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A NEW MODEL FOR THE M/G/1 RETRIAL QUEUE

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ABSTRACT

This article deals with a new model for the M/G/1 retrial queue. We consider the process (M(t),N(t)) where M(t) is the total number of arrivals from the last departure until time t and N(t) is the number of customers in orbit at time t. We obtain the generating function together with a recurrent formula for factorial moments in the steady state. Limit behavior under heavy traffic is also studied. We use this process to get an estimator of the parameter of retrial and obtain its accuracy by solving some linear differential equations. We also give some numerical examples.

KEYWORDS

Retrial queues, factorial moments, heavy traffic, direct estimator, accuracy.

INTRODUCTION

A retrial queue system is characterized by the following feature: if the server is free at the time a call arrives, the customer starts

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to be served immediately and leaves the system after the completion of the service (these customers are identified as primary calls); if the channel is busy the customer is assumed to try again thus forming a source of repeated calls which we call an orbit. In the M/G/1 retrial queue primary calls arrive according to a Poisson process at the rate λ and each customer in orbit reapply for service according to independent Poisson distribution at an unknown rate μ. The service time distribution function, B(x), is assumed to be the same for both primary and repeated calls. The input flow of primary calls, the intervals between repetitions and the service time are taken to be mutually independent.

This system has been studied by the markovian process $(C(t),N(t),\xi(t))$ where C(t) represents the number of busy channels (for single-channel queues C(t)=1 or 0 according to whether the channel is busy or free), N(t) is the number of sources of repeated calls and the supplementary variable $\xi(t)$, when C(t)=1, representing the time elapsed while the call is being served. Through this process G.Falin (see [3]) obtain a direct estimator of traffic intensity with its accuracy.

However when trying to estimate the parameter of retrial this model is insufficient because only effective arrivals (i.e. primary calls and repeated calls who find the service free) are taken into account. In section-1 we generalized the above model using a new process in the M/G/1 retrial queue, which was introduced by M.Martin and G.Falin (see [5]) in the M/M/1 retrial queue. In section-2 we obtain an expression for factorial moments, which are used in section-4 to get the variance of an estimator of μ. In section-3 we study the asymptotic behavior of (M_t,N_t) under heavy traffic and when μ tends to 0. In the last section we calculate, numerically, some factorial moments of our process for different service time distributions, together with some graphics about estimations and confidence intervals of the retrial parameter.

1. THE MODEL

Before describing the model we need some previous research given by Keilson (See [4]) of the classical model which will be of use in our analysis. Let p_{0n} and p_{1n} be the probabilities

$$p_{0n}=Pr[C_t=0, N_t=n] \; ; \qquad p_{1n}=Pr[C_t=1, N_t=n]$$

in the steady state with associated partial generating functions given by

$$P_0(z) = \sum_{n=0}^{\infty} p_{0n}z^n = (1-\rho)\exp\left\{ \frac{\lambda}{\mu}\int_1^z \frac{1 - \beta(\lambda-\lambda u)}{\beta(\lambda-\lambda u) - u}\,du\right\} \qquad (1)$$

$$P_1(z) = \frac{1 - \beta(\lambda-\lambda z)}{\beta(\lambda-\lambda z)- z}P_0(z) \qquad (2)$$

where $\beta(s)$ is the Laplace transform of B(x). Moreover, the distribution of the number of sources N_t has a generating function given by

$$P_N(z) = (1-\rho)\,\frac{1-z}{\beta(\lambda-\lambda z)-z}\,\exp\left\{ \frac{\lambda}{\mu}\int_1^z \frac{1-\beta(\lambda-\lambda u)}{\beta(\lambda-\lambda u)-u}\,du\right\} \qquad (3)$$

and expectation and variance given by

$$E(N_t) = \frac{\lambda^2}{1 - \rho}\left(\frac{\beta_1}{\mu} + \frac{\beta_2}{2}\right)$$

$$Var(N_t) = \frac{\lambda^3\beta_3}{3(1-\rho)} + \frac{\lambda^3\beta_2}{2\mu(1-\rho)^2} + \frac{\lambda^4\beta_2^2}{4(1-\rho)^2} + \frac{\lambda\rho}{\mu(1-\rho)} + \frac{\lambda^2\,\beta_2}{2(1-\rho)}$$

where β_i is the i-th initial moment of the service time and $\rho=\lambda\beta_1$ is the traffic intensity. We introduce now a more complex markovian description of the system.

We consider the process $(M(t),N(t),\xi(t))$ where $M(t)$ is the total number of arrivals from the last departure time. That is, when $M(t)=0$ then the channel is free at time t but if $M(t)=m\geq 1$ then the channel is busy at time t and during the elapsed service time exactly m customers arrived (both primary and repeated) into the system (including the customer being served). We now define the probabilities p_{0n} and $p_{mn}(x)$ in the steady state as:

$$p_{0n} = Pr[M_t=0, N_t=n]=Pr[C_t=0, N_t=n]$$

$$p_{mn}(x)dx = Pr[M_t=m, N_t=n, x<\xi_t<x+dx]$$

The process $(M(t),N(t),\xi(t))$ is markovian and in a general way we can obtain the equations of statistical equilibrium

$$\left.\begin{aligned} (\lambda+n\mu)p_{0n} &= \sum_{m=1}^{\infty} \int_0^{\infty} p_{mn}(x)b(x)dx \\ p'_{mn}(x) &= -(\lambda+n\mu+b(x))p_{mn}(x) + (1-\delta_{1m})(\lambda p_{m-1,n-1}(x)+n\mu p_{m-1,n}(x)) \end{aligned}\right\} \quad (4)$$

where $b(x)=(1-B(x))^{-1}dB(x)$ be the instantaneous service rate given that the time elapsed $\xi(t)$ during service time is equal to x and where δ_{1m} is one if m=1 and zero otherwise. The initial conditions associated with (4) are given by

$$p_{mn}(0) = \begin{cases} \lambda p_{on}+(n+1)\mu p_{0n+1} , & m=1 \\ 0 , & m\geq 2 \end{cases}$$

