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Linking the problems of estimating and allocating unconditional capital

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Abstract

This paper addresses two problems related to determining the unconditional capital required by a credit portfolio: Estimating it using Monte Carlo simulation and allocating it among the different risk units that form the portfolio. By elaborating on a tractable analytical framework, we propose a new simulation algorithm and a new allocation method. Both contributions rely on the conditional loss distributions and share the same core idea. We discuss their optimality, consistence and practical advantages. In an empirical study based on American data, we show the remarkable gains in efficiency achieved by the former and the improvement in the standard variance-covariance allocation provided by the latter.

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Linking the problems of estimating and allocating unconditional capital

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Abstract

This paper addresses two problems related to determining the unconditional capital required by a credit portfolio: Estimating it using Monte Carlo simulation and allocating it among the different risk units that form the portfolio. By elaborating on a tractable analytical framework, we propose a new simulation algorithm and a new allocation method. Both contributions rely on the conditional loss distributions and share the same core idea. We discuss their optimality, consistence and practical advantages. In an empirical study based on American data, we show the remarkable gains in efficiency achieved by the former and the improvement in the standard variance-covariance allocation provided by the latter.

Keywords: default risk, capital estimation, capital allocation, unconditional measurement, conditional measurement

JEL Classification: C58, G17, G21, G32

1. Introduction

The literature regarding default risk pays great attention to the underlying analytical framework, whose sophistication has grown considerably in the last few decades, Koopman and Lucas (2005). In spite of its importance, however, defining the analytical framework is the beginning, not the end, of the measurement process attempted to estimate the amount of capital required. Thus, several practical issues surface once the risk profile and the modeling options are set, which a bank must deal with in order to effectively obtain and use the capital that emanates from the analytical framework. They appear at the calculation step, when the capital is to be obtained, and at the management step, when the capital is to be used. In this paper, we study two of these practical capital problems: Its estimation via Monte Carlo simulation and its allocation among the different risk units forming the portfolio.

Regarding Monte Carlo capital estimation, it is a simple, effective and also widely used method to estimate the capital figure of large portfolios, Glasserman and Li (2005) and BCBS (2009). It is more versatile than the standard analytical approximations, like those discussed by Glasserman (2004b), although it usually involves a significant computational cost. An effective, efficient simulation method reduces the variability of the estimate without introducing any bias or generating a subsequent computational cost higher than that of the standard Monte Carlo simulation.

The objective of capital allocation is to assign a fraction of the portfolio's aggregated capital to each risk unit included in it, Kalkbrener (2005). As opposed to the stand-alone capital, allocated capital accounts for the diversification generated within the portfolio and is employed for uses like Risk Adjusted Return On Capital (RAROC) and Economic Value Added (EVA) analysis, see Stoughton and Zechner (2007). A good allocation method captures the influence of each risk unit on the aggregated capital and is is sensitive to the primitive sources of dependence and diversification, especially regarding the tail of the loss distributions.

Both problems are major concerns for banks and have attracted significant attention in the literature, as observed in the overviews of Glasserman (2004a) and Mausser and Rosen (2007), respectively. Due to their different natures, they are also usually addressed separately. That is, Monte Carlo capital estimation is a technical issue related to efficiency optimization, while capital allocation is a conceptual decision that is more sensitive to the bank's risk management policies. Consequently, they do not affect each other and are hence not expected to share a common underlying principle.

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However, our proposal challenges this preconception. We present a new simulation algorithm and a new allocation method, which are both inspired by the same core idea. To this end, we focus on unconditional capital and formulate the unconditional loss distribution as the equally weighted mixture of a varied set of conditional loss distributions, as in Ferrer et al. (2014).

For Monte Carlo capital estimation, we propose a new simulation algorithm based on the idea of changing the equally weighted mixture by an unequal one, in which the conditional loss distributions related to recession periods receive a higher weight. By doing so, the values in the tail of the unconditional loss distribution are generated more often, which therefore improves the efficiency of the estimation. We derive the optimal unequal weights and discuss the good practical properties of the algorithm, including its ease of calibration and implementation. Based on American data, the empirical analysis shows cyclicality in the optimal mixture weights and a remarkable improvement in the efficiency of the Monte Carlo capital estimation.

For capital allocation, we propose an unconditional allocation formed as a weighted average of conditional allocations. Such conditional allocations are obtained under another subjacent method, so that our approach can be understood as a meta-allocation method. Following the proposed simulation algorithm, the conditional allocations related to recession scenarios obtain a higher relative weight. We prove that our approach satisfies the condition of complete allocation and discuss its singularity with respect to the existing capital allocation literature. The empirical analysis shows that it enhances the standard variance-covariance allocation without harming its practical properties.

As mentioned below, a remarkable feature of these solutions is that, although they can be implemented independently, they are nonetheless connected. Thus, the optimal mixture vector to be used in the efficient simulation algorithm also serves as a weighting function for the conditional allocations. Two seemingly unrelated problems are therefore linked coherently under the idea that the conditional loss distributions that are more involved in the determination of the unconditional capital must also be more involved in its allocation.

From a more general point of view, both contributions outline the versatility offered by the conditional loss distributions for unconditional matters, this time in terms of two practical problems.

The rest of the paper is set up as follows. Section 2 presents the analytical framework on which the rest of the paper relies. Sections 3 and 4 introduce and discuss the proposed efficient Monte Carlo simulation algorithm and the proposed allocation method, respectively. Section 5 presents the empirical results, and Section 6 provides a conclusion. Appendices A and B contain the proofs for the two results obtained in the body of the paper.

2. Framework

2.1. Loss model

Time is measured discretely at regular intervals that, for simplicity, match the time horizon used by the bank to measure the default risk of its credit portfolio[1]. In practice, this time horizon is usually a year, but no specific assumptions about it are made. We also assume that the only source of loss for the bank is its credit portfolio.

As a starting point, we consider a standard default-event, constant-exposure, conditional-independence portfolio model; see Gordy (2000) and Frey and McNeil (2003). This model is formulated as:

$$L = \sum_{j=1}^{N} L^j = \sum_{j=1}^{N}\sum_{i=1}^{M^j} L^{ji} = \sum_{j=1}^{N}\sum_{i=1}^{M^j} Ber^{ji}\left(F^j\right) e^{ji} \quad (1)$$

where N is the number of risk units, formed by homogeneous groups of debtors, e^{ji} is the net exposure of each debtor, $e^{ji} > 0$, and Ber^{ji} is the Bernoulli random variable that takes a value of 1 if the debtor defaults and 0 otherwise. M^j is the number of debtors assigned to the risk unit j, with $M = \sum_{j=1}^{N} M^j$ being the total number of debtors of the portfolio. L^{ji}, L^j and L are, respectively, the loss distributions of debtor i from risk unit j, the loss distribution of the risk unit j and the loss distribution of the entire portfolio, respectively. F^j is the probability of default (PD) of risk unit j during the time horizon. It is a continuous random variable with support in $(0\ 1)$, with $\boldsymbol{F} = \left(F^1, ..., F^N\right)$ being the continuous multivariate PD random vector with support in $(0\ 1)^N$.

