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FROM G/G/1 RETRIAL QUEUES**

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# LARGE SAMPLE INFERENCE FROM G/G/1 RETRIAL QUEUES

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## Abstract

We consider a general  $G/G/1$  retrial queue where retrials can be non Markovian. We obtain asymptotically Gaussian consistent estimators for an unknown  $k$ -dimensional parameter assuming that the distribution functions of the variables involved are known. We consider distinct levels of information which can be interpreted as different disciplines of service. We analyze the problem of impatient customers in a  $G/G/1$  queue as a particular case. We also give some explicit estimators for Markovian queues.

**Keywords:**  $G/G/1$  retrial queue; maximum likelihood estimation; general retrials.

## 1. INTRODUCTION

In communication networks customers do not usually wait in a queue when the service area is busy. They leave and try again some time later. We say that they join a new flow of repeated arrivals (calls)

which is different from the primary flow. More precisely, if a customer finds the service area busy he "goes to an orbit" wherefrom he retries his call until he finds the service free. If the service area is free the customer begins to be served immediately leaving the system after he has finished his service. In recent years many articles have been published (see the survey by G.Falin [5]) dealing with retrial queues from a theoretical point of view so as to obtain distributions or transforms of some random variables. In this article we analyze retrial queues from a statistical viewpoint.

Falin et al. [7] construct integral estimators of the rate of retrial  $\mu$  in an  $M/G/1$  queue with exponential retrials by finding a Markovian process summarizing the system. Warfield and Foers ([8] and [9]) use a Bayesian approach to estimate the traffic intensity for different models of Teletraffic in a Markovian context where primary and repeated flows are subsumed into a homogeneous Poisson process with a different arrival rate. We deal with the statistical aspects of the estimation of some unknown parameters in the less restrictive  $G/G/1$  queue with general retrials where the random variable time ( $w$ ) between the retrials made by each customer does not have to be exponentially distributed. To do this we make some assumptions that we now describe.

We observe a  $G/G/1$  retrial queue over the time interval  $(0, T]$ . We assume that at  $t=0$  the first customer arrives at an empty system and  $T$  (the stopping time) is some positive random variable. Suppose that  $n=n(T)$  arrivals,  $m=m(T)$  complete services and  $r=r(T)$  retrials have been observed during the interval  $(0, T]$ . Let  $\{u_i; 1 \leq i \leq n\}$  and  $\{s_j; 1 \leq j \leq m\}$  be the interarrival and service times observed during  $(0, T]$ , and let  $\{w_k; 1 \leq k \leq r\}$  be the time elapsed between the retrials made by each

particular customer (interretrial times). Suppose that these vectors are three independent sequences of absolutely continuous, independently and identically distributed random variables (iid) with density functions given by

$$dF(u; \theta) = f(u; \theta) du, \quad u \in (0, \infty), \quad \theta \in \Theta \subset \mathbb{R}^p$$

$$dG(s; \phi) = g(s; \phi) ds, \quad s \in (0, \infty), \quad \phi \in \Phi \subset \mathbb{R}^q$$

$$dH(w; \eta) = h(w; \eta) dw, \quad w \in (0, \infty), \quad \eta \in \mathcal{H} \subset \mathbb{R}^v$$

respectively. Let  $u^*$ ,  $s^*$  and  $w^* = \{w_1^*, \dots, w_{X(T)}^*\}$  be the interarrival, service and interretrial incomplete times observed at time  $t=T$ . The stochastic process  $\{X(t); t \geq 0\}$  represents the number of customers in orbit at time  $t$ . Under the above hypotheses the likelihood function of the sample can be written as

$$L_T(\theta, \phi, \eta) = \left( \prod_{i=1}^n f(u_i; \theta) \right) \left( \prod_{j=1}^m g(s_j; \phi) \right) \left( \prod_{k=1}^r h(w_k; \eta) \right) \mathcal{H}(u^*, s^*, w^*; \theta, \phi, \eta) \quad (1)$$

where  $\mathcal{H}(u^*, s^*, w^*; \theta, \phi, \eta)$  is the likelihood corresponding to the right censored variables  $u^*$ ,  $s^*$  and  $w^*$ .