Let

$$P(y,z;x) = \sum_{m=1}^{\infty} y^m \sum_{n=0}^{\infty} p_{mn}(x)z^n$$

be the generating function of $p_{mn}(x)$ then, from (4) we have

$$\frac{\partial}{\partial x} P(y,z;x) = -(\lambda-\lambda yz+b(x))P(y,z;x) + \mu z(y-1)\frac{\partial}{\partial z}P(y,z;x) \qquad (5)$$

$$P(y,z;0) = \lambda y P_N(z)$$

Solving this differential equation we get

$$P(y,z;x)=\lambda y P_N\left(ze^{-\mu(1-y)x}\right)(1-B(x))\exp\left\{\frac{\lambda yz}{\mu(1-y)}(1-e^{-\mu x(1-y)})-\lambda x\right\} \qquad (6)$$

if we now neglect the elapsed time then the generating function in the steady state of (M_t, N_t) is

$$P(y,z)=\lambda y\int_0^{\infty} P_N\left(ze^{-\mu(1-y)x}\right)(1-B(x))\exp\left\{\frac{\lambda yz}{\mu(1-y)}(1-e^{-\mu x(1-y)})-\lambda x\right\}dx \qquad (7)$$

while for M_t alone this function is

$$P_M(y)=\lambda y\int_0^{\infty} P_N\left(e^{-\mu(1-y)x}\right)(1-B(x))\exp\left\{\frac{\lambda y}{\mu(1-y)}(1-e^{-\mu x(1-y)})-\lambda x\right\}dx \qquad (8)$$

Due to the analytical form of the generating functions given in expressions (7) and (8), a direct computation of the moments of (M_t, N_t) is extremely tedious involving the repeated use of the L'Hôpital rule. However, using expressions (5) and (6) the problem can be simplified by finding a recursive formula for factorial moments of M_t given the factorial moments of N_t. We do this in the next section.

2. FORMULAS FOR FACTORIAL MOMENTS

We wish to obtain an expression for factorials moments given by

$$F(m,n)\equiv \frac{\partial^m}{\partial y^m}\frac{\partial^n}{\partial z^n}P(y,z)\Big|_{y=1,z=1}$$

which can be calculated using

$$F(m,n;x) = \frac{\partial^m}{\partial y^m}\frac{\partial^n}{\partial z^n}P(y,z;x)\Big|_{y=1,z=1}$$

because

$$F(m,n)=\int_0^\infty F(m,n;x)dx$$

If we now differentiate equation (5) m times with respect to y and n times with respect to z, at the point y=1, z=1, we get:

$$\frac{\partial}{\partial x}F(m,n;x) = \lambda nmF(m-1,n-1;x)+(m\lambda+nm\mu)F(m-1,n;x)+n\lambda F(m,n-1;x) + \\ + m\mu F(m-1,n+1;x)-b(x)F(m,n;x) \qquad (9)$$

with initial condition given by

$$F(1,n;0)=\lambda\phi_n \ ; \qquad F(m,n;0)=0,\ \ m>1$$

where

$$\phi_n=\frac{d^n}{dz^n}P_N(z)\Big|_{z=1}$$

are the factorial moments of the number of customers in orbit. Now, $F(0,n;x)$ can be easily calculated from (6) setting y=1 and differentiating n times with respect to z at the point z=1. Thus we obtain

$$F(0,n;x)=\lambda\left[\sum_{p=0}^{n}\binom{n}{p}\phi_{n-p}(\lambda x)^p\right](1-B(x)) \qquad (10)$$

Since ϕ_n and $F(0,n;x)$ are known , we can find $F(m,n;x)$ from equation (9) and hence all factorial moments $F(m,n)$, $m\geq1$. In particular we have

$$E[M_t]=F(1,0)=\lambda\beta_1+(\lambda^2+\mu\lambda\phi_1)\frac{\beta_2}{2}+\mu\lambda^2\frac{\beta_3}{6} \qquad (11)$$

$$E[M_tN_t]=F(1,1) = \lambda\beta_1+\lambda^2\beta_2+\left[\frac{\lambda^3}{6}+\lambda^2\frac{\lambda+\mu}{6}\right]\beta_3+\frac{\mu\lambda^3}{8}\beta_4+\mu\lambda\frac{\beta_2}{2}\phi_2 + $$
$$+\left[\left\{(\lambda+\mu)\lambda+\mu\lambda^2\right\}\frac{\beta_2}{2}+\mu\lambda^2\frac{\beta_3}{3}\right]\phi_1$$

$$Var(M_t) = \frac{\beta_2}{2}\left[2\lambda^2+2\mu\lambda\phi_1\right]+\frac{\beta_3}{3}\left[\lambda^3+\mu\lambda^2\phi_1+2\lambda^2\mu+(\lambda+\mu)\lambda\mu\phi_1+\mu^2\lambda\phi_2\right] +$$
$$+\frac{\beta_4}{12}\left[\mu\lambda^3+2\lambda^3\mu+\lambda^2\mu^2+\mu^2\lambda^2\phi_1\right]-E(M_t)^2+E(M_t),$$

where $\phi_1=E(N_t)$, $\phi_2=Var(N_t)+E(N_t)^2-E(N_t)$. Note that we have only a general scheme to calculate the moments. We show now the following algorithm.

<u>Step-1</u> From (1) and (2) we have that $\phi_n=P_0^{n)}(1)+\mu\lambda^{-1}P_0^{n+1)}(1)$, where $P_0^{i)}(1)$ is the i-th derivative of $P_0(z)$ at z=1, which can be obtained recursively from

$$-(i\lambda\mu\beta_1-i\mu)P_0^{i)}(1) = \mu\sum_{k=0}^{i-2}\binom{i}{k}\lambda^{i-k}\beta_{i-k}P_0^{k+1)}(1) +$$
$$+\lambda\sum_{k=0}^{i-1}\binom{i}{k}\lambda^{i-k}\beta_{i-k}P_0^{k)}(1) \qquad (12)$$

i=2 to n, beginning with the values $P_0(1)=1-\lambda\beta_1$ and $P'_0(1)=\lambda^2\beta_1\mu^{-1}$ (this formula is easily calculated differentiating successively in formula (1)).