The loss experienced by the bank during the time horizon is then a fraction of $E = \sum_{j=1}^{N}\sum_{i=1}^{M^j} e^{ji}$. For simplicity, the traditional distinction between expected loss—to be absorbed by provisions—and unexpected loss—to be absorbed by equity—vanishes, and the focus is placed on the total volume of losses the bank is determined to absorb with its own resources, which we denote as *capital*.

Hence, the bank's objective is to hold capital η, $0 \leq \eta \leq E$, large enough to guarantee a pre-defined solvency condition related to L. We consider the standard condition in which η is the Value at Risk (VaR) of L at the coverage level u, $0 < u < 1$. That is, η is such that:

$$u = P\left(L \leq \eta\right) \quad (2)$$

for a given, pre-defined, u.

Different $\boldsymbol{F}$ distributions result in different L distributions. We focus on the case in which the bank is interested in obtaining an unconditional capital estimate, that is, one offering a picture of the portfolio risk profile that is not subjected to any economic scenario. This is the type of measurement that is also followed in the

[1] However, our analysis can easily be extended to the multi-period framework, Duffie et al. (2007).

regulatory framework, BCBS (2011), and requires considering an unconditional PD distribution, which we denote as $\boldsymbol{F}^* = \left(F^{*1}, ..., F^{*N}\right)$.

We consider the unconditional PD model proposed by Ferrer et al. (2014). That is, $\boldsymbol{F}^*$ is defined as the equally weighted mixture, or simply, mixture, of the conditional PD distributions $\boldsymbol{F}_t = \left(F_t^1, ..., F_t^N\right)$ included in a *time window* large enough to properly capture the long-term behavior of the portfolio's default risk. This time window should include both recessions and expansions to avoid biases. We frequently refer to it throughout the paper as the *full-business-cycle* time window to emphasize this alleged feature. We denote it by the stint $t = 1, ..., T$, so that it has length T. Therefore, $\boldsymbol{F}^*$ is the mixture of the conditional distributions $\boldsymbol{F}_t$, $t = 1, ..., T$. This approach is significantly different from the traditional static formulations, like the model of Vasicek, Vasicek (2002), which underpins the regulatory framework.

The conditional PD distributions that form $\boldsymbol{F}^*$ reflect the prevailing economic environment in t. They are identified on the basis of the vector $\boldsymbol{h}_t = \left(h_t^1, ..., h_t^N\right)$ representing the hazard rates of the portfolio, where h_t^j is the realized default rate for risk unit j during period t. It follows that $\boldsymbol{F}_t \sim \boldsymbol{H}_t$, with $\boldsymbol{H}_t = \left(H_t^1, ..., H_t^N\right)$ being the conditional distribution of $\boldsymbol{h}_t$ in t, given all available information up to $t-1$. This means that identifying $\boldsymbol{F}_t$ usually requires previously identifying a dynamic model for $\boldsymbol{h}_t$, see Pesaran et al. (2006).

As noted by Heitfield et al. (2006), the loss model presented in Eq. 1 has two sources of uncertainty: Systemic risk, which is generated by $\boldsymbol{F}^*$, and idiosyncratic risk, which is generated by the M Bernoulli variables. The latter are independent once the former has been realized, hence the name "conditional independence".

However, we only focus on systemic risk in this paper, which makes the analysis clearer without undermining the generality of the results obtained. Thus, the loss model in Eq. 1 is reformulated to assume infinitely fine-grained exposures, Gordy (2003), so that there is no need to characterize debtors individually. L is then defined as follows:

$$L = \sum_{j=1}^{N} L^j = \sum_{j=1}^{N} e^j F^{*j} \tag{3}$$

with $\boldsymbol{e} = \left(e^1, ..., e^N\right)$ being the vector of total exposure of each risk unit, $e^j > 0$.

2.2. Monte Carlo capital estimation

The basic Monte Carlo estimation of η involves the following steps for the model described in Eq. 3:

1. Simulating the vector $\boldsymbol{F}^* = \left(F^{*1}, ..., F^{*N}\right)$ G times.
2. Using the previous values to obtain G realizations of L.
3. Estimating the u percentile of the L realizations obtained in the previous step.

Since u usually takes a high value, like[2] 99.9 in the regulatory case, the value of G required to properly estimate η is also high. This results in a significant computational cost, even if no idiosyncratic risk is taken into account[3].

In order to mitigate this cost, several efficient simulation strategies have been put in place, Glasserman et al. (2008). Their objective are to increase the precision of the estimation of η for a given u, although they are usually designed to achieve the opposite: An efficient estimation of u for a given η. This is due to the poor tractability of the percentile function and gives rise to the next general process:

1. Obtaining, via standard Monte Carlo simulation, analytical approximation or any other heuristic, an initial estimate of η, η^0.
2. Designing an efficient simulation algorithm to estimate u^0, $u^0 = P\left(L \leq \eta^0\right)$.
3. Using the simulation algorithm previously designed to estimate η, given u.

The problem of estimating u^0 given η^0 can be seen as the problem of estimating the mean of K, which is a random variable taking a value of 1 if $L \leq \eta^0$ and 0 otherwise, therefore satisfying $u^0 = E\left[K\right]$. Then, the objective of the efficient simulation algorithm is to minimize the variance of the classic estimate of $E\left[K\right]$, $\phi = {}^1\!/_G \sum_{c=1}^{G} \kappa^c$, where κ^c takes a value of 1 if $l^c \leq \eta^0$ and 0 otherwise, with l^c being one of the G simulated values of L.

One efficient simulation technique is the *importance sampling* method, Glasserman and Li (2005). In short, this method is based on the idea of replacing the distribution $\boldsymbol{F}^*$ by another continuous distribution: $\boldsymbol{B}^* = \left(B^{*1}, ..., B^{*N}\right)$. $\boldsymbol{B}^*$ more often generates values in the tile of L and therefore improves the estimation of η.

