UNIVERSIDAD COMPLUTENSE DE MADRID

FACULTAD DE CIENCIAS MATEMÁTICAS

DEPARTAMENTO DE MATEMÁTICA APLICADA

TESIS DOCTORAL

Free Boundary Problems:
Qualitative Behaviour and Control Results

Problemas de Frontera Libre:
resultados cualitativos y de control

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

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Madrid, 2014

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Tesis doctoral realizada por:

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Madrid, 5 de Marzo de 2014

*To my family, who
won't understand it
for sure, but will
appreciate the effort.*

Acknowledgements

First of all, I want to express my gratitude to my supervisors, Prof. Jesús Ildefonso Díaz and Prof. Ángel Manuel Ramos del Olmo for having introduced me to the world of research, supported me and advised me in these years and patiently helped me to put through this thesis. I am also very thankful to Prof. Jean-Michel Coron for the very nice year spent at the "Laboratoire Jacques-Louis Lions" in Paris under his wise guidance and huge knowledge but especially for his kindness. My last stage in the European project FIRST (Fronts and InteRfaces in Science and Technology) has been at the "Universität Zürich" and I would like to thanks Prof. Michel Chipot for his friendly hospitality. I really appreciated the great efficiency of the referees, Prof. Michel Chipot, Prof. Benjamin Ivorra and Prof. José Francisco Rodrigues.

Many are the people that I have met during my European tour in these three years and that I want to thank. Going back at the beginning of this experience, I shall start with the two "pals" who have shared with me many good and bad moments in Madrid: Alex and Paris. Equally, I have to give many thanks to my friends in Complutense, Javi and Gonzalo, for teaching me what Paddle is and for the nice after-matches in front of tapas y cañas. To Giancarlo and his nice family for being very kind to me. To Stefano, better known as "Il gobbo", for the many funny moments in the streets of Madrid. I remember with great pleasure all the time spent with my flatmates Matteo and Jorge and the very gifted and funny football players Mathilde and el juventino Jorge.

I really enjoyed my year in Paris also because I found a really lovely accommodation. I am sincerely indebted with the family Canzani, Pajaro, Fernanda, Valentino and Juliette and all the other students or wanderers who have been living or just spent few days in the house during that period. I will never forget my friends and mates at LJLL who have been able to create such a good environment both for working and enjoying the time. In particular my office mates of the prestigious second floor, J.P., Malik, Giacomino, Juliette, Kho Casimir, André, Sylvain and the never missing participants of the round tables of the Friday evening, Charles, Mamadou, Pierre, Nastasia and Nicole.

Even though they are simple physicians they are very nice. So I want to say thanks also to the friends of the "Observatoire", Alex, bobo Jes, Andi and to the maîtresse du broderie Clerline.

Zurigo tecnicamente è nella Svizzera tedesca ma nel nostro dipartimento la lingua più parlata era senza dubbio l'italiano. Cominciando dall'Acquario vorrei ringraziare il Ferra, grande avversario tennistico, e il Capitano (conosciuto anche sotto il nome di Giacomo) perché prima o poi l'impiccagione del venerdì verrà approvata. Il trascinatore di folle Albo, gli irriducibili della corsa, Luca, Ale, Sara e il futuro maratoneta Salvatore. Ringrazio i quasi italiani Patrick e Giuseppe, il vez Michele, Marta e tutto il gruppo di chimica, Glo, Anastasia, Erica, Cri e Fabio. Many thanks go also to Jussuf, Marko, Urs and all the participants at the football meetings. Un pensiero va alla "popolazione" di Schwamendingenstrasse 90, a Kirta e alle sue fantastiche colazioni. Un ringraziamento caloroso lo rivolgo in particolar modo a Elia e Fabiola. Non mi dimenticherò mai poi del guaglione Nicola e delle nostre molte chiaccherate in hoch Deutsch.

I cannot forget to include all the others ESRs with whom I have met all around Europe in conferences and workshops. My deepest gratitude goes especially to Oni, Eduard, Rene, Tetiana and Hussein.

Un grande ringraziamento a Ludovica che è stata parte di un bellissimo percorso fatto insieme e che mi ha aiutato tanto a crescere.

Vorrei ringraziare i due amici di una vita, quelli che mi è costato molto lasciare per partire per questa avventura. Matteo, perché Holly e Benji in confronto sono due sconosciuti e anche poco affiatati, e il Desk, anche lui ora felicemente disperso nel mondo. Un pensiero va anche a tutte le persone che hanno fatto parte della mia vita e che spero continuino a esserlo. Il gruppo delle elementari: Simo, Carlos e il Dona. Mesner, Betto e il suo humor all'inglese, i siculi Simone e Marco con un piano B sempre in tasca e tutto il mondo scout che è una parte fondamentale della persona che sono. Un ringraziamento va anche a una persona da poco conosciuta e davvero in gamba, la Fra.

Ma in particolar modo ringrazio tutta la mia famiglia per esserci sempre stata, soprattutto nel momento del bisogno.

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Introducción

El objetivo de esta memoria es explorar los diferentes aspectos de la Teoría de Control, área de investigación que abarca problemas y técnicas de muy diversa índole. Dentro de los problemas de la Teoría de Control se distinguen principalmente los problemas de control óptimo y los de control exactos. El primer tipo de problemas tiene su origen en el cálculo de variaciones, que surge de problemas muy conocidos, como el problema de la Brachistocrona. El segundo se centra principalmente en el control de las trayectorias de los sistemas dinámicos con el fin de lograr un estado específico, abarcando problemas tales como *controlabilidad exacta, controlabilidad a cero, controlabilidad aproximada o estabilización*, y sus muchas variaciones. Para la descripción de tales problemas de control, genéricamente se precisa de tres elementos esenciales: las ecuaciones que describen la dinámica, las variables de entrada (cantidades disponibles a fin de controlar el sistema) y las salidas observadas (cantidades que deben ser controladas u optimizadas).

En la teoría de control pueden distinguirse dos tipos principales de sistemas: sistemas de dimensión finita y sistemas de dimensión infinita. En esta memoria, nos centraremos en este segundo tipo problemas, marco habitual para procesos descritos por ecuaciones en derivadas parciales (EDPs), y de los que se tratarn numerosos ejemplos. En particular, desempeñará un papel central en esta tesis una propiedad específica que presentan algunas soluciones de las EDPs: la frontera libre, importante campo de estudio en matemáticas, con numerosas contribuciones, e intrínsecamente conectada con el carácter no lineal de las EDPs, aspecto que no presenta la teoría lineal. Todos los problemas que se discuten en este trabajo son, de hecho, de naturaleza no lineal.

La expresión general de una EDP que utilizamos en esta tesis viene dada por

$$\frac{\partial \psi(y)}{\partial t} - \operatorname{div} a(y, \nabla y) + f(y) = u(x,t), \tag{1}$$

donde ψ, a, f, g son diferentes funciones estructurales que varían según el captulo estudiado, e y es la solución. La ecuación (1) puede ser de tipo parabólico o elíptico dependiendo de si ψ es o no la función nula, es decir si $\psi \equiv 0$. Las condiciones de contorno y los datos iniciales que se acoplen a la ecuación (1) también dependerán del problema en estudio.

La mayor parte de los resultados de esta memoria están relacionados con documentos elaborados por el propio autor, en colaboración con sus directores y otros especialistas, si bien en algunas ocasiones, la versión que aquí se presenta mejora la versión ya publicada. A continuación se listan las a dichos artículos según el capítulo en que se tratan:

(Capítulo 1):

· J.I. Díaz, T. Mingazzini and Á. M. Ramos, On the optimal control for a semilinear equation with cost depending on the free boundary, NETWORKS AND HETEROGENEOUS MEDIA Vol. 7, Number 4, December 2012.

(Capítulo 2):

· J.-M. Coron, J. I. Díaz, A. Drici, T. Mingazzini, Global Null Controllability of the 1-Dimensional Nonlinear Slow Diffusion Equation, Chinese Annals of Mathematics, 34B(3), 2013, 333-344.

· A. Drici, T. Mingazzini, Feedback Stabilization of the 1-Dimensional Porous Medium Equation. (Presentado).

(Capítulo 3):

· J.I. Díaz, T.Mingazzini, A criterion on the boundary non-diffusion or expansion of the support for some reaction-diffusion free boundary problems, or how the free boundary approaches to the boundary. (En prepración).

Capítulo 1. Empezamos por el modelado de un proceso decisional sobre la política de una industria. En esta situación, el propietario tiene que decidir la forma de regular la velocidad de producción que podemos suponer conectada directamente a la cantidad de sustancias contaminantes generadas durante la actividad. También suponemos que la contaminación se descarga a través de una tubería en un estanque de agua donde hay un área especial B llamada el *área protegida* que no puede ser infectada por alguna ley de regulación: el castigo por cualquier infracción a la presente restricción se castiga con una multa proporcional al volumen vertido. El problema requiere decidir cómo regular la velocidad de producción con el fin de obtener el mejor equilibrio entre las ganancias obtenidas por una alta tasa de producción y las pérdidas debidas a la multa que debe pagarse por la contaminación de B.