<u>Step-2</u> Let G(m,n;x) such that F(m,n;x)=(1-B(x))G(m,n;x). Then

$$G(m,n;0) = \begin{cases}\lambda\phi_n, & \text{if } m=0,1;\ n\geq 0\\ 0, & \text{if } m\geq 2\end{cases}$$

$$G(0,n;x) = \lambda\sum_{p=0}^{n}\binom{n}{p}\phi_{n-p}\ (\lambda x)^p \tag{13}$$

Hence $G(0,n;x)$ is a polinomial with degree n, whose coefficients can be written as

$$G_i(o,n)=\lambda^{i+1}\binom{n}{i}\phi_{n-i}\quad ,0\le i\le n \tag{14}$$

which are calculated using step-1.

Step-3 Equation (9) becomes in

$$\frac{\partial}{\partial x}G(m,n;x) = \lambda nmG(m-1,n-1;x)+(m\lambda+nm\mu)G(m-1,n;x) + \\ + n\lambda G(m,n-1;x)+m\mu G(m-1,n+1;x) \tag{15}$$

then

$$G(m,n;x) = \int_0^x\left\{\lambda nmG(m-1,n-1;u)+(m\lambda+nm\mu)G(m-1,n;u)+n\lambda G(m,n-1;u)\right\}du + \\ + \int_0^x m\mu G(m-1,n+1;u)du + G(m,n;0) \tag{16}$$

By induction respect to m and n in above formula we get, when $m>0$,

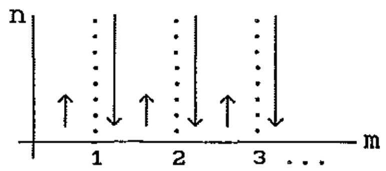
$$G(m,n;x) = \sum_{i=0}^{2m+n} G_i(m,n)x^i$$

where $G_i(m,n)$ verifies the relation

$$iG_i(m,n) = \lambda mnG_{i-1}(m-1,n-1)+(\lambda m+\mu nm)G_{i-1}(m-1,n)+ \\ + \lambda nG_{i-1}(m,n-1)+m\mu G_{i-1}(m-1,n+1),\quad i\ge 1 \tag{17}$$

$$G_0(m,n)=G(m,n,0) \tag{18}$$

Now with the help of $G(m,n,0)$ and $G_i(o,n)$ we can obtain recursively for each m,n all the coefficients $G_i(m,n)$ for all $i\le 2m+n$ in the following order



<u>Step-4</u> Finally from definition of G(m,n;x) we have

$$F(m,n) = \sum_{i=0}^{2m+n} G_i(m,n)\,\frac{\beta_{i+1}}{i+1}, \quad m>0$$

In section-5 we calculate numerically the factorial moments in different situations.

3. LIMIT THEOREMS

In the previous section it has not been possible to obtain an explicit expression for the joint probability of (M_t, N_t) in the steady state. Even in the M/M/1 case the generating function does not reveal the nature of the distribution p_{mn}. However in some domains of the system parameters we can approximate it by any classical distribution.

With this goal in mind we now study the asymptotical behavior of the joint distribution when some parameters tend to certain limit values. Consider first the case of heavy traffic when the arrival rate λ increases in such a way that $\rho=\lambda\beta_1\to 1-o$. It is known (see Falin [1]) that

$$\Psi(s)\equiv\lim_{\rho\to1-0} P_N\left[\exp\left\{-s(1-\rho)N_t\right\}\right]=\left(1+\frac{\beta_2}{2\beta_1^2}s\right)^{-1-\frac{2\beta_1}{\mu\beta_2}} \tag{19}$$

that is, under heavy traffic the queue length N_t has asymptotically a Gamma distribution. Taking this into account we can now establish our main result:

Theorem 1

If the main M/G/1 type retrial queue is in the steady state and $\beta_2 < \infty$ then

$$\lim_{\rho\to1-0} E\left[\exp\left\{-r(1-\rho)M_t, -s(1-\rho)N_t\right\}\right] =$$
$$= \int_0^\infty \left(1+\frac{\beta_2}{2\beta_1^2}(s+\mu rx)\right)^{-1-\frac{2\beta_1}{\mu\beta_2}} \frac{1-B(x)}{\beta_1}dx$$

and

$$\lim_{\rho\longrightarrow1-0} E\left[\exp\left\{-r(1-\rho)M_t\right\}\right] = \int_0^\infty \left(1+\frac{\beta_2}{2\beta_1^2}\mu rx\right)^{-1-\frac{2\beta_1}{\mu\beta_2}} \frac{1-B(x)}{\beta_1}\, dx$$

i.e. the scaled variables $((1-\rho)M_t,\ (1-\rho)N_t)$ are asymptotically dependent and $(1-\rho)M_t$ has a randomized gamma distribution with a random parameter $\beta_2\mu x(2\beta_1^2)^{-1}$, where x represents the residual life of the service time distribution.

