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On the $Q(M)$ depolarization metric

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Abstract

In this work, we have derived a depolarization metric, named $Q(M)$ here, from the nine bilinear constraints between the 16 Mueller-Jones matrix elements, reported previously by several authors following different approaches. This metric $Q(M)$ is sensitive to the internal nature of the depolarization Mueller matrix and does not depend on the incident Stokes vector. $Q(M)$ provides explicit information about the inner 3×3 internal matrix. Four bounds are associated to $Q(M)$ for a totally depolarizing, partially depolarizing, non-depolarizing diattenuating or partially depolarizing, and non-depolarizing non-diattenuating optical system, respectively. To our best knowledge, $Q(M)$ is the unique depolarization metric that provides such information in one single number.

Depolarization is understood as the loss in the degree of polarization of incident light from an optical system. The depolarization concept and different metrics proposed to measure it have been the theme of a recently, notorious, increasing interest [1–8]. Two of the main metrics are the depolarization index $\mathrm{DI}(M)$ [1,2] and the degree of polarization, $\mathrm{DoP}(M, S)$ [5,9]. The depolarization index is applied directly to the Mueller matrix and its physical limits are associated to a totally depolarizing system ($\mathrm{DI}(M) = 0$), to a partial depolarizing system ($0 < \mathrm{DI}(M) < 1$) and to a non-depolarizing system ($\mathrm{DI}(M) = 1$). The degree of polarization metric is usually applied directly to a beam of light, which is the most common case, but it can be calculated also for a previously determined Mueller matrix and by considering, in addition, an appropriate incident Stokes vector on the system. An appropriate incident Stokes vector is a polarization state which has a maximum output for the Mueller matrix under consideration. The $\mathrm{DoP}(M, S)$ has the same limits than the depolarization index.

In this work, we now show that a depolarization metric, named $Q(M)$ here, can be obtained from the nine bilinear constraints between the 16 elements of the Mueller-Jones matrix [10–13] and that $Q(M)$ provides explicit information about the inner 3×3 internal matrix.

The nine bilinear constraints between the Mueller-Jones matrix elements have been reported by several authors by using different approaches [10–13]. These relations basically assert that the 16 elements of the Mueller matrix are related through

$$\begin{aligned}
m_{00}^2 &\geqslant m_{10}^2 + m_{20}^2 + m_{30}^2 \\
m_{01}^2 &\geqslant m_{11}^2 + m_{21}^2 + m_{31}^2 \\
m_{02}^2 &\geqslant m_{12}^2 + m_{22}^2 + m_{32}^2 \\
m_{03}^2 &\geqslant m_{13}^2 + m_{23}^2 + m_{33}^2 \\
m_{00}m_{01} &\geqslant m_{10}m_{11} + m_{20}m_{21} + m_{30}m_{31} \\
m_{00}m_{02} &\geqslant m_{10}m_{12} + m_{20}m_{22} + m_{30}m_{32} \\
m_{00}m_{03} &\geqslant m_{10}m_{13} + m_{20}m_{23} + m_{30}m_{33} \\
m_{01}m_{02} &\geqslant m_{11}m_{12} + m_{21}m_{22} + m_{31}m_{32} \\
m_{01}m_{03} &\geqslant m_{11}m_{13} + m_{21}m_{23} + m_{31}m_{33} \\
m_{02}m_{03} &\geqslant m_{12}m_{13} + m_{22}m_{23} + m_{32}m_{33}
\end{aligned} \tag{1}$$

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The nine bilinear constraints are obtained when the system does not depolarizes (equality holds into Eq. (1) and the zero element, $0 = m_{00}^2 - m_{10}^2 - m_{20}^2 - m_{30}^2$, is added to the last three quadratic elements of Eq. (1) [10–13]:

$$\begin{aligned}
&m_{01}^2 - m_{11}^2 - m_{21}^2 - m_{31}^2 + m_{00}^2 - m_{10}^2 - m_{20}^2 - m_{30}^2 = 0\\
&m_{02}^2 - m_{12}^2 - m_{22}^2 - m_{32}^2 + m_{00}^2 - m_{10}^2 - m_{20}^2 - m_{30}^2 = 0\\
&m_{03}^2 - m_{13}^2 - m_{23}^2 - m_{33}^2 + m_{00}^2 - m_{10}^2 - m_{20}^2 - m_{30}^2 = 0\\
&m_{00}m_{01} - m_{10}m_{11} - m_{20}m_{21} - m_{30}m_{31} = 0\\
&m_{00}m_{02} - m_{10}m_{12} - m_{20}m_{22} - m_{30}m_{32} = 0\\
&m_{00}m_{03} - m_{10}m_{13} - m_{20}m_{23} - m_{30}m_{33} = 0\\
&m_{01}m_{02} - m_{11}m_{12} - m_{21}m_{22} - m_{31}m_{32} = 0\\
&m_{01}m_{03} - m_{11}m_{13} - m_{21}m_{23} - m_{31}m_{33} = 0\\
&m_{02}m_{03} - m_{12}m_{13} - m_{22}m_{23} - m_{32}m_{33} = 0
\end{aligned} \tag{2}$$

We define the depolarization metric, $Q(M)$, as the quotient of the sum of the four quadratic right-hand side inequalities by the sum of the four quadratic left-hand side inequalities of Eq. (1):

$$Q(M) \equiv \frac{\sum_{j=1,k=0}^{3} m_{jk}^2}{\sum_{k=0}^{3} m_{0k}^2}. \tag{3}$$

