Initial field and energy flux in absorbing optical waveguides. I. Theoretical formalism

Maria Luisa Calvo* and Vasudevan Lakshminarayanan

School of Optometry, University of California, Berkeley, California 94720

Received May 19, 1986; accepted December 15, 1986

An exact formulation of the electromagnetic field striking an optical waveguide is presented and compared with the decomposition relationship used in the literature. The expression for the critical angle is derived as a natural consequence of this formulation. Equations for the total fraction of power confined within a waveguide are derived and analyzed for the special case of absorbing waveguides. An explicit expression is derived for the fraction of energy confined within a waveguide supporting two sets of modes.

1. INTRODUCTION

The confinement of energy within dielectric optical waveguides for both monomode and multimode behavior has been widely studied.[1–3] The generalization for the fraction of confined power can be obtained by including the absorption characteristics of the waveguide material. Two possible cases exist: (1) absorbing core with absorbing cladding and (2) absorbing core and nonabsorbing cladding. For both cases homogeneous and nonhomogeneous profiles can be studied. For case (2), direct and simple formalisms can be obtained through the geometrical optics approximation,[3] which does not work for case (1) for obvious reasons. An electromagnetic theory is needed to analyze the importance of attenuation on modal propagation within absorbing waveguides. In earlier studies by Snyder *et al.*,[4,5] two assumptions are made for the waveguide: (a) it is treated as if it were nonabsorbing in terms of the modal field, and (b) the absorption coefficient is considered a small parameter. A first approximation for the ratio of the real and imaginary parts of the complex modal wave number is then used in the resultant eigenvalue equation. The influence of the absorption in distribution of power within a single pulse has also been studied.[6]

In this paper we present an alternative treatment by formulating the incoming electromagnetic wave incident upon the entrance pupil of the waveguide as an exact three-dimensional scattering integral equation. To extend the formalism to the case of an absorbing optical waveguide, we make assumption (a) given above but treat assumption (b) as only an approximation of the more general case of relatively strong absorption. We assume a generalized refractive-index profile. In Section 2 the total field is established as the sum of the contribution of the initial scalar field and the field scattered by the waveguide. This generalized field formalism approximately equals the initial one as a superposition of discrete bound modes for weakly guiding waveguides. From the exact expression for the initial field, the acceptance angle for the penetration of the light ray into the waveguide is derived. The general formulation for the fraction of confined power in an absorbing waveguide is then calculated (Section 3), and the simple case of a waveguide supporting a set of fundamental modes is studied (Section 4). Here, some approximations are considered that would allow one to compare the results with those obtained for the dielectric case. These calculations are extended to an absorbing waveguide supporting two sets of confined modes in Section 5. In particular, attention has been paid to the contribution to the fraction of confined power within the absorbing core, where the dependence on the axial direction of the waveguide appears as an oscillatory term, and the implications of this term are discussed. We conclude with a short summary, discussing the applications of this formalism to optical waveguides whose absorbing properties may play an important role in the propagation of the confined energy.

2. MATHEMATICAL FORMULATION OF THE INITIAL FIELD AND PHYSICAL INTERPRETATION

Consider an initial field Ψ_{in} representing a monochromatic linearly polarized beam of light projected normally onto the entrance pupil of an optical waveguide of length Z_0, lying along the Z axis ($0 < Z < Z_0$). By following the usual decomposition, we can write[3,7]

$$\Psi_{in} = \sum_{\alpha} c_\alpha e_\alpha + \text{scattered field}, \tag{1}$$

where e_α is the modal field and the coefficients c_α are the amplitudes of the projections of Ψ_{in} of e_α (in the sense of the Fourier series in a Hilbert space).

For simplicity, let $Z_0 = +\infty$, and let the scattered field be equal to zero. This second assumption implies that all the incident energy is transmitted along the waveguide under strict confinement conditions. Equation (1) reduces to

$$\Psi_{in} = \sum_{\alpha} c_\alpha e_\alpha. \tag{2}$$

In this section we shall obtain an exact, explicit representation for the initial field Ψ_{in} and compare it with the representation given by Eq. (2).

As is customary, we assume that Ψ_{in} satisfies the scalar differential wave equation as it propagates in the waveguide

$$(\nabla^2 + \varepsilon_c K^2)\Psi_{in} = 0, \tag{3}$$

0740-3232/87/061037-06$02.00

where ∇^2 is the Laplacian operator, K is the wave vector, $K^2 = (\omega/c)^2$, and ω is the angular frequency. For simplicity, we will write the vector $\mathbf{x} = (x, y, z)$ as $(\mathbf{x}, z)$.

The total field (the incoming field, the field propagating along the waveguide, and the scattered field) satisfies

$$[\nabla^2 + \varepsilon_c K^2]\Psi(x) = \begin{cases} 0, & -\infty < z < 0 \\ -K^2(\hat{\varepsilon}(\mathbf{x}) - \varepsilon_c)\Psi(x), & 0 < z < +\infty \end{cases}, \tag{4}$$

with

$$\hat{\varepsilon}(\mathbf{x}) - \varepsilon_c \begin{cases} = 0 & \mathbf{x} \text{ outside } \Omega \\ \neq 0 & \mathbf{x} \text{ inside } \Omega \end{cases},$$

where Ω is the aperture cross section, $\hat{\varepsilon}(\mathbf{x})$ is a complex dielectric permittivity of the waveguide, and ε_c is the dielectric permittivity of the surrounding cladding. Note that the permittivity function is assumed to be a constant along the Z direction, and hence there is only a bidimensional $(\mathbf{x})$ dependence.