Proof. Let

$$\Psi(r,s;x) = \lim_{\rho\to1} P\left(\exp\left\{-r(1-\rho)M_t\right\},\ \exp\left\{-s(1-\rho)N_t\right\};\ x\right)$$

$$\Psi_N(s) = \lim_{\rho\to1-0} P_N\left(\exp\left\{-s(1-\rho)N_t\right\}\right)$$

Under heavy traffic it is not necessary to take into account the probability $p_0=Pr[M_t=0]$ because as ρ tends to 1 p_0 tends to 0 (see formula (1)). Then

$$\Psi(r,s) = \int_0^\infty \Psi(r,s;x)dx = \lim_{\rho \to 1} E\left[\exp\left\{-r(1-\rho)M_t, -s(1-\rho)N_t\right\}\right]$$

Setting $z=e^{-s(1-\rho)}$ and $y=e^{-r(1-\rho)}$ in (8) we get, as ρ tends to 1,

$$\Psi(r,s) = \lim_{\rho\to 1} \int_0^\infty \lambda P_N\left(\exp\left\{-s(1-\rho)-\mu[1-\exp\{-r(1-\rho)\}]x\right\}\right)(1-B(x))dx$$

When ρ is closed to 1, $1-e^{-r(1-\rho)} \cong r(1-\rho)$, then we can write

$$\Psi(r,s) = \lim_{\rho\to 1} \int_0^\infty P_N\left(\exp\{-(s+\mu rx)(1-\rho)\}\right)(1-B(x))\beta_1^{-1}dx$$

Using formula (19) we get

$$\Psi(r,s) = \int_0^\infty \Psi_N(\mu rx+s)\frac{1-B(x)}{\beta_1}dx$$

and when s=0 it becomes

$$\Psi_M(r) = \int_0^\infty \Psi_N(\mu rx)\frac{1-B(x)}{\beta_1} dx$$

■

Remark 1. The previous expression for $\Psi_M(r)$ give us a relation between M_t and N_t: under heavy traffic, $M_t=\mu\tau N_t$, where τ represents the residual life of the service time distribution. This result is intuitively clear because the primary calls and their retrials during the elapsed time are finite variables, so they disappear when we multiply by $(1-\rho)$ and only retrials generated by customers in orbit at the beginning of the service are taking into account. Thus, M_t can be considered as the number of customers in orbit multiplied by the retrial intensity $\mu\tau$.

Theorem 2

If the main M/G/1 type retrial queue is in the steady state and $\beta_2 < \infty$ then as $\mu \to 0$ the variables

$$\left(M_t,\ \mu^{1/2}N_t - \frac{\lambda\rho}{(1-\rho)\mu^{1/2}}\right)$$

are asymptotically independent and the joint characteristic function $\Phi(r,s)$ is given by

$$\Psi(r,s) = \Psi_N(s)\left\{(1-\rho)+\rho e^{ir}\int_0^\infty \exp\left\{\frac{\lambda(e^{ir}-1)x}{1-\rho}\right\}\frac{1-B(x)}{\beta_1}dx\right\}$$

where

$$\Psi_N(s)=\lim_{\mu\to 0} E\left[\exp\left\{\left(\mu^{1/2}N_t - \frac{\lambda\rho}{(1-\rho)\mu^{1/2}}\right)is\right\}\right]$$

is a Gaussian distribution with variance

$$\frac{\lambda^3\beta_2+2\lambda\rho-2\rho^2}{2(1-\rho)^2}$$

(see [2]). In the M/M/1 case the asymptotical distribution of M_t is geometric with parameter ρ.

<u>Proof</u>. The characteristic function

$$\Psi(r,s) = \lim_{\mu\to 0} E\left[\exp\left\{irM_t\right\},\ \exp\left\{is\left(\mu^{1/2}N_t - \frac{\lambda\rho}{(1-\rho)\mu^{1/2}}\right)\right\}\right]$$

can be expressed in terms of P(y,z) as

$$\left(P(e^{ir},\ e^{i\mu^{1/2}s}) + P_0(e^{i\mu^{1/2}s})\right)\exp\left\{\frac{-is\rho\lambda}{(1-\rho)\mu^{1/2}}\right\}$$

where P(y,z) and $P_0(z)$ are given by (7) and (1) respectively. By (2) and (3) we get

$$P_0(z)=\frac{\beta(\lambda-\lambda z)-z}{1-z}P(z)$$

and setting $z=\exp\{i\mu^{1/2}s\}$ we have

$$\lim_{\mu\to 0} P_0\left(e^{\,i\mu^{1/2}s}\right)\exp\left\{\frac{-is\rho\lambda}{(1-\rho)\mu^{1/2}}\right\} = \Psi_N(s)(1-\rho)$$

Setting now $z=\exp\{i\mu^{1/2}s\}$ and $y=\exp\{ir\}$ in (7) and multiplying by

$$\exp\left\{\frac{-is\rho\lambda}{(1-\rho)\mu^{1/2}}\right\}$$

we get, after some algebra,

$$\lim_{\mu\to 0} P\left(e^{ir},\ e^{i\mu^{1/2}s}\right)\exp\left\{\frac{-is\rho\lambda}{(1-\rho)\mu^{1/2}}\right\} =$$

$$= \rho e^{ir}\Psi_N(s)\int_0^\infty \exp\left\{\frac{1}{1-\rho}\lambda x(e^{ir}-1)\right\}\beta_1^{-1}(1-B(x))dx.$$

Hence

$$\Psi(r,s) = \Psi_N(s)\left[(1-\rho)+\rho e^{ir}\int_0^\infty \exp\left\{\frac{1}{1-\rho}\lambda x(e^{ir}-1)\right\}\beta_1^{-1}(1-B(x))dx\right]$$

■

Remark 2. In this case, when the service is busy, M_t can be expressed as $1+\Delta$ where the variable Δ has a randomized Poisson distribution with the random parameter $\lambda x(1-\rho)^{-1}$ where x is the residual life for the service time distribution. This result is intuitively clear because when μ is closed to 0, M_t is the sum of two independent Poissons variables: the primary calls $P(\lambda x)$ and the retrials of the customers in orbit at the beginning of the service $P(\lambda\rho x/(1-\rho))$ (see expression for the mean number of customers in orbit).

Remark 3. It is easy to prove that if B(x) is an exponential distribution we have

$$\Phi(r,s) = \Phi_N(s) \frac{1 - \rho}{1 - \rho e^{ir}}$$

that is M_t has a geometric distribution with parameter ρ.