Obviously, the use of $\boldsymbol{B}^*$ instead of $\boldsymbol{F}^*$ induces a bias that has to be eliminated. To do so, if l^c has been generated by the realization $\boldsymbol{b}^{*,c} = \left(b^{*1,c}, ..., b^{*N,c}\right)$ of $\boldsymbol{B}^*$, then l^c obtains a *correction factor* ω^c:

$$\omega^c = \frac{\phi^{\boldsymbol{F}^*}\left(\boldsymbol{b}^{*,c}\right)}{\phi^{\boldsymbol{B}^*}\left(\boldsymbol{b}^{*,c}\right)} \tag{4}$$

where $\phi^{\boldsymbol{F}^*}$ is the N-dimensional density function of $\boldsymbol{F}^*$ and $\phi^{\boldsymbol{B}^*}$ is the N-dimensional density function of $\boldsymbol{B}^*$. That is, the correction factor ω^c of each simulated value l^c, $c = 1, ..., G$, is inversely proportional to the relative importance that it receives in the simulation under $\boldsymbol{B}^*$.

These correction factors, normalized to one, $\dot{\omega}^c = {}^{\omega^c}\!/_{\sum \omega^c}$, are used in the percentile function. Thus, η is the simulated value satisfying that the sum of the normalized

[2]Throughout the paper, coverage values are expressed in percentage points.

[3]If idiosyncratic risk is considered, as in the model of Eq. 1, the computational cost is much higher, since M ranges between hundreds of thousands and millions for a standard commercial bank.

correction factors for the simulated values lower than it equals[4] u,

$$\sum_{l^c \le \eta} \dot{\omega}^c = u \tag{5}$$

That is, instead of having a weight $1/G$ in the percentile function, each simulated value obtains its normalized correction factor $\dot{\omega}^c$.

2.3. Capital allocation

The allocated, or diversified, capital ς^j, $j = 1, ..., N$, is obtained as $\varsigma^j = \eta r^j$, with $\boldsymbol{r} = \left(r^1, ..., r^N\right)$, $0 \le r^j \le 1$ being the capital allocation vector.

While Monte Carlo capital estimation refers to the efficient simulation literature, the problem of capital allocation is strongly influenced by that of defining a proper measure of risk according to some reasonable principles[5], Artzner et al. (1999). Thus, some desirable properties have been defined for $\boldsymbol{r}$, Denault (2001) and Kalkbrener (2005). The most important is $\sum_{j=1}^{N} r^j = 1$, so that the capital is completely allocated and the sum of the diversified capitals equals the aggregated capital. Other properties, like diversification, $\varsigma^j \le \eta^j$, $j = 1, ..., N$, or continuity, meaning that small changes in $\boldsymbol{e}$ result in small changes in $\boldsymbol{r}$, are usually checked *a posteriori*.

Based on the Euler principle, Tasche (2007),

$$\varsigma^j = \frac{\partial \eta}{\partial e^j} e^j \tag{6}$$

two popular allocation methods are the variance-covariance method (VC) and the expected shortfall method (ES), see Kalkbrener et al. (2004) and Tasche (2007).

The VC method is related to the use of VaR as a risk measure and assumes a decomposition of L^j as $L^j = \alpha^j + \beta^j L + \varepsilon^j$, with $\boldsymbol{\varepsilon} = \left(\varepsilon^1, ..., \varepsilon^N\right)$, $E\left[\boldsymbol{\varepsilon}\right] = \boldsymbol{0}$. Ideally, $\boldsymbol{\varepsilon}$ is normally distributed although the method is used regardless of whether this condition holds. The diversified capital is then obtained as $E\left[L^j / L = \eta\right]$ by substituting α^j and β^j with their OLS estimates:

$$\varsigma^j = (\eta - \mu) \frac{C\left[L^j, L\right]}{V\left[L\right]} + \mu^j \tag{7}$$

where μ is the mean of L, μ^j is the mean of L^j and $V\left[\cdot\right]$ and $C\left[\cdot, \cdot\right]$ are, respectively, the variance and covariance operators. Therefore,

$$r^j = \frac{(\eta - \mu) C\left[L^j, L\right]}{\eta V\left[L\right]} + \frac{\mu^j}{\eta} \tag{8}$$

The ES method is related to the use of expected shortfall as a risk measure and focuses on the tail dependence of the stand-alone loss distributions L^j, $j = 1, ..., N$, which is considered a relevant source of capital diversification, Zhou (2010). The value of r^j is given by

$$r^j = \frac{E\left[L^j / L > \eta\right]}{E\left[L / L > \eta\right]} \tag{9}$$

which takes advantage of the fact that $E\left[L/L > \eta\right] = \sum_{j=1}^{N} E\left[L^j / L > \eta\right]$.

Both methods satisfy the condition of complete allocation with $0 \le r^j \le 1$, $j = 1, ..., N$. VC is simple and intuitive, but it may result in a poor treatment of the tile dependence. On the other hand, ES better captures tail dependence, but it may demand a costly estimation, since $E\left[L^j / L > \eta\right]$ is the mean of an extreme tile, Yamai and Yoshiba (2005).

3. Proposed efficient simulation algorithm

3.1. Definition

All the conditional distributions $\boldsymbol{F}_t$, $t = 1, ..., T$, that form $\boldsymbol{F}^*$ receive the same weight, $1/T$, which forms the vector $\boldsymbol{v} = (1/T, ..., 1/T)$ of equal mixture weights. This vector can be interpreted as the percentage of the G simulations of $\boldsymbol{F}^*$ to be drawn from each conditional PD distribution, or, equivalently, the realizations of L that come from each conditional distribution L_t, $t = 1, ..., T$.

The idea behind the proposed simulation algorithm is to replace the vector $\boldsymbol{v}$ with an alternative vector $\boldsymbol{w} = (w_1, ..., w_T)$, with $0 \le w_t \le 1$ and $\sum_{t=1}^{T} w_t = 1$. $\boldsymbol{w}$ would form $\boldsymbol{F}^*$ as an unequally weighted mixture of conditional PD distributions, and then L as an unequally weighted mixture of conditional loss distributions. This is a more general formulation than the original one, since $\boldsymbol{v}$ is, in fact, a particular case of $\boldsymbol{w}$.

The objective is, then, to choose $\boldsymbol{w}$ in order to ensure a gain in the efficiency of the estimation of u^0—and, therefore, in the estimation of η—with respect to the standard Monte Carlo simulation, which uses $\boldsymbol{v}$. Proposition A states the optimal $\boldsymbol{w}$ for a given number of simulations, G.

Proposition A.

Let L be the loss distribution of Eq. 3 with $\boldsymbol{F}$ given by the mixture of the conditional PD distributions $\boldsymbol{F}_t$, $t = 1, ..., T$, with vector of weights $\boldsymbol{w} = (w_1, ..., w_T)$, and let K be a random variable taking a value of 1 if $L \le \eta^0$ and 0 otherwise, with $0 < \eta^0 < \sum_{j=1}^{N} e^j$. If $\left\{\kappa^1, ..., \kappa^G\right\}$ is a collection of G realizations of K, then the vector $\boldsymbol{w}$ that minimizes the variance of the classic estimate of $E\left[K\right]$, $\phi = 1/G \sum_{c=1}^{G} \kappa^c$, is given by

$$w_t = \frac{\sigma_t}{\sum_{t=1}^{T} \sigma_t} \tag{10}$$

where σ_t is the standard deviation of K_t, a random variable taking a value of 1 if $L_t \le \eta^0$ and 0 otherwise, with

[4] As also happens with the standard percentile estimation, interpolation between adjacent values may be necessary.