We can relate $Q(M)$, Eq. (3), with other well-established metrics by taking into account the definitions of the depolarization index [1,2,5], DI(M),

$$0 \leqslant \mathrm{DI}(M) = \frac{\sqrt{\sum_{j,k=0}^{3} m_{jk}^2 - m_{00}^2}}{\sqrt{3}m_{00}} \leqslant 1, \tag{4}$$

the diattenuation parameter [5,8,13], $D(M)$,

$$0 \leqslant D(M) = \sqrt{m_{01}^2 + m_{02}^2 + m_{03}^2}/m_{00} \leqslant 1, \tag{5}$$

and the polarizance parameter [5,8,13], $P(M)$,

$$0 \leqslant P(M) = \sqrt{m_{10}^2 + m_{20}^2 + m_{30}^2}/m_{00} \leqslant 1, \tag{6}$$

and substituting Eqs. (4)–(6) into Eq. (3), the $Q(M)$ depolarization metric, Eq. (3), can be expressed as

$$\begin{aligned}
0 \leqslant Q(M) &= \frac{\sum_{j=1,k=0}^{3} m_{jk}^2}{\sum_{k=0}^{3} m_{0k}^2} = \frac{3[\mathrm{DI}(M)]^2 - [D(M)]^2}{1 + [D(M)]^2}\\
&= \frac{\left\{\sum_{j,k=1}^{3} m_{jk}^2\right\}/m_{00}^2 + [P(M)]^2}{1 + [D(M)]^2} \leqslant 3.
\end{aligned} \tag{7}$$

By using the physically realizable bounds associated to the depolarization index and the diattenuation and the polarizance parameters, Eqs. (4)–(6), respectively, the physically realizable limits for $Q(M)$ can be established in the following way. The lower limit, $Q(M) = 0$, corresponds to a totally depolarizing system; $0 < Q(M) < 1$ denotes a partially depolarizing system; $1 \leqslant Q(M) < 3$ represents a non-depolarizing diattenuating system if $\mathrm{DI}(M) = 1$ or a partially depolarizing system if $\mathrm{DI}(M) < 1$; and the upper limit, $Q(M) = 3$, represents a totally non-depolarizing non-diattenuating optical system. Even when the depolarization index and the diattenuation and the polarizance parameters are well known, the functional relationship through which they are related by $Q(M)$, Eq. (7), provides further physical information than that offered by the depolarization index and by the degree of polarization. This statement can be clearly understood by noting that $Q(M)$ is the unique metric that can identify when a Mueller matrix has associated a pure or non-diattenuating Jones matrix, which happens when $Q(M) = 3$ (a non-depolarizing system can be associated to a diattenuating or to a non-diattenuating Jones matrix). The upper limit of both, $\mathrm{DI}(M)$ and $\mathrm{DoP}(M, S)$, is associated to a non-depolarizing system; but these metrics cannot distinguish if the system is diattenuating or non-diattenuating. On the other hand, the physical realizability of $Q(M)$ is ensured by the physical realizability of the depolarization index and the diattenuation and the polarizance parameters, as can be observed from the bounds fixed to all of them. It can be observed from Eq. (7) that $d(M)/m_{00}^2 \equiv \sum_{j,k=1}^{3} m_{jk}^2/m_{00}^2$ is the normalized contribution from the 3×3 diagonal part of the Mueller matrix [8,13].

From Eq. (7) it follows that the single numeric values associated to $Q(M)$ can be directly related with the metric values associated to $d(M)$, as can be noted from the following examples:

(a) If $\mathrm{DI}(M) = 0 \Rightarrow D(M) = P(M) = 0 \Rightarrow Q(M) = 0 \Rightarrow d(M) = 0$;
(b) If $\mathrm{DI}(M) = 1$ & $D(M) = P(M) = 0 \Rightarrow Q(M) = 3$ or $d(M)/m_{00}^2 \Rightarrow d(M) = 3m_{00}^2$;
(c) If $\mathrm{DI}(M) = 1$ & $D(M) = 0, P(M) = 1 \Rightarrow Q(M) = 3$ or $d(M)/m_{00}^2 + 1 \Rightarrow d(M) = 2m_{00}^2$;
(d) If $\mathrm{DI}(M) = 1$ & $D(M) = 1, P(M) = 0 \Rightarrow Q(M) = 1$ or $d(M)/2m_{00}^2 \Rightarrow d(M) = 2m_{00}^2$;
(e) If $\mathrm{DI}(M) = 1$ & $D(M) = P(M) = 1 \Rightarrow Q(M) = 1$ or $\left(d(M)/m_{00}^2 + 1\right)/2 \Rightarrow d(M) = m_{00}^2$.

Note carefully that $Q(M)$, Eq. (7), provides explicit information of the internal nature of the Mueller matrix. In this sense, the depolarization index does not provide a clearer relationship between their single number value metric and the 3×3 internal matrix.

In conclusion, it can be said that the $Q(M)$ is a depolarization metric that can be derived from well-established relationships, like the nine bilinear constraints between the elements of the Mueller matrix. Expression (7) is an interesting relation among other definitions of the depolarization index as $\mathrm{DI}(M)$, $D(M)$, and $P(M)$. $Q(M)$ provides explicit information about the inner 3×3 internal matrix. Four bounds are associated to $Q(M)$ for a non-depolarizing, partially depolarizing, non-depolarizing diattenuating (if $\mathrm{DI}(M) = 1$) or partially depolarizing (if $\mathrm{DI}(M) < 1$), and non-depolarizing non-diattenuating optical system, respectively. To our best knowledge, $Q(M)$ is the unique depolarization metric that provides such information in one single number.

This relation can open a future exploration of the internal nature of the Mueller matrix for depolarizing systems.

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