4. A DIRECT ESTIMATOR FOR THE PARAMETER OF RETRIAL

Using the results obtained in previous sections we can get an estimator of parameter of retrial and its accuracy, extending the methods given by M.Martín and G.Falin (see [5]) in the M/M/1 retrial queues to M/G/1. We assume the queue in steady state, then from formula (11) we can consider γ_T given by

$$\gamma_T = \frac{12(1-\rho)\alpha_T - 6\lambda^2\beta_2 - 12\rho(1-\rho)}{2\lambda^2(1-\rho)\beta_3 + 3\lambda^3\beta_2^2} \tag{20}$$

where

$$\alpha_T = \frac{1}{T}\int_0^T M(t)dt$$

as an unbiased estimator of the retrial rate μ (note that, since M(t) is ergodic, $\lim_{T\to\infty}\alpha_T = E(M_t) = \bar{M}$ a.s.). From (20) the variance of γ_T is given by

$$Var(\gamma_T) = \left(\frac{12(1-\rho)}{2\lambda^2(1-\rho)\beta_3 + 3\lambda^3\beta_2^2}\right)^2 Var(\alpha_T)$$

then, we need to obtained an expression for $Var(\alpha_T)$. It is well known that the variance can be expressed in the form

$$\frac{2}{T^2}\int_0^T (T-t)R(t)dt$$

where $R(t)=E[M(t)M(0)]-E[M(t)]E[M(0)]$ is the covariance function of the process $M(t)$. Since $M(t)$ is ergodic, the above expression can be written as

$$\frac{2}{T}V + o(1/T) \tag{21}$$

where

$$V=\int_0^\infty R(t)dt$$

Writing $R(t)$ as the integral in $x\in(0,\infty)$ of $R(t;x)$ which is defined in the same way as in previous sections and putting $R(t;x)$ in terms of transition probabilities we have that

$$R(t;x)=\sum_{m=1}^{\infty} m \sum_{n=0}^{\infty} \sum_{m'=1}^{\infty} m' \sum_{n'=0}^{\infty} p_{m'n'}(p_{mn}(t;x/m',n')-p_{mn}(x))$$

where $p_{m'n'}$ and $p_{mn}(x)$ are the stationary probabilities defined at the beginning of the paper and

$$p_{0n}(t/m',n')=Pr[M(t)=0,N(t)=n/M(0)=m',N(0)=n'],\quad m=0,\quad t>0$$

$$p_{mn}(t;x/m',n')dx=Pr[M(t)=m,N(t)=n,x<\xi(t)<x+dx/M(0)=m',N(0)=n'],\quad m\geq 1,\quad t>0$$

$$p_{0n}(0/m',n')=\delta((0,n),(m',n')),\quad m=0,\quad t=0$$

$$p_{mn}(0;x/m',n')=\delta((m,n),(m',n'))p_{mn}(x),\quad m\geq 1,\quad t=0$$

are the transition probabilities ($\delta((a,b),(a',b'))=1$ if $a=a'$ and $b=b'$; zero otherwise). Now, the constant V is

$$V=\int_0^\infty \int_0^\infty R(t;x)dtdx$$

To get V we will not calculate V directly, we will use some transformations of the Chapman-Kolmogorov equations and we will obtained V by solving some linear differential equations. The Chapman-Kolmogorov equations in terms of transition probabilities are

$$\frac{dp_{0n}(t/m',n')}{dt} = -(\lambda+n\mu)p_{0n}(t/m',n') + \int_0^\infty \sum_{m=1}^\infty p_{mn}(t;x/m',n')b(x)dx$$

$$\frac{\partial p_{mn}(t;x/m',n')}{\partial t} + \frac{\partial p_{mn}(t;x/m',n')}{\partial x} = -(\lambda+n\mu+b(x))p_{mn}(t;x/m',n') +$$
$$+ \left[\lambda p_{m-1,n-1}(t;x/m',n') + n\mu p_{m-1,n}(t;x/m',n')\right](1-\delta_{1m})$$

$$p_{1n}(t;0/m',n') = \lambda p_{0n}(t/m',n') + (n+1)\mu p_{0,n+1}(t/m',n')$$

$$p_{mn}(t;0/m',n') = 0, \quad \text{if } m\geq 2$$

(22)

After calculating Laplace transform in variable t, equations (22) become

$$-\delta((0,n),(m',n'))+sp^*_{0n}(s/m',n') = -(\lambda+n\mu)p^*_{0n}(s/m',n') +$$
$$+ \int_0^\infty \sum_{m=1}^\infty p^*_{mn}(s;x/m',n')b(x)dx$$

$$-\delta((m,n),(m',n'))p_{mn}(x)+sp^*_{mn}(s;x/m',n')+\frac{\partial p^*_{mn}(s;x/m',n')}{\partial x} =$$
$$= \left[\lambda p^*_{m-1,n-1}(s;x/m',n')+n\mu p^*_{m-1,n}(s;x/m',n')\right](1-\delta_{1m}) -$$
$$- (\lambda+n\mu+b(x))p^*_{mn}(s;x/m',n')$$

$$p^*_{1n}(s;0/m',n') = \lambda p^*_{0n}(s/m',n') + (n+1)\mu p^*_{0,n+1}(s/m',n')$$

$$p^*_{mn}(s;0/m',n') = 0, \quad \text{si } m\geq 2$$

(23)

where $p^*_{..}(s;./m',n')$ are the corresponding Laplace transforms. Now, dividing the stationary Chapman-Kolmogorov equations given by (4) by s, subtracting them to equations (23) and taking the limit when $s \longrightarrow 0$, we get

$$-\delta((0,n),(m',n'))+p_{0n}= -(\lambda+n\mu)V_{0n}(m',n') + \int_0^\infty \sum_{m=1}^\infty V_{mn}(x/m',n')b(x)dx$$

$$-\delta((m,n),(m',n'))p_{mn}(x)+p_{mn}(x)+\frac{dV_{mn}(x/m',n')}{dx} =$$

$$= \left[\lambda V_{m-1,n-1}(x/m',n')+n\mu V_{m-1,n}(x/m',n')\right](1-\delta_{1m}) -$$