[5] Banks, however, may use different risk measures for obtaining and allocating capital, as noted by BCBS (2009). Therefore, the choice of VaR for the Monte Carlo efficient simulation analysis does not constrain our discussion of the capital allocation problem.

L_t being the loss distribution of Eq. 3 with $\boldsymbol{F}_t$ as the PD distribution.

Proof.

See Appendix A.

■

As mentioned in Section 2, the replacement of $\boldsymbol{v}$ by $\boldsymbol{w}$ induces a bias that has to be eliminated through the consideration of correction factors. Thus, since the conditional loss distribution L_t receives a weight w_t instead of $v_t = {}^1\!/T$, its correction factor is

$$\omega_t = \frac{1}{Tw_t} \tag{11}$$

Thus, the realized value l_t^c, $t = 1, ..., T$, $c = 1, ..., G_t$, $G_t = Gw_t$, of L receives the normalized correction factor

$$\dot{\omega}_t^c = \frac{1}{Tw_tG} \tag{12}$$

since then

$$\sum_{t=1}^{T}\sum_{c=1}^{G_t}\dot{\omega}_t^c = \sum_{t=1}^{T} G_t\frac{1}{Tw_tG} = \sum_{t=1}^{T} Gw_t\frac{1}{Tw_tG} = 1 \tag{13}$$

Therefore, implementing the proposed efficient simulation algorithm requires the following process:

1. Obtaining, via standard Monte Carlo simulation, analytical approximation or any other heuristic, an initial estimate of η, η^0.
2. Estimating, for each conditional loss distribution L_t, $t = 1, ..., T$, the weight w_t, which in turn requires estimating σ_t.
3. Simulating G_t values of each distribution L_t, with[6] $G_t = Gw_t$.
4. Assigning the normalized correction factor $\dot{\omega}_t^c$ to each simulated value l_t^c, $t = 1, ..., T$, $c = 1, ..., G_t$.
5. Estimating η, which is the simulated value satisfying that the sum of the normalized correction factors for the simulated values lower than it equals u.

3.2. Discussion

Three points can be drawn about the proposed efficient simulation algorithm.

First, it is different from the standard solutions proposed in the literature, like the exponential twisting or the mean shifting mechanisms, Glasserman et al. (2008), since it preserves the conditional PD distributions $\boldsymbol{F}_t$, $t = 1, ..., T$, and the structure of $\boldsymbol{F}^*$ as a mixture, albeit with different weights. That is, it involves a change of measure in a discrete distribution, rather than a continuous one, which simplifies the obtention of the optimal calibration.

On the other hand, the simulation process under the proposed algorithm is almost identical to the standard Monte Carlo case, with the only difference being the use of an unequal allocation of simulations among the T conditional loss distributions. This feature facilitates the implementation of the algorithm.

Second, it is easy to apply to a more general loss model including idiosyncratic risk, like the model presented in Eq. 1. In that case, the algorithm operates in the same way, with the total number of simulations G allocated among the conditional loss distributions L_t, $t = 1, ..., T$, according to $\boldsymbol{w}$. This vector would also be calibrated in terms of the standard deviation of K_t, with L_t comprising both systemic and idiosyncratic risk this time. In fact, our method can be used in conjunction with an efficient simulation strategy for the idiosyncratic risk, like the Berry-Esseen inequality, Frey et al. (2008).

Third, only Step 2 demands a specific calculation. However, this step can be merged with Step 1, so that with a reduced initial Monte Carlo simulation of L under $\boldsymbol{v} = ({}^1\!/T, ..., {}^1\!/T)$, both η^0 and σ_t, $t = 1, ..., T$, can be obtained.

Since the tile of L is formed by conditional distributions L_t related to recession periods, the weights w_t, $t = 1, ..., T$, are expected to present a cyclical evolution, being higher than ${}^1\!/T$ in recessions and lower than it in expansions. This alleged cyclicality makes $\boldsymbol{w}$ suitable for uses other than efficient simulation, since it signals expansions and recessions in terms of the portfolio's default risk. In fact, due to its normalized condition, $\sum_{t=1}^{T} w_t = 1$, it can be employed to obtain weighted averages of metrics collected during the full-business-cycle time window. This use is especially appealing if the objective is to define a long-term metric capturing conditional information and being sensitive to the cyclicality of default risk. In the next section we explore this idea by applying $\boldsymbol{w}$ to the problem of capital allocation.

4. Proposed capital allocation method

4.1. Definition

The allocation vector $\boldsymbol{r}$ is usually directly defined in terms of L and the stand-alone loss distributions L^j, $j = 1, ..., N$, as happens with the VC and ES allocation methods described in Section 2. The model chosen for $\boldsymbol{F}^*$, however, allows for a more structural approach.

Thus, we propose an alternative allocation vector, $\boldsymbol{r}^* = \left(r^{*1}, ..., r^{*N}\right)$, based on the conditional allocation vectors $\boldsymbol{r}_t$, $t = 1, ..., T$. These vectors are obtained using other subjacent allocation method that is applied to the conditional aggregated capital η_t and is derived from the conditional loss distributions L_t and $\boldsymbol{L}_t = \left(L_t^1, ..., L_t^N\right)$.

To aggregate the vectors $\boldsymbol{r}_t$, $t = 1, ..., T$, we explore the approach suggested at the end of Section 3. That is, r^{*j} is defined as follows:

$$r^{*j} = \sum_{t=1}^{T} w_t r_t^j \tag{14}$$

[6] Obviously, since G_t must be an integer, it may be necessary to round Gw_t.

where $\boldsymbol{w}$ is the vector of optimal weights of the proposed efficient simulation algorithm used for the estimation of η.

Therefore, the proposed allocation vector for the unconditional capital is formed as a weighting average of the conditional allocation vectors related to the full-business-cycle time window. The weighting function comes from an efficient simulation algorithm and the conditional allocation vectors are generated by a subjacent allocation method. Proposition B shows that $\boldsymbol{r}^*$ satisfies the condition of complete allocation if the subjacent method itself satisfies it.

Proposition B.

