibits a singular behavior. The latter characteristics do not correspond to the expected shape for a typical LSF function. To verify whether Eq. (11) represents a LSF, one may calculate the MTF and the ESF associated with Steel's LSF. By using the definitions,⁹ one may prove that these functions di-

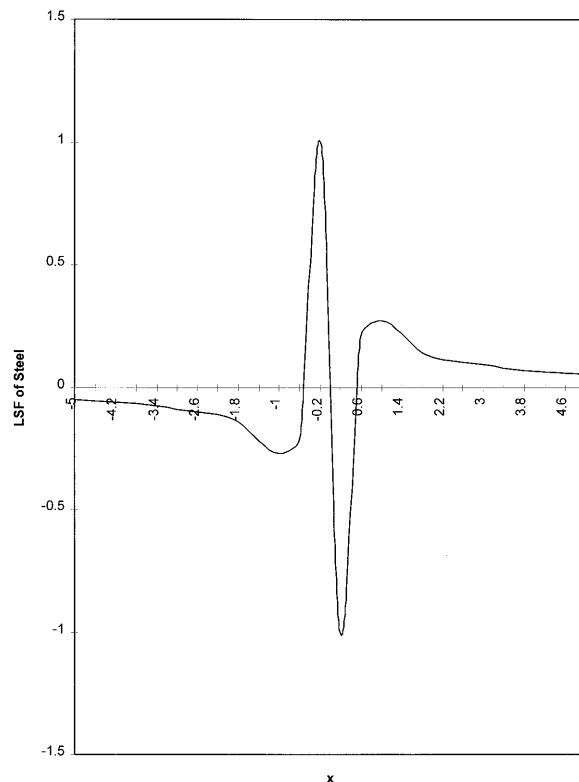


Fig. 3. LSF representation as given by Steel [see Eq. (11)]. Units are as in Fig. 2.

verge. To calculate the ESF from the LSF of Steel, one has to solve the integral of the LSF for the interval $(-\infty, x)$. In particular, one has to calculate the integral of the first term for the interval $(-\infty, x)$, but the result of this integral is $\log(x)$. Obviously, $\log(-\infty)$ diverges, and it is not possible to define the ESF associated with the LSF of Steel. Similar problems are found if one calculates the MTF associated with the LSF of Steel. Again it is not possible to define the MTF since a singularity appears in the expression.

In the analysis of the general formulation [see Eq. (10)], the main goal is to obtain a series expansion for which a higher speed of convergence is found in terms of the Bessel functions. By using the property,¹⁰

$$1 = J_0(x) + 2 \sum_{n=1}^{\infty} J_{2n}(x), \quad (12)$$

and by substituting $J_0(x)$ into Eq. (10), we found that

$$\text{LSF}(u) = 6 \sum_{n=1}^{\infty} \frac{n^2}{4n^2 - 1} \frac{J_{2n}(2u)}{u^2}. \quad (13)$$

This expression appears to have a higher speed of convergence, since Eq. (13) needs only three terms in the series to converge ($1 \leq n \leq 3$) and the time required to compute is 1.48 s, when one is operating as in the precedent computation conditions of Eq. (9). Figure 4 shows both numerical behaviors of Eqs. (11) and (13).

Obviously, the LSF given by Eq. (13) shows a

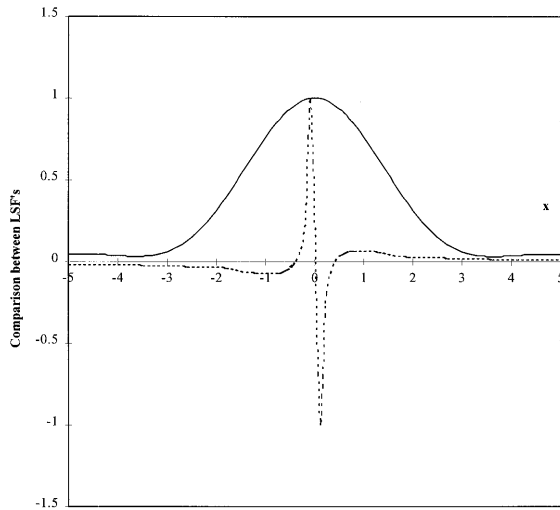


Fig. 4. Comparison between the correct LSF [see Eq. (13)] and the LSF given by Steel [see Eq. (11)]. Units are as in Fig. 2.

