# Perspectives on 1/*f* noise in quantum chaos



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# Perspectives on $1/f$ noise in quantum chaos

**R A Molina[1], A Relaño[2], J Retamosa[2], L Muñoz[2,3], E Faleiro[3] and J M G Gómez[2]**

[1]Instituto de Estructura de la Materia - CSIC, Serrano, 123, 28006 Madrid, Spain.
[2] Departamento de Física Atómica, Molecular y Nuclear, Universidad Complutense de Madrid, E-28040 Madrid, Spain.
[3] Departamento de Física Aplicada, E. U. I. T. Industrial, Universidad Politécnica de Madrid, E-28012 Madrid, Spain.

E-mail: `molina@iem.cfmac.csic.es`

**Abstract.** The power spectrum of the $\delta_n$ statistic of quantum spectra presents $1/f^\alpha$ noise. For chaotic systems $\alpha = 1$ while for regular systems $\alpha = 2$. Although the transition from regularity to chaos is non universal, for a wide variety of systems with a mixed phase space the value of $\alpha$ is intermediate between 1 and 2 and can be related to the fraction of regular or chaotic orbits in the total phase space. This statistic can be a very useful tool for the analysis of experimental spectra, specially in the case of missing levels or spectral sequences with mixed symmetries.

## 1. Introduction

The study of energy level fluctuations is a basic tool in understanding quantum chaos. The pioneering work of Berry and Tabor showed that the spectral fluctuations of a quantum system whose classical analogue is fully integrable are well described by Poisson statistics [1] meaning that the energy levels behave as uncorrelated random variables. On the other hand, Bohigas *et al.* conjectured that the fluctuation properties of quantum systems which in the classical limit are fully chaotic coincide with those of Random Matrix Theory (RMT) [2] showing strong level repulsion and long-range correlations. The Bohigas-Giannoni-Schmit conjecture has been explained in the context of semiclassical periodic-orbit theory [3].

In the last decade a new approach to quantum chaos and spectral fluctuations has been proposed [4]. The main idea behind this approach is to consider the sequence of energy levels as a discrete time series and study level correlations using the tools of time series analysis. The analogy between the energy spectrum and a discrete time series is established in terms of the $\delta_n$ statistic, defined as the deviation of the $(n+1)$th level from its mean value. In terms of unfolded energy levels

$$\delta_n = \sum_{i=1}^{n} (s_i - \langle s \rangle) = \epsilon_{n+1} - \epsilon_1 - n, \qquad (1)$$

where $s_i = \epsilon_{i+1} - \epsilon_i$ and $\epsilon_i$ is the $i$-th unfolded level. The unfolded energy levels are defined using the average accumulated level density $\overline{N}(E)$ as $\epsilon_i = \overline{N}(E_i)$. This mapping is needed to remove the main trend defined by the smooth part of the level density in order to compare the statistical properties of the spectral fluctuations of different systems or different parts of the

 

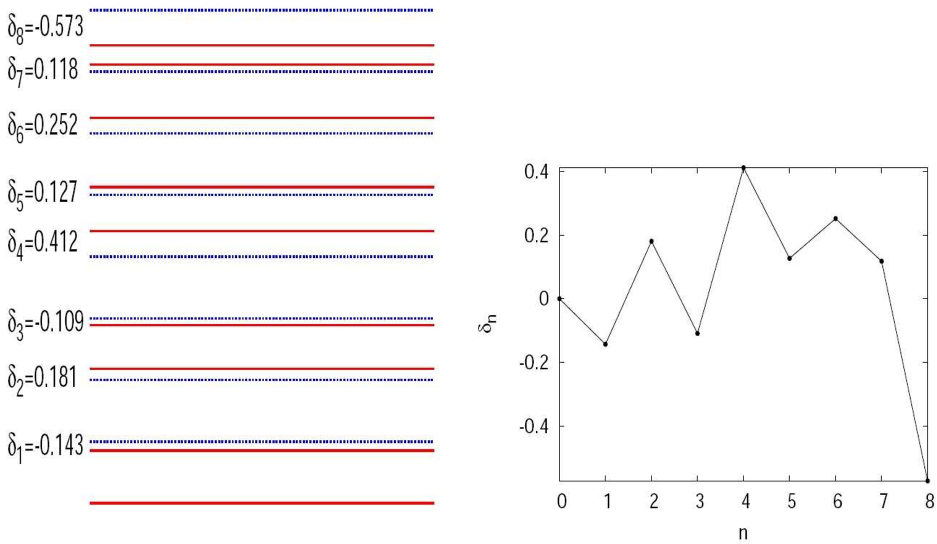


**Figure 1.** Example of how to build the statistic $\delta_n$ from the unfolded spectra. In the top panel we show an spectrum with the unfolded energies ordered in the y-axis. The red full lines is the real spectrum while the blue broken lines mark an hypothetical evenly distributed spectrum. The difference between the position of the red and blue lines is the $\delta_n$ showed in the left panel as a function of the level position $n$. Taking this level ordering $n$ as a pseudotime the $\delta_n$ positions can be considered as the time series of the position of a Brownian particle moving in one-dimension.