We use (1) to get consistent estimators for the parameters  $\theta$ ,  $\phi$  and  $\eta$  under different levels of information about the orbit. In the next section we consider a more general case where we assume we can observe  $r(T)$ ,  $\{X(t); 0 < t \leq T\}$  and also the random vector  $\{w_k; 1 \leq k \leq r\}$ , i.e. each customer in orbit is "marked" and we know when he repeats his call. However complete information is not usually available. It is then proper to consider some reductions in the amount of sample information on retrials. In section three we consider the case of quasi-complete information where customers are not assumed to be marked and in

consequence we cannot tell who makes a retrial. In section four we consider the case of partial information where we assume that we only observe the orbit at the time when a departure occurs. In the last section we give some examples of different service disciplines and interpret the  $G/G/1$  queue with impatient customers as a  $G/G/1$  queue with retrials. We also give some explicit estimators for the Markovian case.

## 2. ESTIMATORS UNDER COMPLETE INFORMATION

When there are no retrials the likelihood function (1) becomes the likelihood given in Basawa and Prabhu [3]. In this section we do an analysis similar to theirs but allowing for retrials. When the conditions

$$n^{-1/2} \frac{\partial}{\partial \theta} \log(1-F(u^*; \theta)) \xrightarrow{P} 0, \quad T \uparrow \infty \quad (2)$$

$$m^{-1/2} \frac{\partial}{\partial \phi} \log(1-G(s^*; \phi)) \xrightarrow{P} 0, \quad T \uparrow \infty \quad (3)$$

$$r^{-1/2} \sum_{k=1}^{X(T)} \left( \frac{\partial}{\partial \eta} \log(1-H(w_k^*; \eta)) \right) \xrightarrow{P} 0, \quad T \uparrow \infty \quad (4)$$

hold (where  $p$  denotes convergence in probability), it is easy to prove that (1) is asymptotically equivalent to

$$L_T^*(\theta, \phi, \eta) = \left( \prod_{i=1}^n f(u_i; \theta) \right) \left( \prod_{j=1}^m g(s_j; \phi) \right) \left( \prod_{k=1}^r h(w_k; \eta) \right). \quad (5)$$

In the article by Basawa and Prabhu the conditions (2) and (3) are

satisfied for a large class of distributions  $F$  and  $G$  even when the queue is not ergodic. If the stochastic process  $\{X(t); t \geq 0\}$  is ergodic and  $T$  is not random, we know from renewal theory that  $1-H(w_k^*; \eta)$  has a limit distribution and hence condition (4) is satisfied provided that  $(\partial/\partial\eta)\log(1-H(w_k^*; \eta))$  is a continuous function. It is also easy to prove that in some particular cases condition (4) is satisfied for other stopping rules. If the process  $\{X(t); t \geq 0\}$  is not ergodic, condition (4) can still be satisfied since the number of retrials by each customer in orbit increases to infinity when  $T \uparrow \infty$ . In this section we assume that conditions (2), (3) and (4) are satisfied.

Taking derivatives in (5) with respect to the unknown parameters, we obtain the maximum likelihood equations

$$\left. \begin{aligned} \sum_{i=1}^n \frac{\partial}{\partial \theta} \log(f(u_i; \theta)) &= 0 \\ \sum_{j=1}^m \frac{\partial}{\partial \phi} \log(g(s_j; \phi)) &= 0 \\ \sum_{k=1}^r \frac{\partial}{\partial \eta} \log(h(w_k; \eta)) &= 0 \end{aligned} \right\} \quad (6)$$



It is well known that under some regularity conditions on the densities  $f$ ,  $g$  and  $h$ , the maximum likelihood estimators (ML) for the parameters  $\theta$ ,  $\phi$  and  $\eta$  exist, are consistent and satisfy equations (6). Let  $\hat{\theta}$ ,  $\hat{\phi}$  and  $\hat{\eta}$  be the ML estimators and suppose that the stability conditions on stopping times

$$\frac{E[n(T)]}{n(T)} \xrightarrow{p} 1, \quad \frac{E[m(T)]}{m(T)} \xrightarrow{p} 1, \quad \frac{E[r(T)]}{r(T)} \xrightarrow{p} 1 \quad \text{as } T \uparrow \infty \text{ a. s.} \quad (7)$$

are satisfied. Then (see [3]), it is easy to prove that  $\hat{\theta}$ ,  $\hat{\phi}$  and  $\hat{\eta}$  are

asymptotically Gaussian and mutually independent.