As in other general scattering phenomena, the total field can be formulated in terms of an integral equation[8] as

$$\Psi(x) = \Psi_{\text{in}} + \int_\Omega \mathrm{d}^2\mathbf{x}' \int_0^{Z_0} \mathrm{d}z' G(\mathbf{x} - \mathbf{x}', z - z') K^2[\hat{\varepsilon}(x') - \varepsilon_c]\Psi(\mathbf{x}'). \tag{5}$$

The function $G(\mathbf{x} - \mathbf{x}', z - z')$ is the standard three-dimensional Green function acting as a perturbation propagator and is defined as[9]

$$G(\mathbf{x} - \mathbf{x}', z - z') = \frac{1}{(2\pi)^3} \int \mathrm{d}^2\mathbf{q}\mathrm{d}q_z \frac{\exp\{i[\mathbf{q}(\mathbf{x} - \mathbf{x}') + q_z(z - z')]\}}{\mathbf{q}^2 + q_z{}^2 - (K^2 + i\eta)}. \tag{5a}$$

The right-hand side of Eq. (5) represents the contribution of the incident field Ψ_{in} plus that scattered by the waveguide. Ψ_{in} propagates freely away from the waveguide; as time evolves, Ψ_{in} interacts with the waveguide, is scattered by it, and eventually penetrates into it under conditions of total internal reflection.

We are interested in characterizing and finding a general expression for the initial field Ψ_{in} such that $\Psi(\mathbf{x})$ exactly equals a superposition of bound propagation modes:

$$\Psi(\mathbf{x}) = \sum_\alpha c_\alpha \phi_\alpha(\mathbf{x}) \exp(i\hat{\beta}_\alpha Z), \tag{6}$$

where c_α is a constant coefficient, $\phi_\alpha(\mathbf{x})$ is an $\mathbf{x}$-dependent modal field, and $\hat{\beta}_\alpha$ is the complex modal propagation constant, which is determined by ϕ_α and K^2 (see below). $\phi_\alpha(\mathbf{x})$ is a finite function that goes rapidly to zero as $|\mathbf{x}| \to \infty$. A similar analysis is given by Snyder.[10]

The corresponding scalar differential wave equation for $\phi_\alpha(\mathbf{x})$ is given by

$$\{\nabla^2 + K^2[\hat{\varepsilon}(\mathbf{x}) - \varepsilon_c]\}\phi_\alpha(\mathbf{x}) = \hat{\chi}_\alpha{}^2\phi_\alpha(\mathbf{x}), \tag{7}$$

with

$$K^2 = \hat{\beta}_\alpha{}^2 - \hat{\chi}_\alpha{}^2. \tag{7a}$$

For the case of optical waveguides with a real absorbing cladding, K^2 is always real; therefore

$$\text{Re}\, \hat{\beta}_\alpha \,\text{Im}\, \hat{\beta}_\alpha = \text{Re}\, \hat{\chi}_\alpha \,\text{Im}\, \hat{\chi}_\alpha,$$

and Eq. (7a) can be written as

$$K^2 = [(\text{Re}\, \hat{\beta}_\alpha)^2 - (\text{Re}\, \hat{\chi}_\alpha)^2] + [(\text{Im}\, \hat{\chi}_\alpha)^2 - (\text{Im}\, \hat{\beta}_\alpha)^2]. \tag{7b}$$

The dependence of $\text{Im}\, \hat{\chi}_\alpha$ and $\text{Im}\, \hat{\beta}_\alpha$ on the absorptive properties of the waveguide will be discussed below.

Alternatively, $\phi_\alpha(\mathbf{x})$ can be expressed as

$$\phi_\alpha(\mathbf{x}) = \frac{-1}{4i} \int_\Omega \mathrm{d}^2\mathbf{x}' H_0^{(1)}(i\hat{\chi}_\alpha|\mathbf{x} - \mathbf{x}'|) K^2[\hat{\varepsilon}(\mathbf{x}') - \varepsilon_c]\phi_\alpha(\mathbf{x}'). \tag{8}$$

$H_0^{(1)}$ is the zero-order Hankel function of first kind that is found in all propagation phenomena involving cylindrical geometry. The Hankel function is exponentially damped for large $|\mathbf{x}|$ ($|\mathbf{x}| \gg |\mathbf{x}'|$, as $\mathbf{x}'$ varies in Ω), since the function depends on $i\hat{\chi}_\alpha|\mathbf{x} - \mathbf{x}'|$.[11,12]

We define $H_0^{(1)}$ as

$$H_0^{(1)}(i\hat{\chi}_\alpha|\mathbf{x} - \mathbf{x}'|) = (1/\pi^2 i) \int \mathrm{d}^2\mathbf{q} \frac{\exp[i\mathbf{q}(\mathbf{x} - \mathbf{x}')]}{\mathbf{q}^2 + \hat{\chi}_\alpha{}^2}. \tag{8a}$$

If we substitute the expression for $\Psi(\mathbf{x})$ [Eq. (6)] into Eq. (5) for $z > 0$, use Eq. (5a) along with Eq. (8), and perform the appropriate integrations and algebraic manipulations, we get

$$\Psi_{\text{in}}(\vec{x}) = -\sum_\alpha c_\alpha \int_\Omega \mathrm{d}^2\mathbf{x}' K^2[\hat{\varepsilon}(\mathbf{x}') - \varepsilon_c]\phi_\alpha(\mathbf{x}') \times \frac{1}{4\pi^2} \int \mathrm{d}^2\mathbf{q} \exp[i\mathbf{q}(\mathbf{x} - \mathbf{x}')] \cdot \Lambda_{\alpha\beta}(\mathbf{q}, Z), \tag{9a}$$

with

$$\Lambda_{\alpha\beta}(\mathbf{q}, Z) = \frac{1}{2\sigma} \frac{e^{i\sigma Z}}{\sigma - \beta}, \qquad \sigma(q) = (K^2 - q^2)^{1/2}.$$