$$-(\lambda+n\mu+b(x))V_{mn}(x/m',n')$$

$$V_{1n}(0/m',n') = \lambda V_{0n}(m',n') + (n+1)\mu V_{0,n+1}(m',n')$$

$$V_{mn}(0/m',n') = 0, \quad \text{si } m\geq 2 \tag{24}$$

where

$$V_{0n}(m',n') = \int_0^\infty \left(p_{0n}(t/m',n')-p_{0n}\right)dt=\lim_{s\to 0}\left(p^*_{0n}(s/m',n')-\frac{1}{s}p_{0n}\right)$$

$$V_{mn}(x/m',n') = \int_0^\infty \left(p_{mn}(t,x/m',n')-p_{mn}(x)\right)dt =$$

$$= \lim_{s\to 0}\left(p^*_{mn}(s,x/m',n')-\frac{1}{s}p_{mn}(x)\right)$$

Multiplying the equations (24) by $m'p_{m'n'}$ and summing in m' y n' we have

$$\bar{M}p_{0n}=-(\lambda+n\mu)V_{0n}+ \int_0^\infty \sum_{m=1}^\infty V_{mn}(x)b(x)dx$$

$$\bar{M}p_{mn}(x)-mp_{mn}(x)+\frac{dV_{mn}(x)}{dx} =-(\lambda+n\mu+b(x))V_{mn}(x) +$$

$$+ \left[\lambda V_{m-1,n-1}(x)+n\mu V_{m-1,n}(x)\right](1-\delta_{1m})$$

$$V_{1n}(0)=\lambda V_{0n}+(n+1)\mu V_{0,n+1}$$

$$V_{mn}(0)=0, \quad \text{si } m\geq 2 \tag{25}$$

where

$$V_{on}=\sum_{m'=1}^{\infty} m' \sum_{n'=0}^{\infty} p_{m'n'} V_{0n}(m',n')$$

$$V_{mn}(x)=\sum_{m'=1}^{\infty} m' \sum_{n'=0}^{\infty} p_{m'n'} V_{mn}(x/m',n').$$

Note also, that $V_{..}(.)$ verifies the following equation (the normalizing condition)

$$\sum_{n=0}^{\infty} V_{0n} + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \int_0^{\infty} V_{mn}(x)dx = 0. \tag{26}$$

Now the constant V is given by

$$\int_0^{\infty} \left(\sum_{m=1}^{\infty} m \sum_{n=0}^{\infty} V_{mn}(x) \right) dx \tag{27}$$

To obtain this constant, we use generating functions in (25) and (26) and we get the equations

$$\bar{M}P_0(z) = -\lambda V_0(z) - \mu z V_0'(z) + \int_0^{\infty} V(1,z;x)b(x)dx \tag{28}$$

$$\bar{M}P(y,z;x) - yP_y'(y,z;x) = (\lambda yz - \lambda - b(x)V(y,z;x) + \mu z(y-1)V_z'(y,z;x) - \frac{dV(y,z;x)}{dx} \tag{29}$$

$$V(y,z;0) = \lambda y V_0(z) + \mu y V_0'(z) \tag{30}$$

$$V_0(1) + \int_0^{\infty} V(1,1;x)dx = 0 \tag{31}$$

where

$$V_0(z) = \sum_{n=0}^{\infty} z^n V_{0n} \quad \text{and} \quad V(y,z;x) = \sum_{m=1}^{\infty} y^m \sum_{n=0}^{\infty} z^n V_{mn}(x)$$

Then, we have that V is given by

$$V=\int_0^{\infty} V'_y(1,1;x)dx \tag{32}$$

Now, $V'_y(1,1;x)$ will be obtained using equations (28), (29), (30) and (31). To simplify calculations we introduce

$$\vartheta(y,z;x)\equiv\bar{M}P(y,z;x)-yP'_y(y,z;x), \quad \vartheta(x)\equiv\vartheta(1,1;x),$$

$$I(\vartheta(x))\equiv\int\bar{\vartheta}(x)dx, \quad I^n(\vartheta(x))=\int I^{n-1}(\vartheta(x))dx \ , \ n=2,3,4,\ldots$$

$$I^n(\vartheta.(\beta))=I^n(\vartheta.(x))\Big|_{x^j=\beta_j} \ ; \ 1\le j\le i$$

$$V^{k)}_{\cdot}(1,1;x)\equiv V^{k)}_{\cdot}(x), \quad i=1,2,3.$$

where $\bar{\vartheta}(x)$ is such that $\vartheta(x)=\bar{\vartheta}(x)(1-B(x))$, i is the degree of the polinomial $I^1(\vartheta.(x))$ and the independent term in $I^n(\vartheta(x))$, n=1,2,3,4, is zero. An explicit formula for $V'_y(x)$ depends on terms which we obtain in the following lemmas:

Lemma 1.

a) $V(x) = [V(0)-I(\vartheta(x))](1-B(x))$

b) $V'_z(x) = [V'_z(0)+\lambda V(0)x-\lambda I^2(\vartheta(x))-I(\vartheta'_z(x))](1-B(x))$

c) $V''_{zz}(x) = [V''_{zz}(0)+2\lambda V'_z(0)x+\lambda^2 V(0)x^2-2\lambda^2 I^3(\vartheta(x)) - 2\lambda I^2(\vartheta'_z(x))-I(\vartheta''_{zz}(x))](1-B(x))$

Proof. (a) Set z=y=1 in (28) and solve. (b) Differentiate with respect to z at z=y=1 and solve. (c) Differentiate twice with respect to z at z=y=1 and solve.

∎

Lemma 2.

a) $V(0) = V'_y(0)=\lambda A-B$

b) $V'_z(0) = (1-\rho)^{-1}[\lambda(2^{-1}\lambda\beta_2+\mu^{-1}\rho)(\lambda A-B)-\lambda\mu^{-1}B-2^{-1}C]$, where

$$A=I^2(\vartheta(\beta)),\qquad B=\bar{M}P'_0(1)+\lambda I^2(\vartheta(\beta))+I(\vartheta'_z(\beta))$$

and

$$C=\bar{M}P''_0(1)+2\lambda^2I^3(\vartheta(\beta))+2\lambda I^2(\vartheta'_z(\beta))+I(\vartheta''_{zz}(\beta)).$$

(We do not need the constant $V''_{zz}(0)$).