**Remark-1.** The above result has been proved by Basawa and Prabhu ([2] and [3]) for a classical  $G/G/1$  queue. They also show that in some particular cases the first two conditions in (7) are satisfied. However the third condition in (7) is not always satisfied as the following example shows. Let  $T=t < \infty$  be a fixed time and let  $\rho$  be the traffic intensity ( $\rho = E[s]/E[u]$ ). If  $\rho \geq 2$  then the mean virtual time in an  $M/G/1$  retrial queue is infinite (see Falin and Fricker [6]). Hence the value of  $E[r(T)]$  is also infinite at that fixed time but  $r(t)$  is finite if  $t < \infty$  and in consequence the third condition in (7) is not satisfied. ■

**Remark-2.** If the density function  $h$  belongs to the exponential family

$$h(w; \eta) = a(w) \exp\{\eta b(w) - c(\eta)\}, \quad w \in (0, \infty),$$

then the unique solution in (6) for  $\eta$  is

$$\hat{\eta} = d^{-1} \left( r(T)^{-1} \sum_{k=1}^r b(w_k) \right)$$

where  $d(\eta) = c'(\eta)$ . Note that  $d'(\eta)$  is the variance of  $b(w)$ . Thus  $d(\eta)$  is a strictly increasing function and  $d^{-1}$  exists. The properties of the estimators  $\hat{\theta}$  and  $\hat{\phi}$  are studied in [2] and [3] under different stopping times rules. We focus next on the estimator  $\hat{\eta}$ . ■

### 3. ESTIMATORS UNDER QUASI-COMPLETE INFORMATION

Assume we cannot observe the interretrial times  $\{w_1, \dots, w_r\}$ . The

only information we have about the orbit is  $\{X(t); 0 < t \leq T\}$ ,  $r \equiv r(T)$  and the instants  $\{z_1, \dots, z_r\}$  where the retrials occur. We denote by  $\{v_i = z_i - z_{i-1}; 1 \leq i \leq r\}$ , where  $z_0 = 0$ , the time between retrials and by  $v_{r+1}^* = T - z_r$  the residual observation. If  $X(z_1^*) = 0$  then  $v_{i+1} = v_i' + y_i$  where  $v_i'$  is the time elapsed between the instant when the orbit becomes occupied and the next retrial and  $y_i$  is the time when the orbit is empty. Hence we consider  $\{v_1, \dots, v_r, v_{r+1}^*\}$  as the time elapsed between retrials when the orbit is not empty.

The distribution of the random variable  $v_i$  depends on the number of customers in the orbit at time  $z_i^-$  and is not memoryless. Hence it becomes quite difficult to use the maximum likelihood theory in this general case. However, if the interretrial times are exponentially distributed, it is easy to see that the likelihood function is

$$L_T(\theta, \phi, \mu) = \mu^r \exp\left(-\mu \sum_{k=1}^e kt_k\right) \left(\prod_{i=1}^n f(u_i; \theta)\right) \left(\prod_{j=1}^m g(s_j; \phi)\right) \mathcal{H}(u^*, s^*; \theta, \phi) \quad (8)$$

where  $e \equiv e(T) = \max\{X(t); 0 < t \leq T\}$ ,  $t_k$  is the sojourn time in state  $k$  during  $(0, T]$  and

$$\mathcal{H}(u^*, s^*; \theta, \phi) = (1 - F(u^*; \theta))(1 - G(s^*; \phi))$$

is the likelihood corresponding to the right censored variables  $u^*$  and  $s^*$ . Note that in this expression there are no residues corresponding to the retrials since they were included in the first term in (8). Thus we do not need to take condition (4) into account. (We use the classical notation  $\mu \equiv \eta$  when the retrials are exponentially distributed).

To obtain straightforward results we assume that the stopping time  $T \equiv T(m)$  is the instant when the  $m$ -th departure occurs (although we would get analogous results for other stopping rules provided the stability



conditions in (7) were satisfied). In this case we have the following asymptotic results:

**Lemma-1.** Assume we observe a G/G/1 retrial queue, with interretrial times exponentially distributed over the time interval (0,T]. Then

$$n/m \xrightarrow{P} \max(1, \rho), \quad m \rightarrow \infty.$$

**Proof.** Let  $X_m$  be the number of customers in the orbit immediately after the  $m$ -th departure occurs. We can then write  $n/m = 1 + X_m^{-1}$ . If  $X(t)$  is ergodic,  $X_m^{-1} \xrightarrow{P} 0$  when  $m \rightarrow \infty$  and hence  $n/m \xrightarrow{P} 1$ . On the other hand if  $X(t)$  is non ergodic we can write

$$\frac{n}{m} = \frac{n(T)}{T} \frac{T}{m} \tag{9}$$

where  $n(t)$ ,  $t \geq 0$ , is a renewal process with the renewals taking place at the times of arrival. Since  $T \rightarrow \infty$  almost surely (a.s.) when  $m \rightarrow \infty$ , it follows from the elementary renewal theorem that

$$n(T)/T \rightarrow (E[u])^{-1} \quad \text{a.s.} \tag{10}$$

But  $T/m$  can be written as

$$\frac{1}{m} \sum_{j=1}^m s_j + \frac{1}{m} \sum_{j=1}^{m-1} I_j \tag{11}$$

where  $I_j$  is the idle period after the  $j$ -th departure. By the strong law of large numbers the first member in (11) tends to  $E[s]$  when  $m \rightarrow \infty$ . Using the non ergodicity of  $X(t)$ ,  $I_j \rightarrow 0$  when  $j \rightarrow \infty$ . Thus the second term in (11) tends to zero. Finally from (9), (10) and (11) we have that  $n/m \xrightarrow{P} E[s](E[u])^{-1} = \rho \geq 1$  and the lemma is proved. ■

**Lemma-2.** If  $\{X(t); t \geq 0\}$  is an ergodic stochastic process and  $X = \lim_{t \rightarrow \infty} X(t)$ ,

then, under the assumptions of the previous lemma,

$$r/T \xrightarrow{P} \mu E[X], \quad m \rightarrow \infty.$$

**Proof.** Let  $r_k$  be the number of retrials made from state  $k$  (i.e. when  $X(t)=k$ ) which are observed during  $(0, T]$ . Then

$$\frac{r}{T} = \frac{1}{T} \sum_{k=1}^e r_k \quad (12)$$

which can be written as

$$\frac{1}{T} \sum_{k=1}^e \sum_{j=1}^{e_k} r_k^j \quad (13)$$

where  $e_k$  is the number of visits the process  $\{X(t); 0 \leq t \leq T\}$  pays to state  $k$  and  $r_k^j$  is the number of retrials made during the  $j$ -th visit to that state. Let  $t_k^j$  be the sojourn time in state  $k$  during the  $j$ -th visit. Then for each  $k$ ,  $\{r_k^j; j \geq 1\}$ , given  $\{t_k^j; j \geq 1\}$ , are independent random variables Poisson distributed at rate  $\mu k t_k^j$ . Since  $X(t)$  is an ergodic process  $e_k \rightarrow \infty$  when  $m \rightarrow \infty$ . Using the weak law of large numbers we have

$$(e_k)^{-1} \sum_{j=1}^{e_k} r_k^j \xrightarrow{P} \mu k t_k^* \quad (14)$$

where  $t_k^*$  is the mean sojourn time in state  $k$  for each visit. On the other hand

$$\frac{e_k}{T} = e_k (t_k)^{-1} \frac{1}{T} \int_0^T I_{\{X(t)=k\}}(t) dt$$

Since  $t_k^j$  are iid random variables, the strong law of large numbers ensures that  $t_k (e_k)^{-1} \rightarrow t_k^*$  a.s. ( $m \rightarrow \infty$ ). Provided that  $T \rightarrow \infty$  a.s. when  $m \rightarrow \infty$ , by the ergodic theorem we have

$$e_k/T \longrightarrow (t_k^*)^{-1} \Pr[X=k] \text{ a.s.} \quad (15)$$

Finally from (13), (14) and (15)  $r/T \xrightarrow{P} \mu E[X]$  as required. ■

**Theorem-1.** Under some regularity conditions on the densities  $f$  and  $g$ , the assumptions made in the two previous lemmas guarantee that the ML estimators  $\hat{\theta}$ ,  $\hat{\phi}$ ,  $\hat{\mu}$  are consistent.