By using the Fourier transform of Eqs. (8) and (8a), we can cast Eq. (9a) into a more useful form as

$$\Psi_{\text{in}}(x) = \sum_\alpha c_\alpha (1/2\pi^2) \int \mathrm{d}^2\mathbf{q} \exp\left\{i\mathbf{q}\mathbf{x} \frac{[\beta + \sigma(q)]}{2\sigma(\mathbf{q})}\right\} \exp[i\sigma(\mathbf{q})Z] \times \left[\int \mathrm{d}^2\mathbf{x}'\phi_\alpha{}'(\mathbf{x}') \exp(-i\mathbf{q}x')\right]. \tag{9b}$$

We shall now consider under what conditions the exact representation [Eq. (9b)] can be approximated by Eq. (2).

The main contribution to the initial field in the above expression is obtained for values of the bidimensional propagation vector $\mathbf{q}$ less than or of the order of the modal constant $\hat{\chi}_\alpha$. On the other hand, when the waveguide optical properties are close to that of the surrounding medium, one has $\hat{\chi}_\alpha{}^2 \ll K^2$. This in fact represents the real behavior of a weak waveguide (for both transmission and absorption properties). Under these conditions, we can set

$$[\beta + \sigma(\mathbf{q})]/2\sigma(\mathbf{q}) \simeq 1,$$

and the exact field gets the approximate simpler form:

$$\Psi_{\text{in}} \simeq \sum_\alpha c_\alpha \phi_\alpha(\mathbf{x}) \exp[i\hat{\beta}_\alpha Z]. \tag{10}$$

By comparing this with Eq. (2), we find that $\phi_\alpha \exp(i\hat{\beta}_\alpha Z) = e_\alpha$.

If the waveguide has a circular cross section with radius R, and $\hat{\varepsilon}(\mathbf{x})$ is constant, the propagation modes are given by

$$\phi_\alpha(\mathbf{x}) = \exp(iM\phi)\begin{cases} C_< J_{|M|}(\hat{\chi}_\alpha'|\mathbf{x}|) & |\mathbf{x}| < R \\ C_> H_{|M|}^{(1)}(i\hat{\chi}_\alpha|\mathbf{x}|) & |\mathbf{x}| > R \end{cases}, \quad (11)$$

where $|M|$ is an integer representing the modal order. Thus $|M| = 0$ represents the monomode behavior in the waveguide, and the modal field is described by the Bessel function of zero order inside the waveguide; the Hankel function $H_0^{(1)}$ represents the modal field outside. $C_>$ and $C_<$ are constants that can be determined by the boundary conditions that, at $|\mathbf{x}| = R$, both ϕ_α and its normal derivative are continuous.

On the other hand, $\hat{\chi}_\alpha'$ and $\hat{\chi}_\alpha$ are complex values, with

$$\hat{\chi}_\alpha{}^2 + \hat{\chi}_\alpha{}^2 = (\omega/c)^2(\hat{\varepsilon} - \varepsilon_c). \quad (12)$$

In terms of the modal parameters,

$$\hat{U}_\alpha = \hat{\chi}_\alpha' R, \qquad \hat{W}_\alpha = \hat{\chi}_\alpha R, \qquad V^2 = \hat{U}_\alpha{}^2 + \hat{W}_\alpha{}^2.$$

According to the values of the V parameters, $\hat{U}_\alpha{}^2 + \hat{W}_\alpha{}^2$ is always real, and hence

$$V^2 = [(\mathrm{Re}\,\hat{U}_\alpha)^2 + (\mathrm{Re}\,\hat{W}_\alpha)^2] - [(\mathrm{Im}\,\hat{U}_\alpha)^2 + (\mathrm{Im}\,\hat{W}_\alpha)^2]. \quad (12a)$$

As the refractive index of the core, $\hat{n}_1$, is complex,

$$\hat{U}_\alpha = R(K^2\hat{n}_1{}^2 - \hat{\beta}_\alpha{}^2)^{1/2}, \quad (12b)$$

with

$$\hat{n}_1{}^2 = n^2(1 - \kappa^2) + 2in^2\kappa, \quad (12c)$$

where κ is the extinction coefficient (see, for example, Ref. 13).

Nevertheless, if the wavelength of the incoming radiation is within the range of the visible spectrum, the imaginary part of $\hat{n}_1{}^2$ is almost negligible, as the frequency is high enough and $2\pi\sigma/\omega$ is very small (one should expect, in general, very small σ values for weakly absorbing media).

Consequently, Eq. (12b) can be written as

$$\hat{U}_\alpha = R[K^2n^2(1 - \kappa^2) - \hat{\beta}_\alpha{}^2]^{1/2}, \quad (12d)$$

and

$$\hat{W}_\alpha = R(\hat{\beta}_\alpha{}^2 - K^2n_2{}^2)^{1/2} \quad (12e)$$

as well. Notice that n_2 is real by assumption.

Finally, we shall use the exact expression for the initial field [Eq. (9b)] to derive a well-known result from geometrical optics: the acceptance angle i_c for the penetration of a light ray into the waveguide.