smooth central peak and sidelobes. Moreover it is positive for all the arguments as expected. It is possible also to calculate the associated ESF and MTF with this LSF. For brevity, we omit the long definitions of the operations. It can be shown that by integrating Eq. (13), we can give the ESF

$$\begin{aligned} \text{ESF}(x) &= \int_{-\infty}^x \text{LSF}(\alpha) d\alpha \\ &= 12 \sum_{n=1}^{\infty} \frac{n^2}{4n^2 - 1} \left\{ \frac{-1}{4n^2 - 1} \right. \\ &\quad + \frac{(x/2)^2}{(2n - 1)x\Gamma(2n + 1)} {}_1F_2 \left[\left(n - \frac{1}{2} \right), \right. \\ &\quad \left. \left. \times \left(n + \frac{1}{2}, 2n + 1 \right), \frac{-x^2}{4} \right] \right\}, \end{aligned} \quad (14)$$

where ${}_1F_2$ is the generalized hypergeometric function.¹⁰ Figure 5 displays the numerical representation of the ESF calculated from the LSF [Eq. (13)].

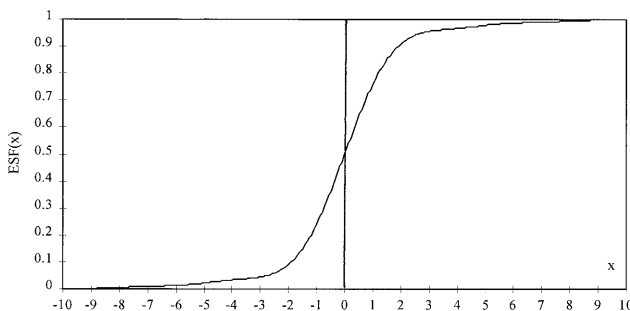


Fig. 5. (a) ESF or edge trace calculated from the correct LSF by integrating Eq. (13). The x represents the spatial coordinate. The ordinate axis represents the edge image radiant flux density (in arbitrary units). The normalization has been obtained with respect to $\text{ESF}(x \rightarrow +\infty)$.

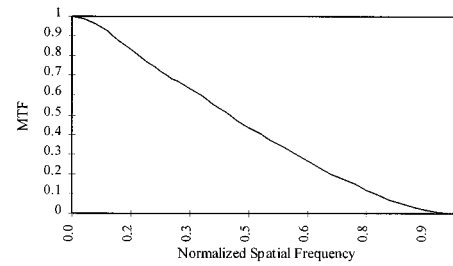


Fig. 6. MTF calculated from the correct LSF by applying the Fourier transform to Eq. (13).

By taking the Fourier transform of Eq. (13), one can give the MTF as

$$\begin{aligned} \text{MTF}(u) &= |\text{TF}[\text{LSF}(x)]| \\ &= \left| 12 \sum_{n=1}^{\infty} \left(\frac{n}{4n^2 - 1} \right)^2 {}_2F_1 \left[-n - \frac{1}{2}, \right. \right. \\ &\quad \left. \left. n - \frac{1}{2}, \frac{1}{2}, \pi^2 u^2 \right] \right|, \end{aligned} \quad (15)$$

where ${}_2F_1$ is the hypergeometric function of the order of 1.¹⁰ Figure 6 displays the corresponding MTF. Both functions show the typical behavior for the ESF and the MTF of a perfect system. This result gives another cue for considering that the proposed LSF fulfills all the desirable properties for a perfect LSF.

4. Conclusions

One can establish that the image of the line source introduced earlier by Steel can be considered as a LSF, since the assumed optical system is diffraction limited by a circular pupil and acts in conditions of incoherent illumination. By reproducing the corresponding calculations, to find the same final expression for the LSF, one finds a major error in Steel's formulation.

We have introduced here an alternative method for calculating the correct expression for the image of a line or LSF in an analogous optical system. A comparison between the two LSF expressions shows dramatic differences. In particular the LSF of Steel does not have the properties necessary to be considered as a LSF:

- Smooth central peak with even symmetry (approximate to a parabolic function).
- Even shape for all the values of the function.
- Sidelobes (these lobes take account of the second-order diffraction).
- Convergent asymptotic behavior.
- Defined MTF and ESF associated with the LSF.

The fulfillment of these main characteristics is necessary for defining the LSF. The expression derived here, represented by Eq. (13) and displayed in Fig. 4, can prove to be the correct fit.

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