**Proof.** The consistence of  $\hat{\phi}$  follows directly from classical maximum likelihood theory. Using lemma-1 we can replace the random variable  $n$  by  $m$  and the consistence of  $\hat{\theta}$  follows from maximum likelihood theory. Solving the first likelihood equation in (6) we get the ML estimator

$$\hat{\mu} = r \left( \sum_{k=1}^m kt_k \right)^{-1} \quad (16)$$

which can be written as

$$\hat{\mu} = \frac{r}{T} \left( \frac{1}{T} \int_0^T X(t) dt \right)^{-1}.$$

This, together with the ergodic theorem and lemma-2, gives us the consistence of  $\hat{\mu}$ . ■

**Theorem-2.** Under the assumptions of theorem-1  $\hat{\theta}$  and  $\hat{\phi}$  are asymptotically Gaussian and

$$m^{1/2}(\hat{\mu} - \mu) \Rightarrow N(0, \mu(E[u]E[X])^{-1}), \quad m \rightarrow \infty \quad (17)$$

where  $(\Rightarrow)$  denotes convergence in distribution,  $\mu$  is the true value of the parameter and  $N$  is a normal distribution. Moreover  $\hat{\theta}$ ,  $\hat{\phi}$  and  $\hat{\mu}$  are asymptotically independent.

**Proof.**  $\hat{\theta}$  and  $\hat{\phi}$  are asymptotically Gaussian because of the combined effect of maximum likelihood theory, lemma-1 and the central limit

theorem for random sums of iid random variables. When we do not consider the residues  $w^* = \{w_1, \dots, w_{X(T)}\}$  in the likelihood function (8), we can make use of the memoryless property of the exponential distribution to obtain

$$\hat{\mu}^* = r \left( \sum_{k=1}^r w_k \right)^{-1} \quad (18)$$

It is easy to prove that condition (4) is satisfied (note that we are assuming that  $X(t)$  is ergodic). Hence  $\hat{\mu}$  and  $\hat{\mu}^*$  are two asymptotically equivalent ML estimators. From the central limit theorem for random sums

$$r^{1/2}(\hat{\mu} - \mu) \Rightarrow \mathcal{N}(0, \mu^2), \quad m \rightarrow \infty \quad (19)$$

On the other hand

$$\frac{r}{m} = \frac{r}{T} \frac{T}{n} \frac{n}{m}.$$

Using lemmas 1 and 2 and the strong law of large numbers we have  $r/m = \mu E[u] E[X]$  and hence (17) follows. The asymptotic independence of the three estimators can be proved easily using the Cramér-Wold device. ■

**Remark-3.** Another way to estimate the rate of retrial  $\mu$  (or a general parameter  $\eta \in \mathbb{R}$ ) is via any estimator  $\hat{X}$  of  $E[X]$ . Since the variables involved in the estimation of  $E[X]$  are not iid, the integral estimator

$$\hat{X} = \frac{1}{T} \int_0^T X(t) dt \quad (20)$$

is the best alternative for this general case. The bidimensional process  $\{X(t), C(t); t \geq 0\}$ , where  $C(t)$  is equal to one if the service area

is busy and zero otherwise, is a regenerative process with regenerations occurring at the instants when the process reaches the state  $(0,1)$ . Let  $Y$  be the random variable "time between regenerations". Using the theory of regenerative processes we have

$$T^{1/2}(\hat{X} - E[X]) \Rightarrow N(0, \sigma^2)$$

where

$$\sigma^2 = (E[Y])^{-1} \text{Var} \left[ Y E[X] - \int_0^Y X(t) dt \right] \quad (21)$$

To obtain  $\sigma^2$  explicitly it is better to proceed as Falin et al. [7] did for the  $M/G/1$  retrial queue (with exponential retrial) rather than calculating it directly from expression (21). To obtain an estimator of  $\mu$  we can use the expression for  $E[X]$ . But that is only known for  $M/G/1$  queues with exponential retrials (see [5]). ■

#### 4. ESTIMATORS UNDER PARTIAL INFORMATION

Suppose that we observe an  $M/G/1$  queue with exponential retrials and that we can distinguish between primary and repeated arrivals at the beginning of each service. The sample information during  $(0, T]$  comprises the  $n$  primary and  $m-n$  repeated arrivals at an empty service, the number of customers in orbit at the instants of departure  $\{X_k = x_k; 1 \leq k \leq m\}$ , the service times  $s_1, \dots, s_m$  and the idle periods  $y_1, \dots, y_m$ . The corresponding likelihood function is proportional to

$$\lambda^{n+K} \mu^{n-m} \exp\left(-\lambda T - \mu \sum_{k=1}^{m-1} x_k y_k\right) \left(\prod_{k=1}^m g(s_k; \phi)\right)$$

and the ML estimators for  $\lambda$  and  $\mu$  are given by

$$\hat{\lambda}_1 = \frac{n+K}{T}, \quad \hat{\mu}_1 = (m-n) \left(\sum_{k=1}^{m-1} x_k y_k\right)^{-1}$$

where  $\lambda=0$  and  $K$  is the total number of arrivals during the busy period. The ML estimator  $\hat{\phi}$  of  $\phi$  is unchanged. The three estimators remain asymptotically Gaussian and mutually independent.