Consider the initial field Ψ_{in} as given by Eqs. (9b) and (11). Since $H_0(i\hat{\chi}_\alpha|\mathbf{x}|)$ decays as $|\mathbf{x}|^{-1/2}\exp(-i\hat{\chi}_\alpha|\mathbf{x}|)$, when $\mathrm{Im}\,\hat{\chi}_\alpha$ is small, for $|\mathbf{x}| \to \infty$, the integral $\int \mathrm{d}^2\mathbf{x}'\phi_\alpha(\mathbf{x}')\exp(-i\mathbf{q}\mathbf{x}')$ can be expected to take appreciable values for $|\mathbf{q}| \leq \mathrm{Re}\,\hat{\chi}_\alpha$ and $|\mathbf{q}| > \mathrm{Re}\,\hat{\chi}_\alpha$ [with $\exp(-i\hat{\chi}_\alpha|\mathbf{x}|) \simeq \exp(-i\,\mathrm{Re}\,\hat{\chi}_\alpha|\mathbf{x}|)$]. Thus the major contribution to the integral over $\mathbf{q}$ in Eq. (9b) comes from $\mathbf{q} \leq \mathrm{Re}\,\hat{\chi}_\alpha$. From geometry, it can be shown that the angle i_c between the Z axis (waveguide axis) which intercepts the $Z = 0$ plane at $\mathbf{x} = 0$ and the wave vector $[\mathbf{q}, \sigma(\mathbf{q})]$ with the largest value of $|\mathbf{q}|$ is given by

$$\sin i_c \simeq |\mathbf{q}|/(\sigma^2(\mathbf{q}) + \mathbf{q}^2)^{1/2}$$
$$= \mathrm{Re}\,\hat{\chi}_\alpha/(\omega/c) = [(\mathrm{Re}\,\hat{n}_1)^2 - n_2{}^2]^{1/2}, \qquad |\mathbf{q}| = \mathrm{Re}\,\hat{\chi}_\alpha. \quad (13)$$

$[(\mathrm{Re}\,\hat{n}_1)^2 - n_2{}^2]^{1/2}$ is the numerical aperture of the absorbing waveguide. From the framework of geometrical optics, we can interpret i_c as the largest acceptance angle. The above expression cannot be derived from the approximate formula (2) but is a direct consequence of the exact representation for Ψ_{in}.

3. FRACTION OF MODAL POWER CONFINED WITHIN AN OPTICAL WAVEGUIDE

Let us define (as has been done previously[2,3,10,14]) the fraction of the power flow of a propagation mode denoted by a double index lm residing within an optical waveguide as

$$\eta_{lm} = \frac{P_{lm}^F}{P_{lm}^{\mathrm{tot}}} = \frac{\text{Fraction of confined power flow}}{\text{Total power flow confined in an infinite area of cross section}}, \quad (14)$$

where l and m are integers with $l, m = 0, 1, 2, 3, \ldots$; η is dependent on the waveguide parameter V.

The power can be calculated by using the complex power density or the Poynting vector. For an electromagnetic field specified by the vectors $\mathbf{E}$ and $\mathbf{H}$, the Poynting vector is defined by[15]

$$\mathbf{S} = \tfrac{1}{2}(\mathbf{E} \times \mathbf{H}^*),$$

where * represents the complex conjugate. The intensity pattern $P(r, \phi)$ in the guide cross section is the real part of the axial component S_z of the Poynting vector and is given by

$$P(r, \phi) = \tfrac{1}{2}\,\mathrm{Re}(S_z) = \tfrac{1}{2}\,\mathrm{Re}(\mathbf{E} \times \mathbf{H}^*) \quad (15)$$

and corresponds to the time-averaged power density flowing along the waveguide. r denotes the radial variable.

We need to determine, for a given bound mode specified by indices lm, the integrals

$$P_{lm}^F = \int_\Omega \mathrm{d}\Omega P_{lm}(r) \quad (16)$$

and

$$P_{lm}^{\mathrm{tot}} = \int_{\Omega \to \infty} \mathrm{d}\Omega P_{lm}(r), \quad (17)$$

where Ω is the cross-sectional area of the waveguide and P_{lm} is the total modal power within an infinite cross section.

Let us consider a monochromatic, linearly polarized field propagating from the remote past incident upon the entrance pupil of the waveguide of radius R. The initial field Ψ_{in} [defined in expression (10)] would give rise to a power flow; by using the scalar-wave approach, we can write the power flow for a particular set of modes in the absorbing waveguide as[8]

$$P_{lm}^F = \mathrm{Re}\int_\Omega \mathrm{d}\Omega\left[\Psi_{\mathrm{in}}{}^{m*}\left(-i\frac{\partial}{\partial Z}\right)\Psi_{\mathrm{in}}{}^{m}\right]. \quad (18)$$

For the radially symmetric case, Eq. (11) gives the propagation modes that satisfy the above relationship.

4. WAVEGUIDE SUPPORTING A SET OF FUNDAMENTAL MODES: MONOMODE BEHAVIOR

In this simple case, let $|M| = 0$ in Eq. (11). Let us define $\alpha = 0$; the initial field then reads as

$$\Psi_{\text{in}}^{0} = C_0\phi_0(\mathbf{x}).$$

From Eq. (14), the fraction of modal power carried within the waveguide is

$$\eta_0(R) = P_0^F/P_0^{\text{tot}}. \tag{19}$$

By using Eqs. (11), (16), and (17), we can arrive at the expressions

$$P_0^F(R) = 2\pi|C_<^{(0)}|^2 \operatorname{Re}\left[\hat{\beta}_0 \int_0^R r\,\mathrm{d}r|J_0(\hat{\chi}_0'r)|^2\right]\exp(-2\operatorname{Im}\hat{\beta}_0 Z) \tag{20}$$

and

$$P_0^{\text{tot}}(R) = 2\pi \operatorname{Re}\left[\hat{\beta}_0|C_<^{(0)}|^2 \int_0^R r\,\mathrm{d}r|J_0(\hat{\chi}_0'r)|^2 + |C_>^{(0)}|^2 \int_R^\infty r\,\mathrm{d}r|H_0^{(1)}(i\hat{\chi}_0 r)|^2\right]\exp(-2\operatorname{Im}\hat{\beta}_0 Z). \tag{21}$$

$\hat{\chi}_0$ and $\hat{\chi}_0'$ are defined according to Eq. (12), and $\hat{\beta}_0$ is the complex modal wave vector. We notice that in Eqs. (20) and (21) the functions J_0 and $H_0^{(1)}$ are complex, as their arguments depend on the complex modal parameters $\hat{\chi}_0'$ and $\hat{\chi}_0$, respectively.