Let $\boldsymbol{r}_t = \left(r_t^1, ..., r_t^N\right)$ *be a vector satisfying* $0 \leq r_t^j \leq 1$ *and* $\sum_{j=1}^N r_t^j = 1$, $t = 1, ..., T$. *If* $\boldsymbol{w} = (w_1, ..., w_T)$ *satisfies* $0 \leq w_t \leq 1$ *and* $\sum_{j=1}^N w_t = 1$, *then the vector* $\boldsymbol{r}^* = \left(r^{*1}, ..., r^{*N}\right)$, *defined as*

$$r^{*j} = \sum_{t=1}^{T} w_t r_t^j \tag{15}$$

satisfies $0 \leq r^{*j} \leq 1$ *and*

$$\sum_{j=1}^{N} r^{*j} = 1 \tag{16}$$

Proof.
See Appendix B.

■

4.2. Discussion

The proposed method can be regarded as a radically different approach to the problem of capital allocation due to (i) the definition of the unconditional allocation vector in terms of the conditional allocation vectors included in the time window, and (ii) given the previous point, their aggregation by means of a weighted average based on an efficient simulation algorithm.

The first point mirrors the approach followed in $\boldsymbol{F}^*$. Thus, while $\boldsymbol{F}^*$ is an unconditional PD distribution built on the conditional PD distributions $\boldsymbol{F}_t$, $t = 1, ..., T$, $\boldsymbol{r}^*$ is an unconditional allocation vector built on the conditional allocation vectors $\boldsymbol{r}_t$, $t = 1, ..., T$. In other words, the unconditional capital η is both obtained and allocated through the conditional loss distributions L_t, $t = 1, ..., T$. This completes a whole family of unconditional capital metrics that are either aggregated, η, stand-alone, η^j, or diversified, ς^j, all of them based on the conditional loss distributions included in the full-business-cycle time window.

The second point establishes a link between two seemingly unrelated default risk issues—the efficient estimation of η via Monte Carlo simulation and its allocation among the portfolio's risk units—under the central idea that the conditional loss distributions that contribute more to the determination of η are also the conditional distributions that contribute more to its allocation.

Both points make the proposed allocation method a novel approach with no precedents in the literature. Still, some analogies can be drawn with other contributions. Thus, Cherny (2009) considers the case of capital allocation based on dynamic risk measures, Riedel (2004). However, the framework is different: Dynamic risk measures are applied to stochastic processes instead of random variables, while the proposed allocation method obtains T static allocation vectors along the time window and integrates them to form a single unconditional allocation vector. Laeven and Goovaerts (2004) also consider conditional allocations, but their study does so with a different purpose, capital optimization, and does not integrate them to form an unconditional allocation vector. Merino and Nyfeler (2004) apply an importance sampling algorithm to the problem of calculating the ES allocation vector, which is different from defining a new allocation method that uses as input a vector derived from an efficient simulation algorithm for the aggregated capital, as happens with our approach.

Three additional points can be drawn about $\boldsymbol{r}^*$.

First, it can be understood as a meta-allocation, in the sense that it serves as a mechanism for applying an allocation method to the unconditional capital by using the conditional loss distributions as an instrument.

Second, since estimating and allocating the unconditional capital are disjointed processes, it can be used regardless of the unconditional PD model that is used to determine η. Thus, it is possible to use a static PD model for the former and then resort to the conditional loss distributions and the proposed method for the latter.

Third, in practice, $\boldsymbol{r}^*$ will allocate the unconditional capital according, mainly, to the behavior exhibited by the hazard rate series during recession periods. The rest of the time window has a lesser influence or even no influence at all.

5. Empirical analysis

5.1. Data, dynamic models and credit portfolios

We use the same set of hazard rate series and the same univariate dynamic models as those employed by Ferrer et al. (2014).

In the case of the hazard rate series, this means using the quarterly charge-off series provided by the FDIC[7] for "Mortgages" (1-4 Family Residential Real Estate Loans), "Business" (Commercial & Industrial Loans to U.S. Addressees), "Credit Cards" (Credit Cards), "Individuals" (Other Loans to Individuals), "Rest" (All Other Loans)

[7] Federal Deposit Insurance Corporation, see *http://www.fdic.gov/*.

and "Lease" (Lease Financing Receivables) between 1991Q1-2010Q4. We consider the stint 1991Q1-2010Q4 as the full-business-cycle time window.

Table 1 summarizes their main statistics and Figure 1 shows the six series together. Hazard rate series are non-stationary and exhibit a cyclical pattern with some degree of heterogeneity.

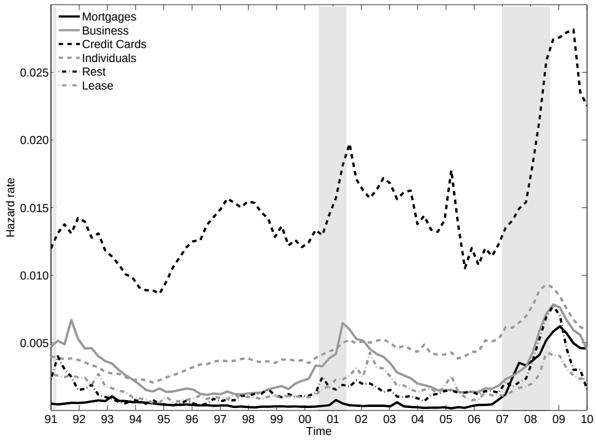

Figure 1: Hazard rates series. Shaded areas indicate U.S. recessions according to NBER (*National Bureau of Economic Research*).

The dynamic model for each hazard rate series is ARIMA with probit link function. That is, a univariate ARIMA model with normal innovations is fitted to each series $x_t^j = N^{-1}\left(h_t^j\right)$. This formulation generates conditional distributions of the form $F_t^j = H_t^j = N\left(X_t^j\right)$, with X_t^j being the conditional distribution of x_t^j given all the available information up to $t-1$. Table 2 summarizes the univariate ARIMA models.

Each hazard rate series represents a stand-alone portfolio formed by a single risk unit, so that there are six portfolios. For simplicity, we assume that the total exposure of each portfolio satisfies $e = 1$, so that $L = eF = F$ has support in $(0\ 1)$.

Since we are studying the allocation of capital, we also need an aggregated portfolio. To this purpose, we formulate the multivariate dynamic model as a Seemingly Unrelated Time Series Equations (SUTSE) structure, Fernández and Harvey (1990), comprising the six ARIMA models, as in Wilson (1997a,b) and Nyström and Skoglund (2006), among others. Similarly, we assume that $\boldsymbol{e} = \left(e^1, ..., e^N\right) = (1, ..., 1)$, so that $L = \sum_{j=1}^{6} L^j = \sum_{j=1}^{6} F^j$ has support in $(0\ 6)$. This choice means that all risk units have the same weight in the portfolio, which is unrealistic but facilitates the allocation analysis.

Therefore, there are six stand-alone portfolios and one aggregated portfolio. We use 99.9 as coverage level, which is the Basel coverage.