<u>Proof</u>. Setting z=y=1 and differentiating equation (29) at that point with respect to y, z and zz we obtain

$$V(0)=V'_y(0)=\lambda V_0(1)+\mu V'_0(1) \tag{33}$$

$$V'_z(0)=\lambda V'_0(1)+\mu V''_0(1) \tag{34}$$

$$V''_{zz}(0)=\lambda V''_0(1)+\mu V'''_0(1) \tag{35}$$

By lemma 1(a) equation (31) is

$$V_0(1)+V(0)\beta_1=A. \tag{36}$$

To find $V(0)$, $V'_z(0)$ and $V'_y(0)$ we need more equations. Differentiating (28) with respect to z at z=1, (note that (28) at z=1 is equation (33)), we have

$$\bar{M}P'_0(1)=-\lambda V'_0(1)-\mu V'_0(1)-\mu V''_0(1)+\int_0^\infty V'_z(x)b(x)dx \tag{37}$$

and using lemma 1(b) together with (34) give

$$\rho V(0)-\mu V'_0(1)=B. \tag{38}$$

Now, through equations (33), (34) and (38) we get $V(0)=V'_y(0)=\lambda A-B$. Differentiating (28) twice respect to z at z=1 we have

$$\bar{M}P''_0(1)=-\lambda V''_0(1)-2\mu V''_0(1)-\mu V'''_0(1)+\int_0^\infty V''_{zz}(x)b(x)dx.$$

Using lemma 1(c) and equations (37) and (35) we calculate $V'_z(0)$. ■

Theorem 3.

$$V'_y(x)=[\lambda I(V)+\mu I(V'_z)-I(\vartheta'_y)+V'_y(0)](1-B(x)) \tag{39}$$

Proof. Differentiating (29) with respect to y at z=y=1 we get

$$\frac{d}{dx}V'_y(x)=-b(x)V'_y(x)+\lambda V(x)+\mu V'_z(x)-\vartheta'_y(x).$$

Solving the above equation and using lemmas one and two, we get expression (39). ■

Integrating $V'_y(x)$; $0\le x<\infty$, we get

$$V = \left(\beta_1+\frac{\lambda\beta_2}{2}+\frac{\lambda\mu\beta_3}{6}\right)V(0)+\frac{\lambda\beta_2}{2}V'_z(0)-\lambda\mu I^4(\vartheta(\beta)) -$$
$$- \lambda I^3(\vartheta(\beta))-\mu I^3(\vartheta'_z(\beta))-I^2(\vartheta'_y(\beta)).$$

Finally note that

$$\vartheta(x)=\bar{M}F(0,0;x)-F(1,0;x), \quad \vartheta'_z(x)=\bar{M}F(0,1;x)-F(1,1;x)$$
$$\vartheta''_{zz}(x)=\bar{M}F(0,2;x)-F(1,2;x), \quad \vartheta'_y(x)=(\bar{M}-1)F(1,0;x)-F(2,0;x)$$

and

$$I^kF(i,j;\beta)=\sum_{n=0}^{2i+1} G_n(i,j)\,\frac{n!}{(n+k)!}\,\beta_{n+k}$$

where $G_n(i,j)$ are the coefficients defined in section two. In the next section we get, numerically, the constant V when B(x) is the determinist distribution and compare them with the exponential case.

5. NUMERICAL RESULTS AND GRAPHICS

In tables below we calculate expectation and variance of M_t, covariance of (M_t,N_t) and the coefficient (vc) of variation of M_t when $\lambda=1$. We consider service time distributions Deterministic, Exponential, Erlang and Hiperexponential To do this we use the algorithm described in section two.

	E(M)	Var(M)	Cov(M,N)	CV(M)	
Deter. ST=.9	6.7117	41.477	33.742	.92074	μ=1
	5.8308	25.364	28.515	.74603	μ=.5
	5.1261	15.063	24.332	.57323	μ=.1
Erlang ν=20/9 NP=2	11.148	211.35	88.532	1.7006	μ=1
	9.0957	113.54	70.196	1.3724	μ=.5
	7.4538	55.772	55.527	1.0038	μ=.1
exp. ν=10/9	16.469	609.47	178.58	2.2470	μ=1
	12.780	297.47	134.70	1.8212	μ=.5
	9.7388	122.20	97.890	1:2884	μ=.1
Hiperexp. NP=2 ν_1=2, ν_2=4	32.852	3168.7	579.52	2.9360	μ=1
	23.376	1327.0	401.93	2.4285	μ=.5
	15.795	409.34	259.85	1.6407	μ=.1

TABLE-I. ST is the service time in the Deterministic case, ν is the rate of service in Exponential and Erlang cases, NP is the number of exponential variables in the Erlang and Hiper-exponential distribution. The traffic intensity is equal to .9 in all cases.

	Deter. ST=.9	Erlang ν=2, NP=4	Exp. ν=2	Hiperexp. ν_1=4, ν_2=4/3, NP=2
E(M)	.80208	1.0078	1.2500	1.5477
Var(M)	1.1364	2.5048	5.0625	10.181
Cov(M,N)	.60937	1.2646	2.2500	3.8031
CV(M)	1.7664	2.4661	3.2400	4.2504

TABLE-II. μ=1 and traffic intensity equal to 1/2.

It is observed that the vc of M_t is strongly dependent to the vc of the service time distribution. For the same values of the parameters, the vc follows the increasing order Determinist, Erlang, Exponential and Hiperexponential. It is also observed that such coefficient decreasing as μ tends to zero (see table-I). However the vc increases as ρ tends to zero and decreases as ρ tends to 1 (see table-II).

We also simulate a M/D/1 and a M/M/1 retrial queue in the interval (0,10000) when the mean service time is equal to 1. We draw some estimations together with the 95% confident intervals for diferent values of μ when λ=0.5, assuming that α_T is asymptotically Gaussian distributed.