Boundary Conditions

For the specific values for the coefficients appearing in Eq. (11) to be obtained, the following boundary conditions need to be applied:

$$C_<^{(0)}J_0(\hat{\chi}_0'R) = C_>^{(0)}H_0^{(1)}(i\hat{\chi}_0 R). \tag{22}$$

If we substitute Eqs. (19)–(21) and perform the appropriate integrations, two different results are obtained:

(a) For energy confined inside the core ($r < R$),

$$\eta_0(r) = \frac{\pi r^2\{[J_0(\hat{\chi}_0'r)]^2 + [J_1(\hat{\chi}_0'r)]^2\}}{\pi R^2\{[J_0(\hat{\chi}_0'R)]^2 + [J_1(\hat{\chi}_0'R)]^2\} + (\pi R/\hat{\chi}_0)[J_0(\hat{\chi}_0'R)]^2}. \tag{23}$$

The above expression represents the flux of energy transmitted along the receptor under strict confinement conditions.

(b) For energy confined in an infinite area ($r > R$),

$$\eta_0(r) \simeq \frac{\pi R^2\{[J_0(\hat{\chi}_0'R)]^2 + [J_1(\hat{\chi}_0'R)]^2\} + (\pi R/\hat{\chi}_0)[J_0(\hat{\chi}_0'R)]^2\{1 - \exp[-2\hat{\chi}_0(r-R)]\}}{\pi R^2\{[J_0(\hat{\chi}_0'R)]^2 + [J_1(\hat{\chi}_0'R)]^2\} + (\pi R/\hat{\chi}_0)[J_0(\hat{\chi}_0'R)]^2}. \tag{24}$$

This expression gives the flux of energy density transmitted inside and outside the waveguide up to a distance r outside the core. Note that as $r \to \infty$, η_0 tends exponentially to 1, which is as expected, since η is a function normalized to unity.

In the above equations, only the real part of the argument of the Bessel function should be considered.

5. WAVEGUIDE SUPPORTING TWO SETS OF MODES: MULTIMODE BEHAVIOR AND ABSORPTION PROPERTIES

Let us consider the case in which the waveguide supports two sets of bound modes: the fundamental and the first set of excited modes, i.e., $\alpha = 0$ and 1, $|M| = 0$ and 1. The initial field for this situation can be written as

$$\Psi_{\text{in}}^{01} = C_0\phi_0(\mathbf{x})\exp(i\hat{\beta}_0 Z) + C_1\phi_1(\mathbf{x})\exp(i\hat{\beta}_1 Z), \tag{25}$$

where C_0 and C_1 are numerical coefficients to be determined and $\hat{\beta}_1$ is the complex modal propagation factor for the first set of excited modes. It is important to remind the reader that Z is the distance that the field penetrates into the waveguide (along the Z axis). Proceeding as was done in the monomode case, we can show that the fraction of energy flow confined within the core can be written as

$$P_{01}^F(\hat{\chi}_0'\hat{\chi}_1', R, Z) = P_{01}^{1F}(\hat{\chi}_0'\hat{\chi}_1', R, Z) + P_{01}^{2F}(\hat{\chi}_0'\hat{\chi}_1', R, Z). \tag{26}$$

The above expression is made up of two factors, one that has Z dependence as a damping term and a second that has Z dependence as an oscillatory term. The explicit expressions for the two contributions are

$$P_{01}^{1F} = 2\pi\left\{\exp(-2\operatorname{Im}\hat{\beta}_0 Z)|C_<^{(0)}|^2 \operatorname{Re}\left[\hat{\beta}_0 \int_0^R r\,\mathrm{d}r|J_0(\hat{\chi}_0'r)|^2\right] + \exp(-2\operatorname{Im}\hat{\beta}_1 Z)C_1^2|C_<^{(1)}|^2 \operatorname{Re}\left[\hat{\beta}_1 \int_0^R r\,\mathrm{d}r|J_1(\hat{\chi}_1'r)|^2\right]\right\} \tag{27}$$

and

$$P_{01}^{2F} = \operatorname{Re}\left\{\hat{\beta}_1 K_2(Z) \int_0^{2\pi} \mathrm{d}\phi\,\exp(i\phi) \int_0^R r\,\mathrm{d}r\overline{J_0(\hat{\chi}_0'r)}J_1(\hat{\chi}_1'r) + \hat{\beta}_0 K_2^*(Z) \int_0^{2\pi} \mathrm{d}\phi\,\exp(-i\phi) \int_0^R r\,\mathrm{d}r\,J_0(\hat{\chi}_0'r)\overline{J_1(\hat{\chi}_1'r)}\right\}. \tag{28}$$

The bars over J_0 and J_1 denote the complex conjugate. K_2 is a complex factor:

$$K_2(Z) = C_1 C_<^{(0)*}C_<^{(1)} \exp[i(\hat{\beta}_1 - \hat{\beta}_0^*)Z]. \tag{29}$$

In what follows we shall consider $C_<^{(0)}$ and $C_<^{(1)}$ real constants for simplicity. Equation (27) represents physically the transmitted energy confined within the core, which is propagated as the two sets of modes. Equation (28) represents an interference term. Its importance is related to the degree of absorption of the waveguide, and the term could also represent the presence of cross talk in cases in which the absorption properties are not negligible.[16]