On the basis of these series, models, and portfolios, we study the behavior of $\boldsymbol{w}$, the gain in efficiency achieved by the proposed simulation algorithm and the performance of the proposed allocation method.

5.2. Results

Table 3 presents the unconditional capital figures estimated by means of standard Monte Carlo simulation with 1,000,000 realizations. Based on these estimates, which are taken as η^0, and on the 12,500 realizations of L generated from each conditional loss distribution, Figure 2 compares, standardized, the hazard rate series and the weights w_t for each stand-alone portfolio. Analogous information for the aggregated portfolio is presented in Figure 3.

Portfolio	η
Mortgages	0.0088
Business	0.0103
Credit Cards	0.0329
Individuals	0.0108
Rest	0.0115
Lease	0.0064
Aggregated	0.0713

Table 3: Unconditional capital for the 99.9 coverage level.

Both figures reveal the cyclical and asymmetric pattern exhibited by w_t, which is equal to zero in most of the time window and significantly higher than $v_t = 0.0125$ in a small group of periods. The fact that positive weights are mainly related to the Great Recession outlines the severity of this crisis with respect to the two previous downturns. It also suggests that $\boldsymbol{w}$ can help identify and rank different economic scenarios, either observed or synthetically defined, according to their severity. It would then be an alternative approach to the problem of assessing the plausibility and severity of adverse scenarios, Breuer et al. (2009).

Obtaining a weight $w_t = 0$ means that all 12,500 realized values of L_t are lower than η^0, so that the standard deviation estimate of K_t equals zero. Obviously, $u_t^0 = E\left[K_t\right] = P\left(L_t \leq \eta^0\right)$ is strictly lower than 1 since L_t has support in $(0\ 6)$, but the value is so close to 1 that $\phi_t = 1$ under the 12,500 sample.

If a conditional loss distribution receives a weight equal to zero, $w_t = 0$, then no realizations are generated from it and therefore there is no way to avoid the induced bias through the correction factor ω_t. In that situation, it is the unconditional coverage level u that has to be adjusted.

Thus, two groups of conditional loss distributions, A and B, can be formed. The conditional loss distributions included in A satisfy $P\left(L_t \leq \eta\right) < 1$, and for those included in B let us assume that $P\left(L_t \leq \eta\right) = 1$. If L_A denotes the mixture of the conditional distributions in A,

Series	*Mean*	*Std. Dev.*	*Min*	*Median*	*Max*	*JB test*	*ADF test*
Mortgages	0.0010	0.0015	0.0002	0.0004	0.0062	0.0010	0.9667
Business	0.0030	0.0018	0.0012	0.0023	0.0078	0.0133	0.3653
Credit cards	0.0148	0.0044	0.0086	0.0138	0.0282	0.0010	0.8222
Individuals	0.0044	0.0016	0.0021	0.0039	0.0093	0.0016	0.7740
Rest	0.0017	0.0014	0.0004	0.0013	0.0077	0.0010	0.1569
Lease	0.0017	0.0009	0.0005	0.0014	0.0043	0.0075	0.2294

Table 1: Main stats for the hazard rate series. The p-value is presented for the Jarque-Bera test (JB test) and the Augmented Dickey-Fuller test (ADF test).

Risk Unit	ρ_1		ρ_2		θ_1		θ_2		$\widehat{\sigma_a}$	*LBQ(16)*	*AIC*	*SBC*
	$\widehat{\rho_1}$	$\widehat{\sigma}_{\widehat{\rho_1}}$	$\widehat{\rho_2}$	$\widehat{\sigma}_{\widehat{\rho_2}}$	$\widehat{\theta_1}$	$\widehat{\sigma}_{\widehat{\theta_1}}$	$\widehat{\theta_2}$	$\widehat{\sigma}_{\widehat{\theta_2}}$				
Mortgages	-	-	0.2120	0.1070	-	-	-	-	0.0670	0.0913	-2.44	-2.38
Business	0.7602	0.1137	-	-	0.4882	0.1456	-	-	0.0435	0.4473	-3.25	-3.16
Credit Cards	-	-	-	-	-	-	-	-	0.0359	0.1160	-3.72	-3.69
Individuals	0.6230	0.1443	-	-	0.5208	0.2090	-0.2886	0.1183	0.0208	0.1218	-4.69	-4.57
Rest	-	-	-	-	-	-	-	-	0.0869	0.9121	-1.98	-1.95
Lease	-	-	-	-	-	-	-	-	0.0712	0.8821	-2.37	-2.34

Table 2: Univariate ARIMA models fitted to x_t^d, $x_t^d = x_t - x_{t-1}$, being $x_t^d = \rho_1 x_{t-1}^d + \rho_2 x_{t-2}^d - \theta_1 a_{t-1} - \theta_2 a_{t-2} + a_t$ and $V[a_t] = (\sigma_a)^2$. $\hat{\beta}$, estimated parameter. $\widehat{\sigma}_{\hat{\beta}}$, estimated standard deviation of $\hat{\beta}$. LBQ(16), Ljung–Box Q test p-value with 16 lags. AIC, Akaike Information Criteria. SBC, Schwarz Information Criteria.

and correspondingly with L_B, then:

$$\begin{aligned} u &= P(L \leq \eta) \\ &= \frac{A^\#}{T} P(L_A \leq \eta) + \frac{B^\#}{T} P(L_B \leq \eta) \\ &= \frac{A^\#}{T} P(L_A \leq \eta) + \frac{B^\#}{T} \end{aligned} \tag{17}$$

where $A^\#$ and $B^\#$ are the cardinals of groups A and B. Therefore,

$$\begin{aligned} P(L_A \leq \eta) &= \frac{T}{A^\#}\left(u - \frac{B^\#}{T}\right) \\ &= \frac{Tu - B^\#}{A^\#} \\ &= \frac{Tu - B^\#}{T - B^\#} \end{aligned} \tag{18}$$

This means that, if η is estimated only by means of L_A instead of L, which is the implication of having weights $w_t = 0$ for $L_t \in B$, then the unconditional coverage level must be adjusted to

$$u' = \frac{Tu - B^\#}{T - B^\#} \tag{19}$$

In other words, the equivalent to u in L is u' in L_A. Since $u \geq \frac{B^\#}{T}$ by construction, u' is always positive. It is also lower than u since

$$\begin{aligned} u' < u \quad &\Leftrightarrow \quad \frac{Tu - B^\#}{T - B^\#} < u \\ &\Leftrightarrow \quad Tu - B^\# < Tu - B^\# u \\ &\Leftrightarrow \quad B^\# > B^\# u \end{aligned} \tag{20}$$

which is satisfied since $u < 1$.