Matemáticamente, esto se traduce en un problema de optimización a través de una función de coste que representa las pérdidas de la industria y que ha minimizarse. El objetivo es decidir si existe una mejor solución. Aunque la teoría de la minimización ya está bien estudiada, adoptamos un funcional que depende de la medida del conjunto de positividad de soluciones de la ecuación de tipo (1). Este tipo de funcional no es común en la literatura a pesar de que es altamente útil. Para la formalización del funcional de coste J, se introduce la siguiente notación. Dada una función $y : \Omega \to \mathbb{R}^+$, definimos

$$\begin{aligned}
&\text{El conjunto de positividad}, \mathcal{S}(y) = \{x \in \Omega \ : \ y(x) > 0\}; \\
&\text{El conjunto nulo}, \mathcal{N}(y) = \{x \in \Omega \ : y(x) = 0\}; \\
&\text{La frontera libre}, \mathcal{F}(y) = \partial \mathcal{S}(y) \cap \Omega,
\end{aligned}$$

donde Ω es un dominio acotado (abierto conectado) de $\mathbb{R}^N$ con $N \geq 1$ que representa el estanque de agua, y $\partial\Omega$ es el borde de Ω. También introducimos la función creciente $G : \mathbb{R}^+ \to \mathbb{R}^+$, tal que $G(0) > 0$. Si denotamos con y_u ($y_u(T, \cdot)$) la solución del problema (1) (caso elíptico y parabólico) con término fuente $u(x)\chi_\omega$ ($u(T, x)\chi_\omega$) y condición de Dirichlet homogénea, la función de coste tiene la forma

$$J(u) = \int_\Omega \chi_{\mathcal{S}(y_u)\cap B}(x) \, \mathrm{d}x + \int_\Omega \frac{1}{G(y_u)} \, \mathrm{d}x. \tag{2}$$

Nuestro problema es encontrar

$$\min_{u \in U_{\mathrm{ad}}} J(u).$$

Un supuesto importante que se refleja en el modelo es la relación entre el tamaño de Ω y el tamaño de ω que representa la sección del tubo de desagüe, siendo este último mucho más pequeño. Esta observación, junto con el supuesto de que la ubicación de ω no está cerca de la orilla ($\partial\Omega$) da sentido al hecho de que la contaminación no llega a la costa o, si lo hace, su densidad es tan baja que puede despreciarse. Este hecho se introduce en el modelo a través de una condición de contorno homogénea de tipo Dirichlet y la formación de la frontera libre con la que el funcional tiene sentido.

Este tipo de problema tiene muchas aplicaciones: para la catálisis en un medio poroso (véase, por ejemplo, Aris [9] y los resultados de homogeneización de Díaz [40] y Conca, Díaz, Liñán y Timofte [31]); para las plantas de desalinización (véase, por ejemplo, Bleninger - Jirka [17] y Díaz, Sánchez, N. S ánchez, Veneros y Zarzo [46]); y para otros muchos problemas relativos al medio ambiente (véase, por ejemplo, [16] y [54]).

Logramos probar que en un subconjunto compacto de $L^\infty(\Omega)$ con la restricción de $u \geq 0$, el funcional J tiene al menos un mínimo. La propiedad de positividad de los controles garantiza que $y_u \geq 0$, dando sentido al modelo.

Tomamos ahora un funcional, J^*, ligeramente diferente, con una estructura similar a J, pero cuyos primeros términos es

$$J_1^* = \int_\Omega \chi_{\mathcal{N}(y_u)\cap B}(x). \tag{3}$$

y J_2^*, que suponemos continuo y creciente en y_u. El problema en este caso es que el funcional $y \mapsto \int_\Omega \chi_{\mathcal{N}(y)\cap B}(x)$ no es semicontinuo inferiormente en $L^1(\Omega)$, por lo que introducimos una condición a la familia de soluciones de $\{y_n\} \subset L^1(\Omega)$ que garantice no sólo la semicontinuidad sino también la continuidad del conjunto vacío. Para ser más precisos, con esta condición, si $y_n \to y$ en $L^1(\Omega)$ luego

$$\int_\Omega |\chi_{\mathcal{N}(y_n)}(x) - \chi_{\mathcal{N}(y)}(x)|\,\mathrm{d}x \to 0, \quad \text{según } n \to \infty.$$

La idea es proporcionar a la familia de funciones con una condición "no-arbitrariamente-plana" cerca del aparato de nivel cero, una herramienta introducida en los trabajos de Caffarelli [24], [25], Brezzi y Caffarelli [23], Phillips [67], Rodrigues [69] y Nochetto [65], [64], entre otros autores. Esta condición impide sustancialmente que los conjuntos de nivel próximos a cero sean demasiado grandes.

Se concluye el primer capítulo con una implementación numérica del problema de minimización del funcional (2) para el problema elíptico cuando (1) tiene la forma

$$\begin{cases} -\Delta y(x) + f(y(x)) = u(x)\chi_\omega & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

con $f(y) = |y|^{q-1}y$, $q \in (0,1)$. Se estudian los casos en los que Ω es un intervalo acotado de $\mathbb{R}$ y cuando se trata de cuadrado cerrado de $\mathbb{R}^2$. Para la función de G que aparece en J_1 elegimos $G(y) = (1 + ky)$ con $k > 0$.

Capítulo 2. En este capítulo nos centramos en la "controlabilidad global a cero" y en la estabilización de la ecuación de los medios porosos (PME) en una dimensión bajo diferentes condiciones de contorno. La PME es una formulación especial de (1) donde $\psi(y) = y^{1/m}$ con $m > 1$, $a(u, \nabla u) = \nabla u$, y $f \equiv 0$. Esta ecuación se puede usar para modelar varios procesos, como la filtración no lineal en medios porosos o la transferencia de calor no lineal.

Subrayamos el carácter degenerado de esta ecuación, que es lo que hace que los problemas de la controlabilidad y estabilización no sean de fácil resolución.

Para probar la controlabilidad global a cero de esta ecuación con apropiados controles al interior y al borde, se aplican diferentes técnicas tomadas de la teoría no-lineal y no degenerada de EDPs ([11]), y de la teoría de control de sistemas de dimensión finita (método de retorno: véase [34] y sus referencias).

En concreto, queremos mostrar cómo una acción combinada de los controles de frontera y de un control interno espacialmente homogéneo puede permitir la extinción de la solución en cualquier prescrito horizonte temporal $T > 0$, para cualesquiera datos iniciales en un cono de un espacio vectorial (*controlabilidad global a cero*). Se prueba la controlabilidad para los siguientes dos problemas de control

$$P_{DD} \begin{cases} y_t - (y^m)_{xx} = u(t)\chi_I(t) & \text{in } (0,1) \times (0,T), \\ y(0,t) = v_0(t)\chi_I(t) & t \in (0,T), \\ y(1,t) = v_1(t)\chi_I(t) & t \in (0,T), \\ y(x,0) = y_0(x) & x \in (0,1), \end{cases} \tag{5}$$

y

$$P_{DN} \begin{cases} y_t - (y^m)_{xx} = u(t)\chi_I(t) & \text{in } (0,1) \times (0,T), \\ (y^m)_x(0,t) = 0 & t \in (0,T), \\ y(1,t) = v_1(t)\chi_I(t) & t \in (0,T), \\ y(x,0) = y_0(x) & x \in (0,1), \end{cases} \tag{6}$$

donde $I := (t_1, T)$ con $t_1 \in (0,T)$, $m \geq 1$ y χ_I es la función indicatriz de I. En ambos problemas, y es la variable de estado y $U_{DN} := (u\chi_I, 0, v_1\chi_I)$, respectivamente $U_{DD} := (u\chi_I, v_0\chi_I, v_1\chi_I)$, las variables de control.

Enfatizamos el hecho de que el control interno $u(t)$ tiene la propiedad de ser independiente de la variable de espacio x y que todos los controles están activos sólo en una parte del intervalo de tiempo. Por otra parte, los sistemas son controlables a cero en tiempo arbitrario, entonces la forma localizada del control $u(t)\chi_I(t)$ (la misma para los controles de frontera) en un subintervalo de $[0,T]$ es más un dificultad enfática que real: sirve principalmente para subrayar que los controles no están activos en la primera parte. Análogamente podría elegirse un intervalo de control $(\underline{t}, \overline{t}) \in (0,T)$ o incluso más generalmente tres intervalos diferentes, uno para cada control v_0, v_1, u,

tal que el intersección de los tres sea no vacía.
El principal resultado de este capítulo es que cada dato inicial que pertenece al espacio $H^{-1}(0,1)$ y sea no negativo, se puede dirigir a cero en cualquier momento utilizando una combinación de los tres controles monodimensionales que aparecen en (5) o los dos en (6).
El enfoque adoptado se basa principalmente en la técnica llamada *método de retorno*, introducida en [32, 33] (ver [34, Captulo 6] para obtener información sobre este método). Más precisamente, se prueba primero la controlabilidad a cero del problema (5) mediante la aplicación de una idea aparecida en [28] (por la controlabilidad de la ecuación de Burger). En un segundo paso se usan argumentos de simetría para obtener el mismo resultado para (6).