Let us discuss the interference term in Eq. (28). As the field is periodic on ϕ, with a period of 2π, integration over ϕ represents the solution for a complete period of the wave. In that case, the contribution P_{01}^{2F} vanishes. If we want to know the magnitude of this Z-dependence term for a given ϕ, that is, for a particular direction in the transverse plane, then no integration is taken over, and we shall omit $\exp(i\phi)$ in the general expression. Even if this discussion seems irrelevant for a single isolated waveguide, it may be interesting to know, for a certain distance r in the transverse plane, the magnitude of its contribution for a given ϕ. In fact, if a second waveguide is located in the neighborhood of the first one, precisely in a direction in which the magnitude of such an interference term is relatively important, then the field of this first waveguide would penetrate into the second one, and cross-talk phenomena could take place.

Consider $\hat{\beta}_0 = \mathrm{Re}\,\hat{\beta}_0 + i\,\mathrm{Im}\,\hat{\beta}_0$ and $\hat{\beta}_1 = \mathrm{Re}\,\hat{\beta}_1 + i\,\mathrm{Im}\,\hat{\beta}_1$, with $\mathrm{Re}\,\hat{\beta}_0$ and $\mathrm{Re}\,\hat{\beta}_1$ representing the transmission properties and $\mathrm{Im}\,\hat{\beta}_0$ and $\mathrm{Im}\,\hat{\beta}_1$ representing the absorption properties.

We define (for fixed $\phi \neq m\pi$, $m = 0, 1, 2\ldots$)

$$P_{01}^{2F} = \mathrm{Re}\left\{\hat{\beta}_1 K_2 \int_0^{\phi_1} \mathrm{d}\phi \exp(i\phi) \int_0^R r\,\mathrm{d}r\, \overline{J_0(\hat{\chi}_0' r)} J_1(\hat{\chi}_1' r) + \hat{\beta}_0 K_2^* \int_0^{\phi_1} \mathrm{d}\phi \exp(-i\phi) \int_0^R r\,\mathrm{d}r\, J_0 \overline{J_1}\right\}, \quad (30)$$

with $\int_0^{\phi_1} \mathrm{d}\phi \exp(i\phi)$ and $\int_0^{\phi_1} \mathrm{d}\phi \exp(-i\phi)$ both acting as constant values.

We are interested in the coefficients

$$\begin{aligned}\hat{\beta}_1 K_2 = C_1 C_<^{(0)} C_<^{(1)} \{ & [\mathrm{Re}\,\hat{\beta}_1 \cos(\mathrm{Re}\,\hat{\beta}_1 - \mathrm{Re}\,\hat{\beta}_0)Z \\ & - \mathrm{Im}\,\hat{\beta}_1 \sin(\mathrm{Re}\,\hat{\beta}_1 - \mathrm{Re}\,\hat{\beta}_0)Z] \\ & + i[\mathrm{Re}\,\hat{\beta}_1 \sin(\mathrm{Re}\,\hat{\beta}_1 - \mathrm{Re}\,\hat{\beta}_0)Z \\ & + \mathrm{Im}\,\hat{\beta}_1 \cos(\mathrm{Re}\,\hat{\beta}_1 - \mathrm{Re}\,\hat{\beta}_0)Z]\} \\ & \times \exp[-(\mathrm{Im}\,\hat{\beta}_1 + \mathrm{Im}\,\hat{\beta}_0)Z] \end{aligned} \quad (31)$$

and

$$\begin{aligned}\hat{\beta}_0 K_2^* = C_1 C_<^{(0)} C_<^{(1)} \{ & [\mathrm{Re}\,\hat{\beta}_0 \cos(\mathrm{Re}\,\hat{\beta}_1 - \mathrm{Re}\,\hat{\beta}_0)Z \\ & + \mathrm{Im}\,\hat{\beta}_0 \sin(\mathrm{Re}\,\hat{\beta}_1 - \mathrm{Re}\,\hat{\beta}_0)Z] \\ & + i[\mathrm{Im}\,\hat{\beta}_0 \cos(\mathrm{Re}\,\hat{\beta}_1 - \mathrm{Re}\,\hat{\beta}_0)Z \\ & - \mathrm{Re}\,\hat{\beta}_0 \sin(\mathrm{Re}\,\hat{\beta}_1 - \mathrm{Re}\,\hat{\beta}_0)Z]\} \\ & \times \exp[-(\mathrm{Im}\,\hat{\beta}_1 + \mathrm{Im}\,\hat{\beta}_0)Z]. \end{aligned} \quad (32)$$

Weakly Guiding and Weak Absorption Properties

Let us consider under what conditions $\hat{\beta}_1 K_2$ and $\hat{\beta}_0 K_2^*$ are negligible. If we consider that the waveguide has low absorption properties, then we can make the approximation $\exp[-(\mathrm{Im}\,\hat{\beta}_1 + \mathrm{Im}\,\hat{\beta}_0)Z] \to 1$ for not very large Z. Also, for weak guiding, $\mathrm{Re}\,\hat{\beta}_1 \simeq \mathrm{Re}\,\hat{\beta}_0$. Special cases and implications of these approximations will be discussed in a forthcoming paper.[17]

Under the above approximations, Eqs. (31) and (32) read as

$$\hat{\beta}_1 K_2 = C_1 C_<^{(0)} C_<^{(1)} \hat{\beta}_1 \quad (33)$$

and

$$\hat{\beta}_0 K_2^* = C_1 C_<^{(0)} C_<^{(1)} \hat{\beta}_0. \quad (34)$$

We note that under the weak absorption and weakly guiding conditions the Z oscillatory dependence in P_{01}^F has been lost and the real power confined within the core depends solely on the radial variable r and the damping term depends solely on Z. No percentage of the confined energy is distributed along the axial direction.