However if we could not distinguish between primary and repeated arrivals, the ML estimators for  $\lambda$  and  $\mu$  would be the unique solution (which exists with probability one) of the likelihood equations

$$\sum_{k=1}^{m-1} [(\lambda + \mu x_k)^{-1} - y_k] + K\lambda^{-1} - S = 0$$

$$\sum_{k=1}^{m-1} [(\lambda + \mu x_k)^{-1} - y_k] x_k = 0$$



where  $S$  is the busy period during  $(0, T]$ . In this case we would not have an explicit expression for the estimators (although they would be consistent and asymptotically normal) but we would be able to obtain numerical estimations. However the estimators  $\hat{\lambda}$  and  $\hat{\mu}$  would not be asymptotically independent.

## 5. SOME PARTICULAR CASES

The telephone network is an intuitive example of a retrial system. Since it is quite difficult to observe the orbit, it is not realistic

to assume either complete or quasi-complete information but we can interpret the model as a classical queue with a particular service discipline. We consider two cases:

1) Customers wait in a "room" until the server calls them after an idle period whose length is chosen by the server (in this case there are no retrials when the service is busy). In J.R. Artalejo and M. Martín [1] these systems are called "queues with priority intervals". We have two alternatives:

- a) the idle period depends on the number of customers waiting in the room, and
- b) the idle period is independent of the number of customers in the room.

2) The server does not decide upon the length of the idle period. The customers are the ones asking for the service following some random variable. We also have two alternatives:

- a) customers do not form a queue in the room (random discipline).
- b) customers form a queue (FIFO discipline).

In case one the server knows the distribution of the idle period and the customer has to solve the inference problem. Note that 1-a above corresponds to the partial information case when retrials are exponentially distributed. We studied this in section four. Note also that both alternatives in case one can be interpreted from the server's viewpoint provided we interpret the length of the idle period as the time needed by the customer to go from the room to the service area.

In case two the server has to infer the distribution of retrials. Note that case 2-a above corresponds to the classical queueing system with retrials we studied in sections two and three.

An alternative viewpoint is to consider the interretrial time in the orbit as an additional service time spent in the service room on secondary activities after which, if the service area is free, the customer again asks for the service. (If the service area is still busy the customer starts another activity).

In all these cases it seems appropriate to assume we have the amount of sample information we supposed we had in the previous sections. We next describe cases 1-b and 2-b.



**ML estimators under cases 1-b and 2-b.**

Case 2-b comprises the cases of both complete and quasi-complete information since it is only the first customer in the orbit who can retry his call. Hence we would get results identical to those we obtained in section-2 if conditions (2), (3), (4) and (7) were to be satisfied.

We next do an analysis similar to that of section three allowing  $H$  to be any general distribution on  $(0, \infty)$ . We assume conditions (2) and (3) hold. Condition (4) now becomes

$$r^{-1/2} \left( \frac{\partial}{\partial \eta} \log(1-H(w^*; \eta)) \right) \xrightarrow{P} 0, \quad T \uparrow \infty$$

which is satisfied if  $T=t$  is fixed. If  $T=T(m)$  is the instant when the  $m$ -th departure occurs this condition is also satisfied at least when  $H$  is an Erlangian distribution. Therefore we have the following lemma,



**Lemma-3.** For any general retrial distribution and under the assumptions of lemma-1

$$n/m \xrightarrow{P} \max(1, \rho^*)$$

where  $\rho^* = \rho + (E[u])^{-1} E[Z^*]$  and  $Z^* = \min(w|w > w^*, u|u > u^*)$ .