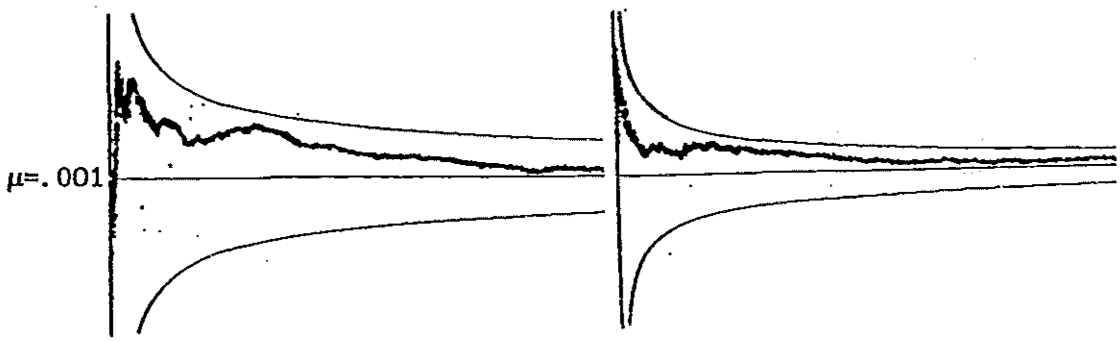


Fig-1. Estimations for μ when λ=.5, μ=.001 and there are 650 customers in orbit at time t=0. The figure on the right corresponds to the exponential case. The one on the left represents the deterministic case.

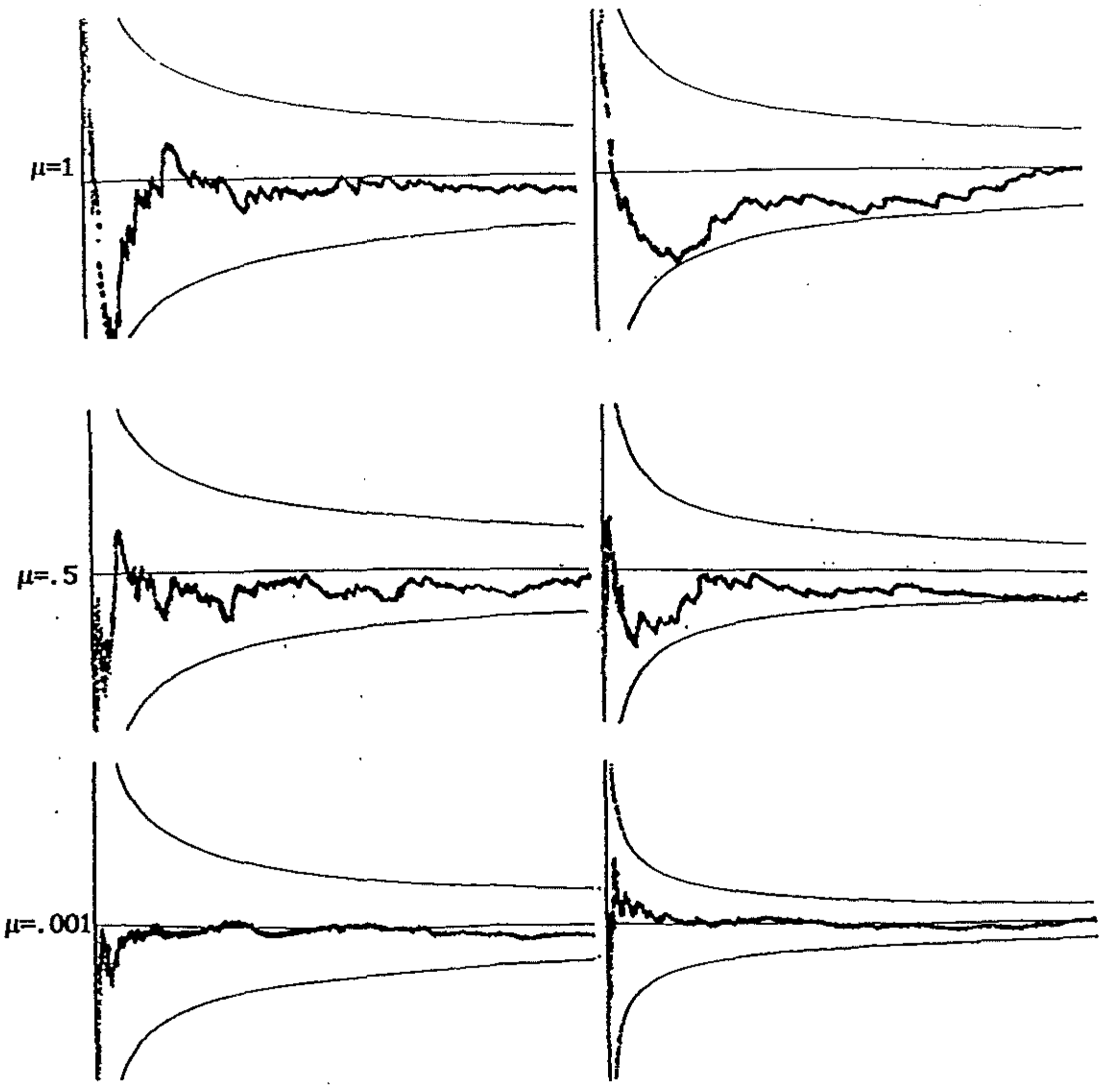


Fig-2. Estimations of parameter μ when the true values of μ are 1, 1/2 and .001 respectively. We consider λ=.5 and that the initial number of customers is the orbit is taken as the integer part of the mean number of customers in orbit in steady state. The figures on the right correspond to the exponential case. The ones on the left represent the deterministic case.

It is observed (see fig-2) that the confident intervals are narrow when μ tends to zero and they increase when μ increases (for exemple when μ=0.5 the confidence interval is about (0.1, 0.9)). Obviously the estimations are strongly dependent of initial number of customers in orbit, which are not observed. In figure-1 we show such dependence when

μ=0.001.

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