For a given ϕ_1, the fraction of energy flow defined by Eq. (26) is now given by

$$\begin{aligned}P_{01}^F = \mathrm{Re}\Big\{ & \hat{\beta}_0 \exp(-2\,\mathrm{Im}\,\hat{\beta}_0 Z)\Big[2\pi |C_<^{(0)}|^2 \int_0^R r\,\mathrm{d}r\, |J_0(\hat{\chi}_0' r)|^2 \\ & + C_1 C_<^{(0)} C_<^{(1)} \int_0^{\phi_1} \mathrm{d}\phi \exp(i\phi) \int_0^R r\,\mathrm{d}r\, J_0 \overline{J_1}\Big]\Big\} \\ & + \hat{\beta}_1 \exp(-2\,\mathrm{Im}\,\hat{\beta}_1 Z)\Big[2\pi |C_<^{(1)}|^2 C_1^2 \int_0^R r\,\mathrm{d}r\, |J_1(\hat{\chi}_1' r)|^2 \\ & + C_1 C_<^{(0)} C_<^{(1)} \int_0^{\phi_1} \mathrm{d}\phi \exp(-i\phi) \int_0^R r\,\mathrm{d}r\, \overline{J_0} J_1\Big]. \end{aligned} \quad (35)$$

From the behavior of the Bessel functions, we know that the products $J_0 J_1^*$ and $J_0^* J_1$ vanish at the origin, and there is only a small modulation in the case of $\mathrm{Re}\,\hat{\chi}_0' \simeq \mathrm{Re}\,\hat{\chi}_1'$. For small r, the terms in Eq. (35) in which $J_0 J_1$ appears can be dropped, implying that there is no mode coupling and that energy propagates along the waveguide supporting two sets of modes only.

By integrating Eq. (35) over ϕ and over r and by making the appropriate approximations, the fraction of energy flow within the core can be obtained as

$$\begin{aligned}P_{01}^F(\hat{\chi}_0', \hat{\chi}_1', R) = \pi R^2\, \mathrm{Re}\{ & \hat{\beta}_0 \exp(-2\,\mathrm{Im}\,\hat{\beta}_0 Z) |C^{(0)}|^2 \\ & \times [J_0^2(\mathrm{Re}\,\hat{\chi}_0' R) + J_1^2(\mathrm{Re}\,\hat{\chi}_0' R)]\} \\ & + \hat{\beta}_1 \exp(-2\,\mathrm{Im}\,\hat{\beta}_0 Z) C_1^2 |C_<^{(1)}|^2 \\ & \times [J_1^2(\mathrm{Re}\,\hat{\chi}_1' R) - J_0(\mathrm{Re}\,\hat{\chi}_1' R) J_1(\mathrm{Re}\,\hat{\chi}_1' R)]. \end{aligned} \quad (36)$$

For very small values of $\mathrm{Im}\,\hat{\beta}_0$ and $\mathrm{Im}\,\hat{\beta}_1$, Eq. (36) turns out to be similar to the equation describing the dielectric case.

This expression gives the energy flow within the waveguide core without absorption, provided that $\hat{\beta}_0$, $\hat{\beta}_1$, $\hat{\chi}_0'$, and $\hat{\chi}_1'$ are real or with negligible imaginary parts. In the above analysis, we have assumed that $C_0 = 1$ for simplicity, and the coefficients $C_<^{(0)}$ and $C_<^{(1)}$ can be derived from boundary conditions as

$$C_>^{(0)} = \frac{J_0(\hat{\chi}_0' R)}{H_0^{(1)}(i\hat{\chi}_0 R)}, \qquad C_>^{(1)} = \frac{J_1(\hat{\chi}_1' R)}{H_1^{(1)}(i\hat{\chi}_1 R)}. \quad (37)$$

Relatively Strong Absorption Properties

Let us consider that the absorption coefficients of the waveguide are not negligible. This implies that $\exp[-(\mathrm{Im}\,\hat{\beta}_1 + \mathrm{Im}\,\hat{\beta}_0)Z] \neq 1$ for any Z length. As before, we assume that $\mathrm{Re}\,\hat{\beta}_1 \simeq \mathrm{Re}\,\hat{\beta}_0$; that is, the two sets of modes are excited under very similar conditions.

Let us write the total absorption coefficient as

$$\alpha_{01} = \mathrm{Im}\,\hat{\beta}_1 + \mathrm{Im}\,\hat{\beta}_0. \quad (38)$$

For a fixed Z_0, the exponential term can be expanded in a series and to a first approximation:

$$\exp(-\alpha_{01} Z) \simeq 1 - \alpha_{01} Z_0, \quad (39)$$

and on substitution into Eqs. (31) and (32) we get

$$\hat{\beta}_1 K_2 = C_1 C_<^{(0)} C_<^{(1)} \hat{\beta}_1 (1 - \alpha_{01} Z_0) \tag{40}$$

and

$$\hat{\beta}_0 K_2{}^* = C_1 C_<^{(0)} C_<^{(1)} \hat{\beta}_0 (1 - \alpha_{01} Z_0). \tag{41}$$