Nuestra versión del método de retorno consiste en la elección de una familia a un parámetro (ε) de trayectorias que es independiente de la variable espacio desde un estado inicial $y \equiv 0$ hasta un estado final $y \equiv 0$, alejándose de cero en un tiempo central. Vamos a utilizar los controles para llegar a la solución de nuestro sistema a una de tales trayectorias, no importa cuál, para algún tiempo positivo más pequeño de T.

A continuación, se analiza un procedimiento simple para estabilizar el estado cero para la ecuación de los medios porosos con condición homogénea al borde de tipo Neumann. Consideramos el siguiente sistema de control

$$\begin{cases} y_t - (y^m)_{xx} = u, & (x,t) \in Q, \\ (y^m)_x(0,t) = 0, & t \in (0,\infty), \\ (y^m)_x(1,t) = 0, & t \in (0,\infty), \\ y(x,0) = y_0, & x \in (0,1), \end{cases} \tag{7}$$

donde $Q := (0,1) \times (0,\infty)$ y $m > 1$. El valor inicial y_0 puede cambiar de signo, entonces, para que (7) sea bien planteado, el término no-lineal y^m ha interpretarse como $|y|^{m-1} y$.

Para $u \equiv 0$, el comportamiento de las soluciones de (7) está bien explicado en [3], donde se muestra que si $y_0 \in L^\infty(\Omega)$, con $\Omega \subset \mathbb{R}^{\mathbb{N}}$ y $N \geq 1$, entonces

$$y(\cdot,t) \xrightarrow{L^p} \frac{1}{|\Omega|} \int_\Omega y_0 \, dx, \quad \text{para } t \to \infty.$$

La convergencia puede darse también en L^∞, si $N = 1$ (para $N > 1$, ha de exigirse que el valor inicial sea estrictamente positivo en $\overline{\Omega}$). En el caso en que el dato inicial tenga media cero, se demuestra la existencia de $C > 0$ tal que para cada $y_0 \in L^\infty(\Omega)$ with $\int_\Omega y_0 \, dx = 0$

$$\|y(\cdot,t)\|_{L^\infty(\Omega)} \leq \frac{C \|y_0\|_{L^\infty(\Omega)} \|y_0\|_{L^{m+1}(\Omega)}^{-1}}{\left(c(m-1)t + \|y_0\|_{L^{m+1}(\Omega)}^{1-m}\right)^{\frac{1}{m-1}}}.$$

Nuestro objetivo es estabilizar el sistema a cero independientemente de los datos iniciales, que se elegirán en un espacio de energía adecuado. Para hacer esto, utilizamos un control interno de tipo *feedback* $u : y \mapsto u(y) \in \mathbb{R}$, $u(t) := u(y(\cdot, t))$ que sea constante en el espacio:

$$u(t) := -\int_0^1 y^m(x,t)\,dx,$$

arriving to the following decay rate for the solution

$$\|y(\cdot,t)\|_{L^{m+1}(0,1)} \leq Ct^{-\frac{1}{m-1}},$$

Concluimos este capítulo con el estudio de la controlabilidad a cero para un tipo de ecuación con diferente no linealidad, cuyo valor de elipticidad no depende del valor de la solución sino de su primera derivada. Para empezar, tratamos el caso de ecuaciones no degeneradas y en un segundo paso se aborda el problema de las ecuaciones de tipo p-laplaciano, mostrando por el sistema no degenerado (8) la controlabilidad local y por (9) la controlabilidad global a cero bajo la hipótesis adicional $y_{0x} \geq 0$.

$$\begin{cases} y_t - a(y_x)_x = u(t) & (x,t) \in Q, \\ y_x(0,t) = 0 & t \in (0,T), \\ y_x(1,t) = v(t) & t \in (0,T), \\ y(x,0) = y_0 & x \in (0,1), \end{cases} \tag{8}$$

$$\begin{cases} y_t - (|y_x|^{p-2}y_x)_x = u_1(t)x + u_2(t) & (x,t) \in Q, \\ y_x(0,t) = v_0(t) & t \in (0,T), \\ y_x(1,t) = v_1(t) & t \in (0,T), \\ y(x,0) = y_0 & x \in (0,1), \end{cases} \tag{9}$$

siendo $Q = (0,1) \times (0,T)$ en ambos sistemas. En el primero paso, el hecho de ser no degenerado viene dado por la función $a : \mathbb{R} \to \mathbb{R}$ que verifica la condición $0 < \mu \leq a'(x)$. La variables de control vienen dados por u, v y u_1, u_2, v_1, v_2.

Capítulo 3. Este capítulo no se centra en aspectos directos de la Teoría de Control, sino que va a estudiar herramientas muy útiles para esta teoría. De hecho, el tema es la existencia y el comportamiento de las soluciones muy débiles de ciertos problemas elípticos y parabólicos semilineales y de su frontera libre cuando los datos son muy irregulares. El punto es que muchas veces cuando se trata de problemas de controlabilidad exacta o problemas de control óptimo, la oportunidad de elegir controles muy irregulares es muy útil porque da muchos más grados de libertad en la explotación del sistema. Específicamente, vamos a estudiar la forma en que se comporta la frontera libre de soluciones de algunas ecuaciones en derivadas parciales dependiendo de la traza de las soluciones. Consideramos los problemas

$$\begin{cases} -Lu + \lambda\beta(u) \ni f & \text{en } \Omega, \\ u = h & \text{sobre } \partial\Omega, \end{cases} \tag{10}$$

y la versión parabólica asociada

$$\begin{cases} u_t - Lu + \lambda\beta(u) \ni f(x,t) & \text{en } Q_T, \\ u = h(t,x) & \text{sobre } \Sigma_T, \\ u(x,0) = u_0(x) & \text{sobre } \Omega. \end{cases} \tag{11}$$

En ambos casos, $L = \operatorname{div}(A\nabla u)$ en un operador uniformemente elíptico y $\beta(u)$ es el gráfico maximal monótono de $\mathbb{R}^2$ tal que $0 \in \beta(0)$, dado por

$$\beta(u) = \lambda\, |u|^{q-1}\, u \tag{12}$$

en el caso de ecuaciones de reacción-difusión; y por

$$\beta(u) = \begin{cases} 0 & \text{para } u < 0, \\ [0,1] & \text{para } u = 0, \\ 1 & \text{para } u > 0, \end{cases} \tag{13}$$

en el caso del problema de obstáculo.

Como se mencionó antes, los problemas, tanto el elíptico como el parabólico, dan lugar a una frontera libre definida como el borde del soporte de la solución.

Estudiamos aquí el comportamiento de la frontera libre cerca del soporte dado al contorno h (respectivamente $h(\cdot,t)$). Empleando la misma notación a la del Capítulo 1, asumimos que

$$\mathcal{S}(h) \subsetneq \partial\Omega,$$

y

$$\mathcal{S}(h(\cdot,t)) \subsetneq \partial\Omega, \text{ for } a.e.\ t > 0.$$

La principal cuestión que se investiga es si el borde de $\mathcal{F}(u)$ está conectado o no con el borde de el soporte del dato h (y pregunta similar para la formulación parabólica).

En particular hemos encontrado algunos criterio suficiente sobre el comportamiento de h cerca del borde de su soporte, asegurando que el borde de $\mathcal{F}(u)$ está en contacto con $\partial\mathcal{S}(h)$. De esta manera, el soporte del dato no se difunde en el borde del dominio, es decir

$$\partial\mathcal{S}(u) \cap \partial\Omega = \partial\mathcal{S}(h).$$

Es lo que podemos llamar la propiedad de *no difusión en el borde del soporte*. Además, queremos dar condiciones suficientes que aseguren el comportamiento cualitativo contrario, es decir, condiciones en h que impliquen que hay una expansión estricta del apoyo $\mathcal{S}(h)$ en el borde $\partial\Omega$. Esto es, queremos saber los casos en los que $\mathcal{F}(u)$ no tiene contacto con $\partial\mathcal{S}(h)$, así

$$\partial\mathcal{S}(h) \subsetneq \partial\mathcal{S}(u) \cap \partial\Omega.$$

Llamamos a este fenómeno propiedad de *expansión en el borde del soporte*. En cierto sentido, esta investigación puede considerarse como continuación natural del estudio de la *no-difusión del soporte* (ver [39] y [5]) en el caso $h \equiv 0$; bajo apropiadas condiciones sobre f cerca de la frontera de su soporte se ha que $\mathcal{S}(u) = \mathcal{S}(f)$. En el caso de problemas parabólicos de frontera libre, el problema radica en estudiar la evolución de la frontera libre para tiempos pequeños (en inglés, *waiting time property*) y recibió una gran atención en los últimos 40 años (ver, por ejemplo, las monografías [70], [8] y sus muchas referencias).

El único papel en la literatura previa sobre tal comportamiento, segn conocimiento del autor, es [48], en el cual se prueba la *expansión en el borde del soporte* para el problema (10) para el caso particular $Lu = \Delta u$, $\beta = u^q$, con h la función de Heaviside y Ω el semi plano $\mathbb{R} \times \mathbb{R}^+$.