**Proof.** Proceeding as we did in lemma-1 we conclude that  $n/m \xrightarrow{P} 1$  when  $X(t)$  is an ergodic process. For the non ergodic case, when  $m$  is large enough, the idle periods after departures are distributed as  $\min(w|w > w^*, u|u > u^*)$  since the orbit is occupied with probability one. Using (9), (10) and (11), the lemma is proved. ■

**Lemma-4.** Under the assumptions of lemma-2,

$$r/T \xrightarrow{P} (E[w])^{-1} Pr[X \geq 1], \quad m \uparrow \infty.$$

Moreover when  $X(t)$  is non ergodic,

$$r/T \xrightarrow{P} (E[w])^{-1}, \quad m \uparrow \infty.$$

**Proof.** The first part can be obtained proceeding as we did in the proof of lemma-2. Let  $T_0$  and  $T_1$  be the time periods before  $T$  when the orbit is empty and busy respectively. If  $X(t)$  is non ergodic,  $T_0$  is finite a.s. when  $m \uparrow \infty$ . Therefore  $T_0/r$  tends to zero as  $m \uparrow \infty$ . The variable  $T_1$  can be written as  $w^* + w_1^* + \dots + w_r^*$  where  $w^* < \infty$  a.s. Finally, using the strong law of large numbers the lemma follows. ■

**Theorem-3.** Let  $\hat{\theta}$ ,  $\hat{\phi}$  and  $\hat{\eta}$  be the ML estimators obtained from (6) and assume that conditions (2), (3) and (4) are satisfied. Then, under some regularity conditions on the density functions  $f$ ,  $g$  and  $h$ , the

estimators  $\hat{\theta}$ ,  $\hat{\phi}$  and  $\hat{\eta}$  are consistent and asymptotically independent. Moreover

$$m^{1/2}(\hat{\eta} - \eta) \Rightarrow N(0, \sigma^2 \xi^{-1}) \quad (22)$$

where

$$\sigma^2 = -E \left[ \frac{\partial^2}{\partial \eta^2} \log(h(w; \eta)) \right]^{-1}$$

and where the constant  $\xi$  is  $\xi = E[u](E[w])^{-1} Pr[X \geq 1]$  in the ergodic case and  $\xi = E[u](E[w])^{-1}$  in the non ergodic case.

**Proof.** We get consistence proceeding as we did in the proof of theorem-1. The result obtained in (22) holds since (7) is satisfied (note that  $T$  is now to be interpreted as the instant of the  $m$ -th departure). Using lemmas three and four and the central limit theorem for random sums, the theorem follows. ■

**Remark-4.** If the densities  $f$  and  $h$  are exponentially distributed with parameters  $\lambda$  and  $\mu$ ,

$$n/m \xrightarrow{P} \max(1, \rho + \lambda(\lambda + \mu)^{-1}) = a$$

$$r/T \xrightarrow{P} \mu \min(1, Pr[X \geq 1]) = \mu b$$

$$m^{1/2}(\hat{\eta} - \eta) \Rightarrow N(0, c)$$

where  $c = \lambda \mu (ab)^{-1}$ . Note that  $\rho + \lambda(\lambda + \mu)^{-1} \geq 1$  if  $X(t)$  is non ergodic. Note also the probability  $Pr[X \geq 1]$  could be obtained explicitly even when  $H$  is a general distribution. We would get analogous results for other stopping times provided condition (7) was satisfied. ■

Case 1-b deals with the case of partial information. When  $f$  and  $h$  are both exponential densities we get estimators which are identical to those of the previous section (but we should set each  $x_k=1$  when  $1 \leq k \leq m-1$ ). When  $h$  is not exponentially distributed the likelihood function depends on the distribution of  $\min(u,w)$  in which case we can do the following approximation. Let  $\delta \equiv \delta(\lambda, \eta)$  denote the probability that the next customer arriving into the service comes from the orbit. Then

$$\delta = \int_0^{\infty} e^{-\lambda w} h(w; \eta) dw$$

If we remove the idle periods when the orbit is occupied, we have the "approximate likelihood"

$$\lambda^k \exp(-\lambda S - \lambda I_q) \delta^r (1-\delta)^{m-r-q}$$

where  $r$  is the number of served customers from the orbit and  $q$  and  $I_q$  denote the number of primary arrivals and the sum of idle periods during  $(0, T]$  respectively when the orbit and the service area are both free. Even when primary arrivals are not Poisson distributed we can at least estimate the probability  $\delta$  by the maximum likelihood method. (Note that this probability can be defined even when the distribution function  $H$  is not absolutely continuous).