$\mathrm{Re}(\hat{\beta}_1 K_2)$ and $\mathrm{Re}(\hat{\beta}_0 K_2{}^*)$ depend linearly on the absorption coefficient, while $\mathrm{Im}(\hat{\beta}_1 K_2)$ and $\mathrm{Im}(\hat{\beta}_0 K_2{}^*)$ exhibit nonlinear behavior. We also derive an expression for Eq. (30):

$$P_{01}^{2F}(\alpha_{01}, Z_0) = C_1 C_<^{(0)} C_<^{(1)} (1 - \alpha_{01} Z_0) \times \mathrm{Re}\left[\hat{\beta}_1 \int_0^{\phi_1} \mathrm{d}\phi \exp(i\phi) \int_0^R r\, \mathrm{d}r\, \overline{J_0(\hat{\chi}_0' r)} J_1(\hat{\chi}_1' r) + \hat{\beta}_0 \int_0^{\phi_1} \mathrm{d}\phi \exp(-i\phi) \int_0^R r\, \mathrm{d}r\, J_0(\hat{\chi}_0' r) \overline{J_1(\hat{\chi}_1' r)}\right]. \tag{42}$$

We note that for particular values of $\mathrm{Im}(\hat{\beta}_1 K_2)$ and $\mathrm{Im}(\hat{\beta}_0 K_2{}^*)$, P_{01}^{2F} can be nonnegligible, even if the effect of the integral term is unimportant, and in that case there is a certain percentage of the energy absorbed along the Z direction. The importance of the oscillatory behavior of P_{01}^{2F} depends on the α_{01} values.

6. SUMMARY

Earlier formulations of the initial field striking the entrance pupil of a waveguide have resorted to simple approximation (decomposition). However, in this paper we used the scalar-wave equation approach and derived an exact expression for the total field that takes into account the incoming field, the field propagating along the waveguide, and the scattered field. This expression was shown to take the approximate form under certain conditions. By using this result, the expression for the acceptance angle for penetration of a light ray into the waveguide was derived. This is a result that cannot be achieved by using the approximate formula. An expression for the fraction of total energy confined in a waveguide was then derived and applied to the problem of a cylindrically symmetric waveguide supporting both zero- and first-order sets of excited modes. We then showed that the fraction of energy flow confined within the core can be decomposed into two factors, one containing a Z dependence as a damping term and the other containing a Z dependence as an oscillatory term. This expression is similar to that shown by Snyder and Pask[18]; however, we have derived explicit representations of these terms. These terms are multiplied by numerical coefficients that depend on the optical properties and dimensions of the waveguide and the way in which the energy flows into the waveguide. The consequences of absorption have also been treated in this paper. A forthcoming paper will discuss in detail the results contained herein by applying them to absorbing waveguides.[17]

ACKNOWLEDGMENTS

This work was supported in part by grant EY03674 from the National Eye Institute, National Institutes of Health, Bethesda, Md. M. L. Calvo was supported by a grant from the U.S.–Spain Joint Committee for Scientific and Technological Cooperation Program. We thank J. M. Enoch for support and encouragement.

* Permanent address, Departamento de Optica, Facultad Ciencias Físicas, Universidad Complutense, 28040 Madrid, Spain.

Reprint requests and correspondence should be sent to V. Lakshminarayanan.

REFERENCES

1. D. Marcuse, *Light Transmission Optics* (Van Nostrand Reinhold, New York, 1972).
2. B. R. Horowitz, "Theoretical considerations of the retinal receptor as a waveguide," in *Vertebrate Photoreceptor Optics*, J. M. Enoch and F. L. Tobey, Jr., eds. (Springer-Verlag, Berlin, 1981), pp. 219–300.
3. A. W. Snyder and J. D. Love, *Optical Waveguide Theory* (Chapman and Hall, New York, 1983).
4. A. W. Snyder, "Leaky-ray theory of optical waveguides of circular cross-section," Appl. Phys. **4**, 273–298 (1974).
5. A. W. Snyder and J. D. Love, "Attenuation coefficient for rays in graded fibers with absorbing cladding," Electron. Lett. **12**, 255–257 (1976).
6. K. F. Barrell and C. Pask, "The effect of cladding loss in graded index fibers," Opt. Quantum Electron. **10**, 223–231 (1978).
7. C. Pask and K. F. Barrell, "Photoreceptor optics. I. Introduction to formalism and excitation in a lens-photoreceptor system," Biol. Cybern. **36**, 1–8 (1980).
8. R. F. Alvarez-Estrada and M. L. Calvo, "Neutron fibres: a possible application of neutron optics," J. Phys. D. **17**, 475–502 (1984).
9. P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), Vol. 1.
10. A. W. Snyder, "Excitation and scattering of modes on a dielectric or optical fiber," IEEE Trans. Microwave Theory Tech. **MTT-17**, 1138–1144 (1969).
11. M. Abramowitz and I. Stegun, eds., *Handbook of Mathematical Functions* (Dover, New York, 1965).
12. R. F. Alvarez-Estrada, M. L. Calvo, and P. Juncos, "Scattering of TM waves by dielectric fibre: iterative and eikonal solutions," Opt. Acta **27**, 1367–1378 (1980).
13. M. Born and E. Wolf, *Principles of Optics*, 6th ed. (Pergamon, New York, 1980).
14. G. Biernson and D. G. Kinsley, "Generalized plots of mode patterns in a cylindrical dielectric waveguide applied to retinal cones," IEEE Trans. Microwave Theory Tech. **MTT-13**, 345–356 (1965).
15. J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975).
16. P. McIntyre, "Cross-talk in absorbing optical fibers," J. Opt. Soc. Am. **65**, 810–813 (1975).
17. V. Lakshminarayanan and M. L. Calvo, "The initial field and energy flux in absorbing optical waveguides. II. Implications," J. Opt. Soc. Am. A (submitted).
18. A. W. Snyder and C. Pask, "Waveguide modes and light absorption in photoreceptors," Vision Res. **13**, 2605–2608 (1973).