Therefore, obtaining weights w_t equal to zero brings the additional advantage of defining η as a less extreme percentile, which benefits the accuracy of the estimation. It also simplifies the application of the proposed allocation method because the group of vectors $\boldsymbol{r}_t$ to be estimated is smaller.

With respect to the simulation algorithm, we run a competing test of both Monte Carlo estimation methods, standard and proposed, as in Merino and Nyfeler (2004) and Glasserman and Li (2005), among others. The comparison consists of the following: Given a portfolio, 50 estimates of η are obtained by means of each simulation algorithm. For each portfolio and method, Table 4 presents the mean and standard deviation of the 50 η estimates, as well as the ratio of standard deviations as an intuitive metric of efficiency gain.

Results show trivial differences between the mean of the 50 estimates obtained by each algorithm, which are only attributable to the Monte Carlo estimation error. More striking is the dramatic reduction in the standard deviation of the 50 η estimates achieved by the proposed simulation method. It is, on average, approximately a fourth of that obtained if no efficient simulation technique is used. For the aggregated portfolio, for example, the reduction in the standard deviation reaches 81%. These results clearly support the proposed method: Remarkable efficiency improvement with no induced bias at a negligible implementation cost.

Regarding capital allocation, we compare the VC allocation applied in its standard formulation and in that given by the proposed method. We also obtain the standard ES allocation for contrasting purposes.

Portfolio	ϕ_η^{ST}	ϕ_η^{PR}	σ_η^{ST}	σ_η^{PR}	$\sigma_\eta^{PR}/\sigma_\eta^{ST}$
Mortgages	0.008826	0.008815	0.000061	0.000019	0.31
Business	0.010312	0.010314	0.000054	0.000012	0.22
Credit Cards	0.032896	0.032887	0.000096	0.000021	0.22
Individuals	0.010754	0.010759	0.000026	0.000004	0.17
Rest	0.011468	0.011460	0.000129	0.000029	0.23
Lease	0.006391	0.006398	0.000053	0.000014	0.26
Aggregated	0.071309	0.071332	0.000651	0.000124	0.19

Table 4: Comparison between the standard and proposed Monte Carlo simulation methods. 50 estimates of the unconditional 99.9 capital requirement are generated by each method. ϕ_η^{ST}, average of the 50 estimates generated by the standard method. ϕ_η^{PR}, average of the 50 estimates generated by the proposed method. σ_η^{ST}, standard deviation of the 50 estimates generated by the standard method. σ_η^{PR}, standard deviation of the 50 estimates generated by the proposed method.

Table 5 presents the three allocation vectors, while Table 6 presents the corresponding diversification ratios, obtained as[8] $\lambda^j = 1 - \varsigma^j/\eta^j$, $j = 1, ..., N$.

Portfolio	VC^{ST}	VC^{PR}	ES
Mortgages	0.0912	0.0917	0.0927
Business	0.1405	0.1333	0.1332
Credit Cards	0.4476	0.4268	0.4250
Individuals	0.1459	0.1286	0.1326
Rest	0.1046	0.1456	0.1444
Lease	0.0702	0.0740	0.0721

Table 5: For the aggregated unconditional 99.9 capital, allocation vectors of the standard variance-covariance method, VC^{ST}, the proposed method using the variance-covariance allocation as subjacent, VC^{PR}, and the standard expected shortfall method, ES.

Portfolio	VC^{ST}	VC^{PR}	ES
Mortgages	0.2608	0.2567	0.2483
Business	0.0258	0.0763	0.0765
Credit Cards	0.0273	0.0723	0.0762
Individuals	0.0311	0.1463	0.1200
Rest	0.3461	0.0895	0.0972
Lease	0.2166	0.1740	0.1955

Table 6: For the aggregated unconditional 99.9 capital, diversification ratios generated by the standard variance-covariance method, VC^{ST}, the proposed method using the variance-covariance allocation as subjacent, VC^{PR}, and the standard expected shortfall method, ES. Diversification ratios are defined as $\lambda^j = 1 - \varsigma^j/\eta^j$, where ς^j and η^j are, respectively, the diversified and stand-alone capital estimates of risk unit j.

Three main points can be derived from these tables.

First, the proposed approach satisfies, as happens with the standard VC and ES methods, the diversification property, since $\lambda^j > 0$, $j = 1, ..., N$,

Second, the proposed method generates an allocation closer to that of the ES method than it does the standard VC method. This can be observed in the mean of the absolute differences between allocation vectors, which is 0.0017 for the former case and 0.0144 for the latter. In terms of the diversification vectors, the values are 0.0144 and 0.0785, respectively. This fact is coherent with the way the proposed method is defined. Thus, similar to the ES method, the proposed allocation focuses, implicitly, on the tile of the loss distributions by defining $\boldsymbol{r}^*$ in terms of the conditional allocations related to the periods that generate such tiles. Moreover, under the standard VC method the allocation vector does not depend on u, since the variance-covariance matrix of $\boldsymbol{L} = \left(L^1, ..., L^N\right)$ does not depend on η. Under the proposed method, however, the allocation always depends on u since $\boldsymbol{w}$ itself depends on it. Therefore, if used as subjacent in the proposed method, the VC allocation is tailored to the particular value of η, as also happens with the ES method.

Third, a consequence of the previous point is that the proposed method generates an allocation more balanced than does the standard VC method. Thus, the standard deviation of the allocation vector is 0.1303 for the former and 0.1406 for the latter. In terms of diversification ratios these values are 0.0720 and 0.1413, respectively. This result is also consistent with Figure 3: Only the weights w_t related to the Great Recession are greater than zero, which means that $\boldsymbol{r}^*$ is formed by the conditional allocation vectors of a time when the hazard rate series present a remarkable similarity, as is shown in Figure 1. Consequently, a more homogeneous allocation of the aggregated diversification[9] is obtained when the VC method is used under our approach.

These points suggest that applying the proposed method with the standard VC allocation as subjacent can be a useful combination. It is a simple way to improve its low tail sensitivity without losing its simplicity and the ease of estimation. In other words, this combination generates an allocation that is similar to the standard ES method in results but to the standard VC method in implementation.

[8] The higher λ^j is, the higher the diversification granted to risk unit j, see Tasche (2007).

[9] Aggregated diversification is $\lambda = 1 - \frac{0.0713}{0.0807} = 0.1165$.