El punto delicado en nuestro estudio es que queremos permitir que el dato sea discontinuo, por lo que la noción de la traza de la soluciń debe tomarse en un marco muy general (algo que, en nuestra opinión, no es discutido suficientemente en [48]. Para ello, seguimos el enfoque propuesto por Haïm Brezis, en un artículo no publicado (en 1972) y a menudo mencionado en la literatura (véase [71], [61] y [42]), aplicable a ecuaciones semilinealeas de segundo orden con valor al borde en $L^1(\partial\Omega)$ (extendido más tarde a medidas sobre $\partial\Omega$).

Podemos probar la existencia y unicidad de forma análoga a la utilizada en [20], [71] y [61], para cada f y h en $L^1(\Omega;\rho) \times L^1(\partial\Omega)$, donde ρ es la función distancia del borde. Además la dependencia continua de los datos y el principio de comparación valen (ver Téorema 3.0.6).

Para el estudio del comportamiento de la solución cerca del borde de $\mathcal{S}(h)$ consideramos el caso particular en que $\Omega = \mathbb{R} \times [0, \infty)$ es el semi plano superior, L tiene coeficientes constantes, no hay término fuente f y $\beta(u)$ viene dado por (12) o (13). En lo que se refiere a los datos de la frontera, estamos interesados en la función h que satisface $h \in L^\infty(\partial\Omega), h(x_1) = 0$ en $(-\infty, 0)$ y $h(x_1) > 0$ en $(0, +\infty)$. Como queremos estudiar el comportamiento cerca de $x = (0, 0)$ y no en todo Ω, supondremos que h es no decreciente y que $h(x_1) = c_+ > 0$ para $x_1 \geq \delta > 0$. En este dominio introducimos la definición de solución limite muy débil por (10), construida como $u = \lim_{n\to\infty} u_n$, donde u_n es la solución del problema truncado, es decir, del problema definido en un rectángulo acotado con valores en la frontera, que es un truncamiento en sí mismo del original. El teorema 3.0.8 da el resultado de existencia y el principio de comparación para este problema.

Nuestro resultado principal para el comportamiento cualitativo de la solución de (10) es:

Teorema 1. *Suponiendo el marco anteriormente explicado, existen cuatro constantes $\underline{C} < \overline{C}$, $\underline{\varepsilon} < \overline{\varepsilon}$ y dos puntos $\underline{x}_{1,\varepsilon}, \overline{x}_{1,\varepsilon} > 0$, tales que:*

i) Si $h(x_1) \geq \overline{C} x_1^{\frac{2}{1-q}}$ para c.t. $x_1 \in (0, \overline{x}_{1,\varepsilon})$ y $h(x) \geq \overline{\varepsilon}$ para c.t. $x_1 \in (\overline{x}_{1,\varepsilon}, +\infty)$,entonces se verifica la propiedad de expansión en la frontera del soporte.

ii) Si $h(x_1) \leq \underline{C} x_1^{\frac{2}{1-q}}$ para c.t. $x_1 \in (0, \underline{x}_{1,\varepsilon})$ y $h(x_1) \leq \underline{\varepsilon}$ para c.t. $x_1 \in (\underline{x}_{1,\varepsilon}, +\infty)$, entonces se cumple la propiedad de no difusión en el borde del soporte.
En ambos casos, $q \in (0, 1)$, cuando β viene dado por (12), y $q = 0$ cuando β viene dado por (13).

Con respecto al problema parabólico, se investiga la existencia y unicidad de soluciones en dominios acotados, la estabilización de las soluciones para $t \to \infty$ a la solución del problema estacionario y el comportamiento cualitativo de las soluciones en un contexto similar al que se ha introducido para el caso elíptico. El concepto de solución muy débil en dominios acotados y el concepto de solución límite muy débil en cilindros con el semiplano como sección transversal es muy semejante a los adoptados para el caso elíptico. Los resultados relativos a la existencia y unicidad, así como el estudio del comportamiento del borde siguen las mismas estrategias que se aplican para el caso elíptico, bajo hipótesis apropiada en el dato inicial.

Se señala que el análisis de la frontera libre estudiado en este capítulo puede ser de gran interés en el estudio de algunos problemas de control óptimo, con aplicaciones al medio ambiente como se plantea en el capítulo 1.

Introduction

This thesis has the objective to explore different aspects of the Control Theory. Control Theory is by now a very huge area of research which includes very different problems and various techniques. Inside the Control Theory one can find mainly two types of problems: Optimal Control problems and Exact Control problems. The first kind has its origin in the calculus of variations, springing from very known problems such as the Brachystochrone problem. The second one focus mainly on the control of trajectories of dynamical systems in order to achieve a specific target. It includes problems which go under the name of *exact controllability, null controllability, approximate controllability, feedback stabilization* and many other variations.

For the description of a generic control problem three elements are essential: the equations describing the dynamics, the input variables (quantities available in order to control the system), the observed outputs (quantities to be controlled or optimized). We can distinguish two main types of systems which are encountered in the Control Theory: finite dimensional systems and infinite dimensional systems. We are interested in the second one which is the normal setting for processes described by Partial Differential Equations (PDEs): we will give many examples of such processes in the rest of this dissertation.

Control theory first scope was to design mechanism that keep certain to-be-controlled variables at constant values against external disturbances that act on a system or changes in its properties. The word system, which will be use often in the text, is a passe-partout term that can mean a physical, chemical, mechanical, biological or economical process but is also used to identify the mathematical models which are behind the process itself. An easy example of a control system from our common experience is our house. Houses are regulated by thermostats so that the inside temperature remains constant, notwithstanding variations in the outside weather conditions or changes in the situation in the house: doors that may be open or closed, number of persons present in a room, activity in the kitchen, etc.

In this thesis a central role is played by a specific property which some solutions of PDEs display: the Free Boundary or Dead Core. This itself is a huge field of study in Mathematics which have seen many contributions. This property is intrinsically connected with the nonlinear character of such PDEs since the linear theory does not present this aspect. All problems we discuss in this work are in fact of nonlinear nature.

The general form of the PDE we encounter in this thesis is

$$\frac{\partial \psi(y)}{\partial t} - \operatorname{div} a(y, \nabla y) + f(y) = g(x,t), \tag{14}$$

where ψ, a, f, g are different structural functions which change according to the chapter, and y is the solution of the equation. Equation (14) can be of parabolic or elliptic type depending on whether or not ψ is the null function, i.e., $\psi \equiv 0$. The boundary conditions and initial data which couple equation (14) will also depend on the problem under study.

Most of the results of this memoir are related to papers produced by this author in collaboration with his advisers and other specialists. In some occasions the version presented here improves the already published version. Here is the list of related papers:

(Chapter 1):

· J.I. Díaz, T. Mingazzini and Á. M. Ramos, On the optimal control for a semilinear equation with cost depending on the free boundary, NETWORKS AND HETEROGENEOUS MEDIA Vol. 7, Number 4, December 2012.

(Chapter 2):

· J.-M. Coron, J. I. Díaz, A. Drici, T. Mingazzini, Global Null Controllability of the 1-Dimensional Nonlinear Slow Diffusion Equation, Chinese Annals of Mathematics, 34B(3), 2013, 333-344.

· A. Drici, T. Mingazzini, Feedback Stabilization of the 1-Dimensional Porous Medium Equation. (Submitted).

(Chapter 3):

· J.I. Díaz, T.Mingazzini, A criterion on the boundary non-diffusion or expansion of the support for some reaction-diffusion free boundary problems, or how the free boundary approaches to the boundary. (In preparation).

Chapter 1. We start by modelling a decision-making process about the policy of an industry. In this situation the owner has to decide how to regulate the production facing the consecutive emission of pollution, which we can suppose directly connected to the amount of polluting substances generated during the activity. We also assume that the pollution is discharged through a pipe in a pond of water where there is a special area B called the *protected area* which cannot be infected due to some regulation law: the penalty for any infringement of this restriction is punished with a fine proportional to the volume invaded. The point is to decide how to regulate the rate of production in order to get the best balance between the gain obtained by a high rate of production and the loss due to the fine which needs to be paid for the contamination of B.

Mathematically, this translates into an optimization problem via a cost functional, which represents the loss of the industry and has to be minimized. The aim is to decide whether a best solution exists. Although the theory of minimization is already well studied, we adopt a functional which depends on the measure of the positivity set of solutions of equation of type (14). This type of functional is not common in the literature though it is highly useful.

For the mathematical formulation of the problem we can consider two main cases, the stationary case and the dynamical case.

In the first one, we assume that the factory has already reached a stable regime of production. The modelling of the reaction-diffusion-absorption of the pollutants in water is done through the nonlinear elliptic boundary-value system

$$\begin{cases} -\operatorname{div} a(y, \nabla y) + f(y) = u(x)\chi_\omega & x \in \Omega, \\ y = 0 & x \in \partial\Omega, \end{cases} \tag{15}$$

where Ω is a bounded domain (open connected) of $\mathbb{R}^N$ with $N \geq 1$ representing the pond of water, $\partial\Omega$ is the boundary of Ω, $a : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ have to satisfy specific structural conditions. The set $\omega \subset \Omega$ is the location of the outlet of the discharge tube. The characteristic function of ω is denoted with χ_ω.