same spectrum. In the language of time series analysis the unfolding mapping is a procedure for making stationary the discrete time series defined by the $\delta_n$, its average and fluctuations not depending on time. In Fig. 1 we show an example of the procedure of building the $\delta_n$ statistic from the sequence of unfolded levels.

There are some differences between the $\delta_n$ statistic and the displacement of a Brownian particle. For example, $s_k - 1 > -1$ as there are no negative spacings by definition. Also the displacement of a Brownian particle usually depends on the past trajectory while the position of the energy levels and consequently the value of $s_k - 1$ depends not only on the lower energy levels, but also on the upper levels so considered as a time series there is a dependence on the past as well as on the future. Nevertheless, in spite of these peculiarities, the analogy exists, and the function $\delta_n$ is the analogue of a Brownian particle displacement in one-dimension.

Once we make the interpretation of the $\delta_n$ as a discrete time series we can use the powerful tools of time series analysis to get information about the dynamics of the quantum system. One of the most important and ubiquitous properties of time series is self-similarity, defined as the invariance of the signal $X(t)$ under a global scale transformation $X(\lambda t) \propto X(t)$. In real physical systems there are always limiting time scales limiting strict self-similarity so this property only appears in a finite range of time scales. To unveil the characteristic time scales of a signal $X(t)$ the Fourier transform can be used. The power spectrum $P(\omega)$ is normally used to avoid working with complex systems. Self-similarity in the frequency space can also appear.

Although the reasons are poorly understood, time series describing many different natural and social phenomena present a power spectrum behaving as

$$P(\omega) \propto \omega^{-\alpha}, \tag{2}$$

where $\alpha \geq 1$ [5, 6]. The signals of this type, generically known as presenting $1/f^\alpha$ noise, have no characteristic time scale, and their correlation time is comparable to the duration of the whole time series. Surprisingly enough in many different systems the power law is maintained over many orders of magnitude and the term "ubiquitous $1/f$ noise" was coined to describe this fact.

## 2. Theoretical derivation

The power spectrum of the $\delta_n$ can be calculated by

$$P_k^\delta = \frac{1}{N}\sum_{p,q}^{N-1} \delta_p \delta_q \exp\left(-\frac{2\pi i k(p-q)}{N}\right), \tag{3}$$

where $N$ is the number of levels in the sequence. In the first work that published results on this quantity [4] it was numerically found that regular systems present $1/f^2$ noise while chaotic systems present $1/f$ noise and this difference was proposed to be a general characterization of the spectral statistics of quantum chaotic systems. Numerical results from the Two Body Random Ensemble were shown also to present $1/f$ noise [7]. Two years later a theoretical derivation of $1/f$ and $1/f^2$ noises in chaotic and regular systems from RMT was reported by our group [8]. The power spectrum of the $\delta_n$ statistic was expressed in terms of the average power spectrum of the fluctuating part of the level density, known as spectral form factor $K(\tau)$.

$$\begin{aligned} \left\langle P_k^\delta \right\rangle &= \frac{N^2}{4\pi^2}\left[\frac{K(k/N)-1}{k^2} + \frac{K(1-k/N)-1}{(N-k)^2}\right] \\ &+ \frac{1}{4\sin^2(\pi k/N)} + \Delta, \ \ \text{for} \ \ N \gg 1, \end{aligned} \tag{4}$$

where $k = 1, 2, ..., N-1$ and $\Delta = -1/12$ for large energy level sequences $N \gg 1$ belonging to the three classical RMT and their interpolations while $\Delta = 0$ for Poisson spectra [8, 9]. Exact analytical expressions for the spectral form factor are known for the Gaussian Ensembles (GE) of RMT and for the Poisson ensemble. The important behavior for our purpose is that $K(\tau) \propto \tau$ for $\tau \ll 1$ for the GE while is constant for the Poisson ensemble. Substituting the values of the spectral form factor for the different ensembles we can obtain the final result for the GE in the small frequency region

$$\left\langle P_k^\delta \right\rangle \sim \frac{N}{2\nu\pi^2 k}, \ \ 1 \leq k \ll N, \quad \nu = 1, 2. \tag{5}$$

where $\nu = 1$ corresponds to the GOE ensemble and $\nu = 2$ to the GUE case. Similarly, for integrable systems and $k \ll d$, Eq. (4) becomes

$$\left\langle P_k^\delta \right\rangle \sim \frac{N^2}{4\pi^2 k^2}, \ \ 1 \leq k \ll N. \tag{6}$$

These expressions show that for small values of $k$ (small frequencies), the fluctuations of the excitation energy exhibit $1/f$ noise in chaotic systems and $1/f^2$ noise in integrable systems. As the frequency approaches the Nyquist frequency [10] $k = d/2$, higher order terms start becoming really appreciable.