#### General Markovian case.

If  $f$ ,  $g$  and  $h$  are exponentially distributed and  $X(t)$  is ergodic, we can get results analogous to the ones we have already obtained using the theory developed by Billingsley [4] for Markovian ergodic processes. Moreover we can relax some independence assumptions on the

variables  $u$ ,  $s$  and  $w$ . In particular, if  $u$ ,  $s$  and  $w$  depend on the number of customers in the orbit, i.e. if, when  $X(t)=k$ , the rates of occurrence are  $\lambda_k$ ,  $\mu_k$  and  $\phi_k$ , we can apply the results given in [4]. In order to do this let  $M(t)$  be the number of retrials made from the last departure till time  $t$ . It is easy to prove that the three dimensional process  $\mathcal{Y}(t)=\{(X(t),M(t),C(t));t\geq 0\}$  is a Markovian process with intensity rates

$$q[(k, i, 1) \rightarrow (k, i+1, 1)] = \mu_k, \quad k \geq 0, \quad i \geq 0$$

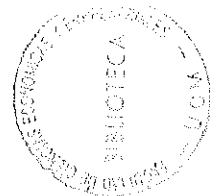
$$q[(k, i, 1) \rightarrow (k+1, i, 1)] = \lambda_k, \quad k \geq 0, \quad i \geq 0$$

$$q[(k, i, 1) \rightarrow (k, 0, 0)] = \phi_k, \quad k \geq 0, \quad i \geq 0$$

$$q[(k, 0, 0) \rightarrow (k, 0, 1)] = \lambda_k, \quad k \geq 0$$

$$q[(k, 0, 0) \rightarrow (k-1, 0, 1)] = \mu_k, \quad k \geq 1$$

$$q[(k, i, j) \rightarrow (k', i', j')] = 0, \quad \text{otherwise.}$$



Assuming that  $\lambda_k = \lambda j_1(k)$ ,  $\mu_k = \mu j_2(k)$  and  $\phi_k = \phi j_3(k)$ , where  $j_1(k)$ ,  $j_2(k)$  and  $j_3(k)$  are known functions, the ML estimators

$$\hat{\lambda} = n \left( \sum_{k=1}^{\infty} j_1(k) t_k \right)^{-1}, \quad \hat{\mu} = r \left( \sum_{k=1}^{\infty} j_2(k) t_k \right)^{-1}, \quad \hat{\phi} = m \left( \sum_{k=1}^{\infty} j_3(k) t_k \right)^{-1}$$

are strongly consistent and, when  $T \uparrow \infty$ ,

$$\Delta(T)^{1/2} \begin{pmatrix} \hat{\lambda} - \lambda \\ \hat{\mu} - \mu \\ \hat{\phi} - \phi \end{pmatrix} \Rightarrow N_3 \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad (\lambda j_1 + \mu j_2 + \phi j_3) \begin{pmatrix} \lambda/j_1 & 0 & 0 \\ 0 & \mu/j_2 & 0 \\ 0 & 0 & \phi/j_3 \end{pmatrix}$$

where  $\Delta(T)$  denotes the number of transitions of the process  $\{\mathcal{Y}(t); 0 \leq t \leq T\}$ ,  $j_i$  is given by

$$j_i = \left( \sum_{k=0}^{\infty} j_i(k) \pi_k \right), \quad i=1, 2, 3$$

and  $\{\pi_k; k \geq 0\}$  are the stationary probabilities of  $\{X(t); t \geq 0\}$ .

Note that for suitable functions  $f_1(k)$ ,  $f_2(k)$  and  $f_3(k)$  we could consider finite orbit capacity or/and  $N$  channels of service.

#### Queues with impatient customers.

Take a classical  $G/G/1$  queue and let  $X(t)$ ,  $t \geq 0$ , be the number of customers in the queue at time  $t$  without including the customer that is being served. Assume that a customer in a queue can leave the system if, after a random time  $w$ , he does not reach the service. That is to say, there are impatient customers. A classical  $G/G/1$  queue with impatient customers can be understood as "an almost  $G/G/1$  retrial queue" where the number of customers in the queue coincides with the number of customers in the orbit. There are only two differences. The first one is that after each time an impatient customer has left the queue, the number of individuals in the process  $X(t)$  decreases while this does not have to happen when a customer makes a retrial. The second difference is that with impatient customers we have censored observations when a customer in the queue goes into the service area.

If the random variable  $w$  is exponentially distributed we can do an analysis similar to that of section-3 using the memoryless property of exponential distributions. However if  $w$  has a general distribution  $H(w; \eta)$ , the likelihood functions (1) and (5) are not equivalent. In that case we would have to consider a problem of right censored variables because of the need to take into account the distribution of residues for all non impatient customers.

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