In the dynamical case, we assume that the factory starts its production from zero (the initial datum is zero) and we analyse the situation at a certain time $T > 0$. The mathematical system we use in this case is the nonlinear parabolic problem

$$\begin{cases} \dfrac{\partial \psi(y)}{\partial t} - \operatorname{div} a(y, \nabla y) + f(y) = u(x,t)\chi_\omega & (x,t) \in Q_T, \\ y = 0 & (x,t) \in \Gamma, \\ y(x,0) = 0 & x \in \Omega, \end{cases} \tag{16}$$

with $Q_T = \Omega \times (0,T)$ for $T > 0$, $\Gamma = \partial\Omega \times (0,T)$ and $\psi : \mathbb{R} \to \mathbb{R}$ a non-decreasing function. In both cases y represents the density of the polluting solution in the water (observed output).

An important assumption which is reflected in the model is the relation between the size of Ω and the size of ω, the latest being much smaller, as it represents the cross area of a tube. This observation together with the assumption that the location of ω is not close to the shore ($\partial\Omega$) gives sense to the fact that the pollution does not reach the coast or, if it does, it is so low in density that it can be neglected. This fact is brought into the model through homogeneous Dirichlet boundary condition. This assumption is also fundamental for the formation of the Free Boundary.

The optimization problem that we set is actually a minimization problem, which can be seen as a minimization of the losses. The minimization functional J is composed by two terms, i.e. $J = J_1 + J_2$. The first term, J_1, measures the part of B invaded by the pollution and can be informally thought as a quantity increasing as the amount of pollution u increases. J_2 represents the losses which derive from the choice of a low rate of production, and so it will be a decreasing function of u. The aim is to find an appropriate u out of a possible bunch of candidates, which minimizes J.

This kind of problem has many applications: catalysis in a porous medium (see, e.g., Aris [9] and the homogenization results of Díaz [40] and Conca, Díaz, Liñán and Timofte [31]), desalination plants (see, for instance, Bleninger-Jirka [17] and Díaz, Sánchez, N. Sánchez, Veneros and Zarzo [46]), other environmental discharge problems (see, e.g. [16] and [54]), etc.

For the formalization of the cost functional J and for further uses we introduce the following notation. Given a function $y : \Omega \to \mathbb{R}^+$ we define:

$$\begin{aligned} &\text{The positivity set, } \mathcal{S}(y) = \{x \in \Omega \ : \ y(x) > 0\}; \\ &\text{The null set, } \mathcal{N}(y) = \{x \in \Omega \ : y(x) = 0\}; \\ &\text{The free boundary, } \mathcal{F}(y) = \partial\mathcal{S}(y) \cap \Omega. \end{aligned} \tag{17}$$

We recall that Ω is a bounded domain, and ω and B are measurable sets. We also introduce the increasing function $G : \mathbb{R}^+ \to \mathbb{R}^+$, such that $G(0) > 0$.

If we denote with y_u either the solution of problem (1.1) with source term $u(x)$ or the one of (1.2) at time T with source term $u(x,t)$, the minimization functional reads as follows

$$J(u) = \int_\Omega \chi_{\mathcal{S}(y_u)\cap B}(x)\,\mathrm{d}x + \int_\Omega \frac{1}{G(y_u)}\,\mathrm{d}x. \tag{18}$$

Our problem is to find

$$\min_{u\in U_{\mathrm{ad}}} J(u),$$

where U_{ad} is the set of admissible controls. We prove that when U_{ad} is any compact subset of the cone $K = (L^\infty_+(Q_T); ||\cdot||_{L^1(Q_T)}) \subset L^1(Q_T)$, we can find a minimum for J.

We then take a slightly different functional, J^*, with similar structure to J but whose first term is

$$J_1^* = \int_\Omega \chi_{\mathcal{N}(y_u)\cap B}(x). \tag{19}$$

and the second term is J_2^* which we assume continuous and increasing in y_u. The problem in this case is that the functional $y \mapsto \int_\Omega \chi_{\mathcal{N}(y)\cap B}(x)$ is not lower semicontinuous in $L^1(\Omega)$.

We introduce a condition on the family of solutions $\{y_n\} \subset L^1(\Omega)$ which guarantees not only the semicontinuity but also the continuity of the null set. To be more precise, under this condition, if $y_n \to y$ in $L^1(\Omega)$ then

$$\int_\Omega |\chi_{\mathcal{N}(y_n)}(x) - \chi_{\mathcal{N}(y)}(x)|\mathrm{d}x \to 0, \quad \text{as } n \to \infty.$$

The idea is to provide the family of functions with a "not-arbitrarily-flat" condition near the zero level set, an instrument which comes from the works of Caffarelli [24], [25], Brezzi and Caffarelli [23], Phillips [67], Rodrigues [69] and Nochetto [65], [64], among other authors. This condition substantially forbids the level sets close to zero to be too big.

We conclude the first chapter with some numerical implementation of the minimization problem of functional (18) for the elliptic problem when (15) has the form

$$\begin{cases} -\Delta y(x) + f(y(x)) = u(x)\chi_\omega & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases} \tag{20}$$

with $f(y) = |y|^{q-1}y$, $q \in (0,1)$. We study the cases when Ω is a bounded interval of $\mathbb{R}$ and when it is a bounded square of $\mathbb{R}^2$. For the function G appearing in J_1 we choose $G(y) = (1 + ky)$ with $k > 0$.

Chapter 2. In this chapter we focus on the “global null controllability” and stabilization of the Porous Medium Equation (PME) in one spacial dimension under different boundary type conditions. The PME is a specific formulation of (14) where $\psi(y) = y^{1/m}$ with $m > 1$, $a(u, \nabla u) = \nabla u, f \equiv 0$. This equation may be used to model different processes, from nonlinear filtration in porous media to nonlinear heat transfer.

We underline the degenerate character of this equation which is what makes the problem of controllability and stabilization not easy at all. Actually, there are really few works in the literature concerning controllability of such type of equations.

To prove the global null controllability for this equation with appropriate boundary and interior controls, we applied different techniques borrowed from the theory of nonlinear, nondegenerate PDEs ([11]) and from the Control Theory for finite dimensional systems (Return Method: see [34] and its references).

Specifically, we want to show how a combined action of boundary controls and a spatially homogeneous internal control may allow the global extinction of the solution (the so-called *global null controllability*) in any prescribed temporal horizon $T > 0$. We prove the global null controllability for the following two control problems

$$P_{DD} \begin{cases} y_t - (y^m)_{xx} = u(t)\chi_I(t) & \text{in } (0,1) \times (0,T), \\ y(0,t) = v_0(t)\chi_I(t) & t \in (0,T), \\ y(1,t) = v_1(t)\chi_I(t) & t \in (0,T), \\ y(x,0) = y_0(x) & x \in (0,1), \end{cases} \tag{21}$$

and

$$P_{DN} \begin{cases} y_t - (y^m)_{xx} = u(t)\chi_I(t) & \text{in } (0,1) \times (0,T), \\ (y^m)_x(0,t) = 0 & t \in (0,T), \\ y(1,t) = v_1(t)\chi_I(t) & t \in (0,T), \\ y(x,0) = y_0(x) & x \in (0,1), \end{cases} \tag{22}$$

where $I := (t_1, T)$ with $t_1 \in (0,T)$, $m \geq 1$ and χ_I is the characteristic function of I. In both problems, y represents the state variable and $U_{DN} := (u\chi_I, 0, v_1\chi_I)$, respectively $U_{DD} := (u\chi_I, v_0\chi_I, v_1\chi_I)$, is the control variable.

We emphasize the fact that the internal control $u(t)$ has the property to be independent of the space variable x and that all the controls are active only on a part of the time interval. However, since the systems are null controllable in arbitrarily fixed time, the localized form of the control $u(t)\chi_I(t)$ (the same for the boundary controls) on a subinterval of $[0,T]$ is more an emphatic than a real difficulty. It serves mostly to underline that the controls are not

active in the first time lapse. In the same way, it could be possible to take a control interval $(\underline{t}, \overline{t})$ with $\underline{t}, \overline{t} \in (0, T)$ or even more generally three different intervals, one for each control v_0, v_1, u, such that the intersection of the three is not empty.
The main results of this chapter is that any non-negative initial data which belongs to the space $H^{-1}(0,1)$ can be steered to zero at any time using a combinations of the three one dimensional controls appearing in (21) or the two in (22).

As already said, the approach followed is mainly based on the so-called *return method* introduced in [32, 33] (see [34, Chapter 6] for information on this method). Our version of the return method consists in choosing a suitable one parameter family of trajectories $a(t)/\varepsilon$, which is independent of the space variable, going from the initial state $y \equiv 0$ to the final state $y \equiv 0$. We shall use the controls to reach one of such trajectories, no matter which one, in some positive time smaller than the final T.