The previous formulas describe not only the $1/f$ or $1/f^2$ behavior correctly, characteristic of small and intermediate frequencies, but also deviations at high frequency due to the $\Delta$ parameter. Also the factor preceding the power law is perfectly accounted for. An important point worth mentioning is that RMT is one of the few examples where $1/f^\alpha$ noise behavior have been theoretically derived. Experimentally, the presence of $1/f$ noise in microwave billiards was demonstrated in [11]. The theoretical expressions as a function of $K(\tau)$ paved the way to further applications of the power spectrum of the $\delta_n$ and of the implications of considering the level sequence as a time series.

## 3. Applications to imperfect spectra

The statistical analysis of spectral fluctuations and the comparison with RMT results requires energy level sequences to be pure, i. e., all the states of a sequence must have the same quantum numbers ($J^\pi$ or $J^\pi T$), and there must be no missing or spurious levels in any sequence. Experimentally, it is very difficult to guarantee that these conditions are fulfiled. Missing or spurious levels and mixed symmetries would induce misleading results about the chaotic or regular nature of the quantum systems involved. It is possible to generalize Eq. (4) to the general case of $l$ different symmetries mixed in the sequence, each with a fraction of total number of $N$ levels $\phi_i$ and each with a fraction of observed levels $f_i$

$$\begin{aligned}\left\langle P_k^\delta \right\rangle &= \frac{N^2}{4\pi^2}\sum_{i=1}^{l}\phi_i f_i\left[\frac{K_i\left(\frac{f_i k}{\phi_i N}\right)-1}{k^2}+\frac{K_i\left(\frac{f_i(N-k)}{d\phi_i}\right)-1}{(N-k)^2}\right]\\ &+ \frac{1}{4\sin^2\left(\pi k/N\right)}+\left\langle f^2\right\rangle\Delta, \end{aligned} \tag{7}$$

where $K_i(\tau)$ is the original spectral form factor of the $i$ pure level sequence mixed in the total one that is being analyzed, and $\left\langle f^2\right\rangle = \sum_{i=1}^{l}\phi_i f_i^2$. It is quite clear that in the limit of very large number of mixed symmetries and/or very few number of observed levels the Poisson result is recovered. In order to use Eq. (7) in practice it is unavoidable to assume a form of the function $K_i(\tau)$. Assuming chaotic RMT result is the most useful case. Our group showed that this expression can be useful to estimate the number of missing levels and mixed symmetries in realistic situations [12]. One way to check this assumption could be to estimate independently the fraction of missing levels from the short range correlations by the $P(s)$ [13, 14] and from long range correlations by the power spectrum. If both results are within statistical errors one could be more certain of the plausability of the original assumption.

## 4. Behavior in mixed systems

One of the most important open problems in the statistical study of quantum spectra is the behavior of the fluctuations in the spectra of mixed systems, systems that are in between integrability and chaos. There is a consensus in the quantum chaos community that this transition is not universal and different systems present different intermediate statistics. As far as the long range correlations are concerned there appear to be two main ways for the transition to develop as a function of a control parameter. In one case, the long range correlations present an intermediate behavior at all scales. Depending on the value of the control parameter the intermediate behavior can be closer to integrability or to chaos. In the other case, there are chaotic correlations but only between levels separated less than some energy difference. For larger energy separations correlations disappear as in the integrable case [15]. The transition parameter controls the value of the correlation energy scale in the spectrum that can be assimilated to the Thouless energy in weakly disordered mesoscopic systems. Of course, systems with all kinds of intermediate behavior between these two extreme cases are possible and found in the literature, complicating the analysis of the spectral statistics of mixed systems.