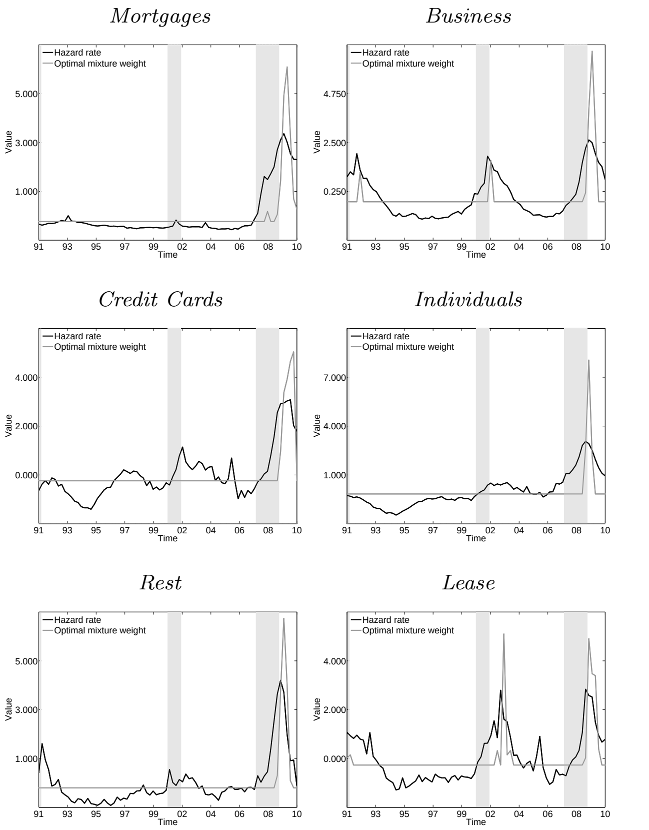

Figure 2: For each stand-alone portfolio, hazard rate series, h_t, and optimal mixture weights series, w_t. The latter is related to the unconditional 99.9 capital estimate. Both series are presented standardized. Shaded areas indicate U.S. recessions according to NBER (*National Bureau of Economic Research*).

6. Concluding remarks

This paper deals with two problems related to the determination of the unconditional capital required by a credit portfolio: its Monte Carlo estimation, and its allocation. Elaborating on the framework of Ferrer et al. (2014) for the unconditional PD distribution, which is formed as an equally weighted mixture of the conditional PD distributions, we have contributed to the existing literature in both problems.

For the Monte Carlo estimation problem, we have proposed a new efficient simulation algorithm inspired by the importance sampling framework. It is based on the idea of substituting the equally weighted mixture with a unequal one in which the conditional PD distributions related to economic downturns receive a higher weight. We have obtained analytically the optimal vector of unequal weights. The empirical analysis has shown a cyclical and asymmetric behavior of the optimal weights and a remarkable gain in efficiency with respect to the standard Monte Carlo estimation.

For the allocation problem, we have proposed a new

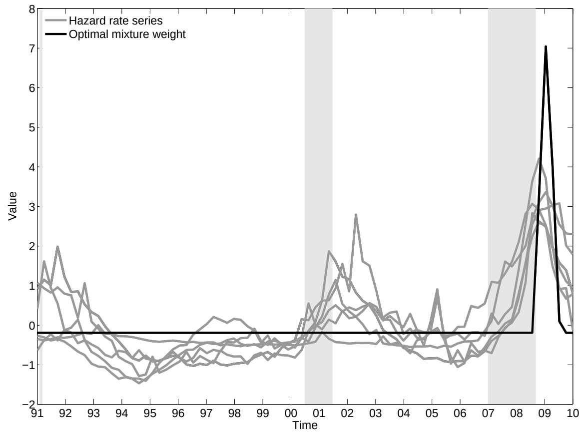

Figure 3: For the aggregated portfolio, hazard rate series, $\boldsymbol{h}_t$, and optimal mixture weights series, w_t. The latter is related to the unconditional 99.9 aggregated capital estimate. All series are presented standardized. Shaded areas indicate U.S. recessions according to NBER (*National Bureau of Economic Research*).

method that forms the unconditional allocation vector as a weighted average of the conditional allocation vectors included in the time window. The conditional allocations are obtained using another subjacent method and the weighting vector is the optimal mixture vector obtained for the proposed efficient simulation algorithm. We have proved that the proposed method satisfies the condition of complete allocation. The empirical analysis has shown an improvement in the variance-covariance allocation when used as subjacent in the proposed method.

Both proposals share the same underlying principle, which can be summarized with the idea that the conditional loss distributions that are more involved in the determination of the unconditional capital figure should also be more involved in its allocation.

As general conclusion, the paper supports the idea of analyzing and measuring unconditional default risk by means of the conditional loss distributions. Thus, the methods proposed are more than a separate, although connected, contribution to their respective problems; they are also evidence of the modeling possibilities that appear when the unconditional measurement is addressed from a conditional perspective. Obviously, neither of them could have been formulated under the static modeling framework.

Appendix A.

Proof of Proposition A.

The optimal weights w_t, $t = 1, ..., T$, are derived from the stratified sampling theory, see Kish (1965).

Since $\boldsymbol{F}$ is a mixture of the conditional distributions $\boldsymbol{F}_t$, $t = 1, ..., T$, L itself is also a mixture of the conditional distributions L_t, $t = 1, ..., T$. We can then formulate the simulation of G values of K as a stratified sampling process over T infinitely populated strata.

Thus, if no costs are considered, basic sampling theory states that the stratification that minimizes the variance of $\phi = {}^1/_G \sum_{c=1}^{G} \kappa^c$ is proportional to the standard deviation of K subjected to each strata; that is, to the standard deviation of K_t, which is σ_t.

In other words, the optimal number of sampled values drawn from each strata, G_t, $t = 1, ..., T$, is given by

$$G_t = G \frac{\sigma_t}{\sum_{t=1}^{T} \sigma_t} \quad \text{(A.1)}$$

Coming back to the mixture framework, this means forming L as the weighted mixture of the conditional loss distributions L_t, $t = 1, ..., T$, with the vector of weights $\boldsymbol{w}$ as follows:

$$w_t = \frac{\sigma_t}{\sum_{t=1}^{T} \sigma_t} \quad \text{(A.2)}$$

■

Appendix B.

Proof of Proposition B.

Since $0 \leq r_t^j \leq 1$, $0 \leq w_t \leq 1$ and $\sum_{j=1}^{N} w_t = 1$, r_t^{*j} satisfies $0 \leq \min_t \left\{ r_t^j \right\} \leq r^{*j} \leq \max_t \left\{ r_t^j \right\} \leq 1$.

On the other hand,

$$\begin{aligned} \sum_{j=1}^{N} r^{*j} &= \sum_{j=1}^{N} \sum_{t=1}^{T} w_t r_t^j \\ &= \sum_{t=1}^{T} \sum_{j=1}^{N} w_t r_t^j \\ &= \sum_{t=1}^{T} w_t \sum_{j=1}^{N} r_t^j \quad \text{(B.1)} \\ &= \sum_{t=1}^{T} w_t \\ &= 1 \end{aligned}$$

■

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