We then pass to analyse a simple procedure to stabilize the zero state for the porous medium equation with homogeneous Neumann boundary condition. We consider the following control system

$$\begin{cases} y_t - (y^m)_{xx} = u, & (x,t) \in Q, \\ (y^m)_x(0,t) = 0, & t \in (0,\infty), \\ (y^m)_x(1,t) = 0, & t \in (0,\infty), \\ y(x,0) = y_0, & x \in (0,1), \end{cases} \tag{23}$$

where $Q := (0,1) \times (0,\infty)$ and $m > 1$. The initial datum y_0 can be of changing sign, so in order to have the well-posedness of the problem (23), the nonlinear term y^m must be intended as $|y|^{m-1} y$.

For $u \equiv 0$, the behaviour of the solutions of (23) is well described in [3] where it is shown that if $y_0 \in L^\infty(\Omega)$, with $\Omega \subset \mathbb{R}^{\mathbb{N}}, N \geq 1$ then

$$y(\cdot,t) \xrightarrow{L^p} \frac{1}{|\Omega|} \int_\Omega y_0 \, dx, \quad \text{for } t \to \infty.$$

This convergence can be even in the L^∞-norm if $N = 1$ (for $N > 1$, one needs the initial data to be strictly positive in $\overline{\Omega}$). Also, different rates of homogenization are proved depending on the mean of the initial data. For the case of zero mean, they showed the existence of $C > 0$ such that for every $y_0 \in L^\infty(\Omega)$ with $\int_\Omega y_0 \, dx = 0$

$$\|y(\cdot,t)\|_{L^\infty(\Omega)} \leq \frac{C \|y_0\|_{L^\infty(\Omega)} \|y_0\|_{L^{m+1}(\Omega)}^{-1}}{\left(c(m-1)t + \|y_0\|_{L^{m+1}(\Omega)}^{1-m}\right)^{\frac{1}{m-1}}}.$$

Our purpose is to stabilize the system to zero independently of the initial data, which will be chosen in a proper energy space. To do that, we will use the internal feedback control $u : y \mapsto u(y) \in \mathbb{R}$, $u(t) := u(y(\cdot,t))$, constant in space, given by

$$u(t) := -\int_0^1 y^m(x,t)\,dx, \tag{24}$$

arriving to the following decay rate for the solution

$$\|y(\cdot,t)\|_{L^{m+1}(0,1)} \le Ct^{-\frac{1}{m-1}}, \tag{25}$$

We conclude this chapter studying the null controllability for a different type of nonlinearity, whose ellipticity value does not depend any more on the value of the solution but of its first derivative. To begin with, we treat the case of non-degenerate equations and in a second moment we address the problem of the p-laplacian equation, showing for the non-degenerate system (26) the local null controllability and for (27) the global null controllability under the assumption that $y_{0x} \ge 0$.

$$\begin{cases} y_t - a(y_x)_x = u(t) & (x,t) \in Q, \\ y_x(0,t) = 0 & t \in (0,T), \\ y_x(1,t) = v(t) & t \in (0,T), \\ y(x,0) = y_0 & x \in (0,1). \end{cases} \tag{26}$$

$$\begin{cases} y_t - (|y_x|^{p-2}y_x)_x = u_1(t)x + u_2(t) & (x,t) \in Q, \\ y_x(0,t) = v_0(t) & t \in (0,T), \\ y_x(1,t) = v_1(t) & t \in (0,T), \\ y(x,0) = y_0 & x \in (0,1). \end{cases} \tag{27}$$

In both systems $Q = (0,1) \times (0,T)$. In the first one, the non-degeneracy is expressed through $a : \mathbb{R} \to \mathbb{R}$ which satisfies the uniform ellipticity condition $0 < \mu \le a'(x)$. The controls in the systems are represented by the functions u, v and u_1, u_2, v_1, v_2 respectively.

Chapter 3. This chapter does not focus on direct aspects of Control Theory but goes to study very useful tools for this field. In fact, the topic is the existence and behaviour of very weak solutions of certain semilinear elliptic and parabolic problems and of their free boundary when the data are very irregular. The point is that many times when dealing with exact controllability problems or optimal control problems, the opportunity to choose very irregular controls is very useful because it gives much many degrees of freedom in operating on the system.

The ultimate goal is to describe the way the free boundary of solutions to some partial differential equations behaves depending on the trace of the solutions. We consider the problems

$$\begin{cases} -Lu + \lambda\beta(u) \ni f & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega, \end{cases} \tag{28}$$

and the associated parabolic version

$$\begin{cases} u_t - Lu + \lambda\beta(u) \ni f(x,t) & \text{in } Q_T, \\ u = h(t,x) & \text{on } \Sigma_T, \\ u(x,0) = u_0(x) & \text{on } \Omega. \end{cases} \tag{29}$$

In both cases, $L = \operatorname{div}(A\nabla u)$ is a uniformly elliptic operator and $\beta(u)$ is a maximal monotone graph of $\mathbb{R}^2$ such that $0 \in \beta(0)$: β is given by

$$\beta(u) = \lambda\, |u|^{q-1}\, u \tag{30}$$

in case of reaction-diffusion equations and by

$$\beta(u) = \begin{cases} 0 & \text{for } u < 0, \\ [0,1] & \text{for } u = 0, \\ 1 & \text{for } u > 0, \end{cases} \tag{31}$$

in case of the Obstacle Problem, for example.

As mentioned before, the above problems give rise to a free boundary defined as the boundary of the support of the solution. In the parabolic case definitions (17) apply to $u(\cdot,t)$. We assume always that

$$\mathcal{S}(h) \subsetneq \partial\Omega,$$

respectively

$$\mathcal{S}(h(t,\cdot)) \subsetneq \partial\Omega, \text{ for } a.e.\ t > 0.$$

To be more precise, our main goal is to find some sufficient criterion on the behaviour of h near the boundary of its support ensuring that the free boundary $\mathcal{F}(u)$ is in contact with $\partial\mathcal{S}(h)$. In this way the support of the datum is not diffused on the boundary of the domain and we would have

$$\partial\mathcal{S}(u) \cap \partial\Omega = \mathcal{S}(h).$$

It is what we can call the *non-diffusion on the boundary of the support* property. In addition, we want to give some sufficient conditions ensuring the opposite qualitative behaviour, i.e., to find conditions on h implying that there is a strict expansion of the support $\mathcal{S}(h)$ on the boundary $\partial\Omega$. In other words, we want to know cases in which $\mathcal{F}(u)$ has no contact with $\partial\mathcal{S}(h)$ and so

$$\mathcal{S}(h) \subsetneq \partial\mathcal{S}(u) \cap \partial\Omega.$$

We call this phenomenon the *expansion on the boundary of the support* property. In some sense, this research can be considered as a natural continuation of the study of the so called *non-diffusion of the support property* (see [39] and [5]) in the case where $h \equiv 0$; under a suitable behaviour of f near the boundary of its support $\mathcal{S}(u) = \mathcal{S}(f)$. In the case of parabolic free boundary problems this question is related with the behaviour of the free boundary for small times (the so called *waiting time property*) and received a great attention in the last 40 years (see, e.g., the monographs [70], [8] and their many references). The only paper in the previous literature about such boundary qualitative behaviour we are aware of is [48] in which they proved the *expansion on the boundary of the support* property for problem (28) in the special case of $Lu = \Delta u$, $\beta = u^q$, h given by the Heaviside function and Ω the half plane $\mathbb{R} \times \mathbb{R}^+$. As we shall see later, this property also holds even for suitable continuous boundary data h.

The delicate point in our study is that we want to allow the boundary datum to be discontinuous and so the notion of the trace of the solution must be taken in a very general framework (something which, in our opinion, is not discussed enough in [48]). We follow the approach proposed by Haïm Brezis, in an unpublished paper (1972) profusingly mentioned in the literature (see [71], [61] and [42]), which holds for semilinear second order boundary value problems with boundary data in $L^1(\partial\Omega)$ (later extended to measures on $\partial\Omega$). The main idea is to multiply by a "regular" test function and to integrate twice by parts, see Definition 3.0.5

We can prove existence and uniqueness, in similar way to [20], [71] and [61], for any f and h in $L^1(\Omega;\rho) \times L^1(\partial\Omega)$, where ρ is the distance function from the boundary. Moreover continuous dependence from the data and comparison principle hold (see Theorem 3.0.6).

For the study of the behaviour of the solution close to the boundary of $\mathcal{S}(h)$ we consider the particular case where $\Omega = \mathbb{R} \times [0, \infty)$ is the upper half plane, the matrix A is constant, there is no source term f and the nonlinear term $\beta(u)$ is given by (30) or (31). For what concern the boundary datum, we are interested in the case of h satisfying, $h \in L^\infty(\partial\Omega), h(x_1) = 0$ on $(-\infty, 0)$ and $h(x_1) > 0$ on $(0, +\infty)$.