There have been two very interesting works calculating the power spectrum of semiclassical systems that transit between integrability and chaos as a function of a control parameter. For the systems analyzed in the intermediate regime $1/f^\alpha$ noise was found in the power spectrum of the $\delta_n$ statistic with $1<\alpha<2$ [16, 17]. From the theoretical formulas for the power spectrum, Eq. (4), it is easy to see that $1/f^\alpha$ implies a similar fractional exponent in the spectral form factor $K(\tau)\propto\tau^\beta$ with $\beta=2-\alpha$, for $\tau\ll 1$. This behavior is quite difficult to justify theoretically. Semiclassical models predict a mixture of $1/f$ and $1/f^2$ behaviors [18] as RMT models also do [15, 19]. Resorting to a RMT model taking into account Chaos Assisted Tunneling (CAT)[20],

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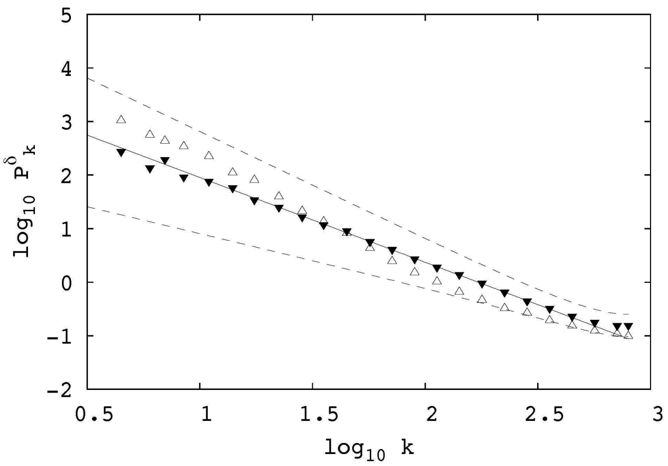


**Figure 2.** $P_k^\delta$ of the RMT model for a mixed system with CAT with an exponential dependence in the energy difference of the variance of the tunneling elements (full triangles) and a constant variance (empty triangles). Comparison with a power law $P_k^\delta = A/k^\alpha$ with $\alpha = 1.58$ (full lines) and the theoretical results for GOE and GDE (broken lines). To obtain the points in the figure a logarithmic binning average was used to reduce the number of points and the fluctuations. 20 matrices of size $N = 3000$ were used for the ensemble average.

recently one of us, A. Relaño, put forward an explanation of the fractional exponent found in mixed systems [21].

We define a RMT model for a mixed system with CAT as

$$H = \begin{pmatrix} GOE & V \\ V & GDE \end{pmatrix}, \tag{8}$$

where GOE and Gaussian Diagonal Ensemble (GDE) represent square diagonal submatrices, whose elements are the eigenvalues of a matrix belonging to the GOE or to GDE, defined as random uncorrelated diagonal matrix. They represent the chaotic energy levels and the regular energy levels of the system, respectively. $V$ determines the tunneling rate between these states. To model CAT in a physically sound way the $V$ matrix is composed by independent Gaussian random variables but the variance is non-constant and depends exponentially in the difference in energies between the regular and the chaotic state connected.

In Fig. 2 we show results for the power spectrum of model of Eq. (8) in the case that the variance of the random elements of $V$ depends exponentially on the difference of energies (full triangles)

$$\sigma_{ij} = \exp\left(-\lambda\left|\bar{E}_i^C - \bar{E}_j^R\right|\right), \tag{9}$$

where $\lambda$ is the control parameter ($\lambda = 30$ in the figure), $\bar{E}_i^C$ and $\bar{E}_j^R$ are the average position of the $i(j)$ energy levels in the chaotic(regular) region respectively. The average position is used instead of the actual position of the energy levels in order to use the same distribution of the matrix elements of $V$ for the whole ensemble of random matrices instead of a matrix-dependent distribution. The full line in Fig. 2 is a linear fit to the power spectrum that results in slope $\alpha = 1.58 \pm 0.01$. Depending on the value of $\lambda$ one can obtain different values of the exponent $\alpha$ going from 2 to 1. The empty triangles in the figure are results for the same model of Fig. (2) but with a constant value of $\sigma = 2\dot{1}0^{-3}$. Comparing to the results for Poisson and GOE (broken lines) one can see that the results go from GOE in large values of $k$ to close to Poisson in small values of $k$.

## 5. Perspectives

In spite of the great success of the RMT with energy-dependent CAT to explain fractional exponents in the power spectrum that, for example, have been calculated in billiards or perturbed oscillators with a mixed classical phase space [16, 17] there are still many opened questions before fully explaining the behavior of the spectral fluctuations in mixed systems. The first question to answer would be if the modeled form of the matrix elements corresponds to reality in models where $1/f^{\alpha}$ behavior appears and if the difference in the energy dependence of the variance of the matrix elements is behind the two kinds of behavior of the intermediate systems as it appears from the RMT model simulations.

As mentioned previously, RMT is one of the few cases where the origin of $1/f$ noise has been theoretically explained in detail. Could some kind of RMT be behind the origin of $1/f$ noise in other complex systems besides quantum chaotic ones? We believe more work is needed in this direction.

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