The reason why we consider boundary data in $L^\infty(\partial\Omega)$ instead of in $L^1(\partial\Omega)$ (remember that now $\partial\Omega$ is unbounded, so $L^\infty(\partial\Omega) \not\subseteq L^1(\partial\Omega)$) is that we know the explicit solution in the unperturbed linear case ($\lambda = 0$, $L = \Delta$, $f \equiv 0$) with boundary data given by the Heaviside function. Such solution is given by

$$u(x_1, x_2) = 1 - \frac{1}{\pi} \arctan\left(\frac{x_1}{x_2}\right) \tag{32}$$

(the result can be found in [48] formula (2.6)). Having at disposal an explicit solution like (3.14) is really useful in the study of the behaviour of general solutions close to the point $x = (0, 0)$. In addition, since our main interest, as already said, is specifically the behaviour near the boundary of the support $\partial\mathcal{S}(h)$ and not in the whole Ω, we shall assume also that h is non-decreasing and that $h(x_1) = c_+ > 0$ for $x_1 \geq \delta > 0$. On an unbounded domain we introduce the notion of limit very weak solution of problem (28) given by $u = \lim_{n\to\infty} u_n$, where u_n is the solution of the truncated problem, i.e., the problem studied on a bounded rectangle with boundary datum which is a truncation itself of the original one. In Theorem 3.0.8 we give the existence result and the comparison principle for this problem.

Our main result on the qualitative behaviour of the solution of (28) is

Theorem 1. *Assuming the setting just explained we have that there exist four positive constants* $\underline{C} < \overline{C}$, $\underline{\varepsilon} < \overline{\varepsilon}$ *and two boundary points* $\underline{x}_{1,\varepsilon}, \overline{x}_{1,\varepsilon} > 0$, *such that :*
i) If $h(x_1) \geq \overline{C} x_1^{\frac{2}{1-q}}$ *for a.e.* $x_1 \in (0, \overline{x}_{1,\varepsilon})$ *and* $h(x) \geq \overline{\varepsilon}$ *for a.e.* $x_1 \in (\overline{x}_{1,\varepsilon}, +\infty)$ *then the expansion on the boundary of the support property holds.*
ii) If $h(x_1) \leq \underline{C} x_1^{\frac{2}{1-q}}$ *for a.e.* $x_1 \in (0, \underline{x}_{1,\varepsilon})$ *and* $h(x_1) \leq \underline{\varepsilon}$ *for a.e.* $x_1 \in (\underline{x}_{1,\varepsilon}, +\infty)$ *then the non-diffusion on the boundary of the support property holds.*
In both cases, $q \in (0, 1)$ *when* β *is given by* (3.8) *and* $q = 0$ *when* β *is chosen as* (31).

Concerning the parabolic problem, we investigate existence and uniqueness of solutions on bounded domain, the stabilization of solutions for $t \to \infty$ to the solution of the stationary problem and the qualitative behaviour of solutions in a context similar to the one already introduced for the elliptic case. The concept of very weak solution on bounded domains and the concept of limit very weak solution on cylinders whose cross section is the half plane are very similar to the ones adopted for the elliptic case. The results concerning existence and uniqueness as well as the study of the boundary behaviour follow the same strategies applied for the elliptic case, under appropriate hypothesis on the initial datum.

We point out that the analysis of the free boundary studied in this chapter can be of great interest in the study of some optimal control problems. Consider, for example, a functional J as the one given in Chapter 1 but with B being a neighbourhood in $\mathbb{R} \times [0, \infty)$ of the interval $(-k, 0) \times \{0\}$, and assume that the control variable this time is the boundary value h, with h as in Theorem 1. Then, in the minimization problem we will not consider those controls h whose growth near $x_1 = 0$ is too fast while we will prefer those with bigger mass at infinity and low growth close to the origin.

Chapter 1

Optimal Control and Free Boundary

This chapter has been written starting from the paper:

· J.I. Díaz, T. Mingazzini and Á. M. Ramos, On the optimal control for a semilinear equation with cost depending on the free boundary, NETWORKS AND HETEROGENEOUS MEDIA Vol. 7, Number 4, December 2012.

1.1 Motivation and formulation of the problem

The topic of this chapter lies in the intersection of three branches of Mathematics: Partial differential equations, optimal control, and free boundary problems. Indeed we design an optimal control problem whose cost functional depends in some way on the support of solutions to quasilinear elliptic-parabolic problems. A crucial role is played by the specific forms of the different non-linearities appearing in the differential equation. In fact, a right balance between them gives rise to the existence of a free boundary, in this case identified with the set separating the null part of the solution from the part where it is different from zero. It is exactly this phenomenon which gives sense to the problem. In a linear elliptic-parabolic problem, for example, or in other cases of non-linear problems, the cost functional would be trivial because the solution possesses an empty null set due to an infinite speed of propagation or because it satisfies a strict comparison principle. Let us take, for example, the heat equation in $\mathbb{R}^n \times \mathbb{R}^+$,

$$y_t - \Delta y = u,$$

with u being any non-negative function and $y(0) \geq 0$, both belonging to suitable spaces in order to have existence and uniqueness. It is a known fact that the solution y will satisfies $y(x,t) > 0$ for all $(x,t) \in \mathbb{R}^n \times (0,\infty)$. Therefore, a functional which depends on $S(y) = \{(x,t) \in \mathbb{R}^n \times (0,\infty) : y(x,t) > 0\}$ or any subsets of it would not have any sense, as it would contribute in the same way for all the solutions whenever $u \geq 0$.

Our interest in this type of problems sprang from physical-chemical problems. We consider a toy model consisting of a factory polluting water, gas or ice. The factory is located next to a pond of water like a lake and it has a discharge tube which ends into the water.

Suppose there is a specific region B inside the water which cannot be polluted because of some reason: imagine some area close to a holiday resort where people are allowed to swim. This area is referred to as a protected region. Every infringement of this restriction is sanctioned with a fine which is proportional to the part of B invaded.

The company has to decide how to regulate the rate of production, which we can suppose directly connected to the amount of polluting substances generated during the activity of the industry, in order to get the best balance between the gain obtained by a high rate of production and the loss due to the fine which needs to be paid for the contamination of B.

For the mathematical formulation of the problem we can consider two main cases, the stationary case and the dynamical case.

In the first one, we assume that the factory has already reached a stable regime of production. The important choice is which should it be. The modelling of the reaction-diffusion-absorption of the pollutants in water is done with the nonlinear elliptic boundary system

$$\begin{cases} -\operatorname{div} a(y,\nabla y) + f(y) = u(x)\chi_\omega & x \in \Omega, \\ y = 0 & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain (open connected) of $\mathbb{R}^N$ with $N \geq 1$ representing the pond of water, $\partial\Omega$ is the boundary of Ω, $a : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies hypothesis (1.5) and $f : \mathbb{R} \to \mathbb{R}$ is a non decreasing function satisfying (1.6). The set $\omega \subset \Omega$ is the location of the outlet of the discharge tube. The characteristic function of ω is denoted with χ_ω,

$$\chi_\omega(x) = \begin{cases} 1 & x \in \omega, \\ 0 & x \in \Omega \setminus \omega. \end{cases}$$

In the dynamical case, we assume that the factory starts its production from zero (the initial datum is zero) and we analyze which is the situation at a certain time $T > 0$. The mathematical system we use in this case is the nonlinear parabolic problem

$$\begin{cases} \dfrac{\partial \psi(y)}{\partial t} - \operatorname{div} a(y, \nabla y) + f(y) = u(x,t)\chi_\omega & (x,t) \in Q_T, \\ y = 0 & (x,t) \in \Gamma, \\ y(x,0) = 0 & x \in \Omega, \end{cases} \tag{1.2}$$

with $Q_T = \Omega \times (0,T)$ for $T > 0$, $\Gamma = \partial\Omega \times (0,T)$ and $\psi : \mathbb{R} \to \mathbb{R}$ a non decreasing function. In both cases y represents the density of the polluting solution in the water.

An important assumption which is reflected in the model is the relation between the size of Ω and the size of ω, the latest being much smaller. The reason is obvious if we think that the size of ω represents the cross area of a tube. This observation together with the assumption that the location of ω is not close to the shore ($\partial\Omega$) gives sense to the fact that the pollution does not reach the coast or, if it does, it is so low in density that it can be neglected. This fact is brought into the model through homogeneous Dirichlet boundary condition.

Another aspect in the model is that ω and B are disjoint: it would be a nonsense to let the discharge tube pour directly inside the protected region. This last fact could be anyway neglected as it doesn't affect the solution of the model.

The optimization problem that we set is actually a minimization problem, which can be seen as a minimization of the losses. The minimization functional J is composed by two terms, i.e. $J = J_1 + J_2$. The first term, J_1, measures the part of B invaded by the pollution and can be informally thought as a quantity increasing as the amount of pollution u increases. J_2 represents the losses which derive from the choice of a low rate of production, and so it will be a decreasing function of u. The aim is to find an appropriate u out of a possible bunch of candidates, which minimizes J.

We point out that this minimum might be not unique. What we have done is to prove the existence of a minimum but we did not investigate the question of uniqueness as it seems to be a too hard point given the high generality of our setting.

The problem under consideration is relevant in many different applied contexts: catalysis in a porous medium (see, e.g., Aris [9] and the homogenization results of Díaz [40] and Conca, Díaz, Liñán and Timofte [31]), desalination plants (see, for instance, Bleninger-Jirka [17] and Díaz, Sánchez, N. Sánchez, Veneros and Zarzo [46]), other environmental discharge problems (see, e.g. [16] and [54]), etc.

For the formalization of the cost functional J and for further uses we introduce the following notation. Given a function $y : \Omega \to \mathbb{R}^+$ we define:

$$\begin{aligned}&\text{The positivity set, } \mathcal{S}(y) = \{x \in \Omega \ : \ y(x) > 0\};\\ &\text{The null set, } \mathcal{N}(y) = \{x \in \Omega \ : y(x) = 0\};\\ &\text{The free boundary, } \mathcal{F}(y) = \partial\mathcal{S}(y) \cap \Omega.\end{aligned} \tag{1.3}$$

We recall that Ω is a bounded domain, and ω and B are measurable sets. We also introduce the increasing function $G : \mathbb{R}^+ \to \mathbb{R}^+$, such that $G(0) > 0$. If we denote with y_u either the solution of problem (1.1) with source term $u(x)$ or the one of (1.2) at time T with source term $u(x,t)$, the minimization functional reads as follows

$$J(u) = \int_\Omega \chi_{\mathcal{S}(y_u)\cap B}(x)\, \mathrm{d}x + \int_\Omega \frac{1}{G(y_u)}\, \mathrm{d}x. \tag{1.4}$$

Our problem is to find

$$\min_{u \in U_{\mathrm{ad}}} J(u),$$

where U_{ad} is the set of admissible controls: It is a compact subset of an energy functional space with the constraint $u \geq 0$. This property of the controls guarantees that $y_u \geq 0$ (see Remark 1.2.3).

Remark 1.1.1. Instead of (1.4) we could have considered a functional whose first term were $-\int_\Omega \chi_{\mathcal{N}(y_u)\cap B}(x)\, \mathrm{d}x$.

1.2 Well-posedness and free boundary

There are many references dealing with degenerate quasilinear equations, their well-posedness and qualitative properties. We recall, without trying to be complete, the monograph of DiBenedetto [47] were the porous medium type equations and the p-laplacian type equations are treated. For the case of doubly nonlinear equations we cite [4],[66] and [27] where the concept of entropy solution was developed. We will refer mainly for the existence and uniqueness to [7] where the authors study triply nonlinear degenerate elliptic-parabolic-hyperbolic equations. In general the concept of entropy solution is useful for the uniqueness of solutions which a weak formulation may not have.

We will give the definition of entropy solution and then a well-posedness result only for problem (1.2). All definitions and results for (1.1) can be obtained upon setting $\psi \equiv 0$, $u(t,\cdot) \equiv \tilde{u}(\cdot)$. First of all we impose the structural assumption on the nonlinearities appearing in (1.1) and (1.2). Let us start with $a : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$:

$$
\begin{aligned}
&a \text{ is continuous with } a(r,0) = 0;\\
&(a(r,\xi) - a(r,\eta)) \cdot (\xi - \eta) \geq 0;\\
&\text{There exist } p \in (1,\infty), C_1 > 0, C_2 \geq 0, \text{ such that}\\
&\qquad a(r,\xi) \cdot \xi \geq C_1 |\xi|^p,\\
&\qquad |a(r,\xi)| \leq C_2(|\xi|^{p-1});\\
&\text{There exists } C \in C(\mathbb{R}^2, \mathbb{R}) \text{ such that}\\
&(a(r,\xi) - a(s,\eta)) \cdot (\xi - \eta) + C(r,s)(1 + |\xi|^p + |\eta|^p)|r - s| \geq 0.
\end{aligned}
\tag{1.5}
$$

All these relations should be valid for all $\xi, \eta \in \mathbb{R}^N, r, s \in \mathbb{R}$. In addition we impose the following restrictions on the remaining terms:

$$
\begin{aligned}
&\psi : \mathbb{R} \to \mathbb{R} \text{ continuous, non-decreasing with } \psi(0) = 0 \text{ and } \psi(r) > 0, r > 0;\\
&f : \mathbb{R} \to \mathbb{R} \text{ continuous, non-decreasing with } f(0) = 0, f(\pm\infty) = \pm\infty;\\
&u \in L^\infty(Q_T).
\end{aligned}
\tag{1.6}
$$

We will use the following notation: for $c, z \in \mathbb{R}$

$$
z_c^+ = (z - c)^+ = \max(z - c, 0), \quad z_c^- = (z - c)^- = -\min(z - c, 0)
$$

$$
H_c^+(z) = \begin{cases} 1, & z > c, \\ 0, & z \leq c, \end{cases} \qquad H_c^-(z) = \begin{cases} 0, & z \geq c, \\ -1, & z < c, \end{cases}
$$

For any set X we define $C_0^\infty(X)$ as the space of infinitely differentiable function whose support is contained in X.

Definition 1.2.1. *An entropy solution of* (1.2) *is a measurable function* $y : Q_T \to \mathbb{R}$ *satisfying*

(P1) $y \in L^\infty(Q_T) \cap L^p(0,T; W_0^{1,p}(\Omega))$*;*

(P2) For all $\xi \in C_0^\infty([0,T) \times \Omega)$

$$
\int_{Q_T} \psi(y)\partial_t \xi - a(y, \nabla y) \cdot \nabla \xi - f(y)\xi \, \mathrm{d}x \, \mathrm{d}t + \int_{Q_T} u\chi_\omega \xi \, \mathrm{d}x \, \mathrm{d}t = 0;
$$

(P3.1) For all $(c,\xi) \in \mathbb{R}^+ \times C_0^\infty([0,T) \times \overline{\Omega})$, $\xi \geq 0$,

$$\int_{Q_T} \psi_c^+(y)\partial_t\xi - H_c^+(y)a(y,\nabla y)\cdot\nabla\xi - H_c^+(y)f(y)\xi \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} H_c^+(y)u\chi_\omega\,\xi\,\mathrm{d}x\,\mathrm{d}t \geq 0. \tag{1.7}$$

(P3.2) For all $(c,\xi) \in \mathbb{R}^- \times C_0^\infty([0,T) \times \overline{\Omega})$, $\xi \geq 0$,

$$\int_{Q_T} \psi_c^-(y)\partial_t\xi - H_c^-(y)a(y,\nabla y)\cdot\nabla\xi - H_c^-(y)f(y)\xi \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} H_c^-(y)u\chi_\omega\,\xi\,\mathrm{d}x\,\mathrm{d}t \geq 0. \tag{1.8}$$

(P3.3) For all $(c,\xi) \in \mathbb{R} \times C_0^\infty([0,T) \times \Omega)$, $\xi \geq 0$, (1.7) *and* (1.8) *hold.*

Proposition 1.2.2. *Assume that* (1.5) *and* (1.6) *hold. For each* $u \in L^\infty(Q_T)$ *there exits one entropy solution to problem* (1.2) *and it is unique in the sense that if* y *and* $\hat{y}$ *are two entropy solutions, then* $\psi(y) \equiv \psi(\hat{y})$ *and* $f(y) \equiv f(\hat{y})$.

Moreover, let us assume again that y *and* $\hat{y}$ *are two solutions corresponding to the data* u *and* $\hat{u}$ *respectively. Then for a.e.* $t \in (0,T)$,

$$\int_\Omega (\psi(y)-\psi(\hat{y}))^+(t) + \int_{Q_t}(f(y)-f(\hat{y}))^+ \leq \int_{Q_t} H^+(y-\hat{y})(u-\hat{u})\chi_\omega, \tag{1.9}$$

where $Q_t = \Omega \times (0,t)$.

The proof of this result can be found in [7].

Remark 1.2.3. From Proposition 1.2.2, we can deduce two interesting properties of solutions. The first one is the positiveness of solutions, which means that if the source term u is non negative then the same property is true for the solution y. To show this, we notice that if we set $u \equiv 0$ we have that the corresponding solution is $y \equiv 0$. Now we take another solution $\hat{y}$ with $\hat{u} \geq 0$. We plug these two solutions in (1.9) and get

$$\int_\Omega (-\psi(\hat{y}))^+(t) + \int_{Q_t}(-f(\hat{y}))^+ \leq \int_{Q_t} H^+(-\hat{y})(-\hat{u})\chi_\omega. \tag{1.10}$$

From the definition of H^+ and the fact that $\hat{u} \geq 0$ we see that the right-hand side of (1.10) is non positive, while the left-hand side is clearly non negative. This implies that both sides are zero. Moreover, as both term in the left-hand side are non negative, each of them is zero. From the properties of ψ we deduce that $\hat{y} \geq 0$.
The second property is the continuous dependence of solutions from the initial data and the source term. Once again we take two general solutions $y, \hat{y}$ with source term $u, \hat{u}$ respectively. We put them in (1.9) twice, firstly ordered in a way and then with exchanged position. We get two inequalities and we sum them up to obtain

$$||\psi(y)(t) - \psi(\hat{y})(t)||_{L^1(\Omega)} + ||f(y) - f(\hat{y})||_{L^1(Q_t)} \leq ||u - \hat{u}||_{L^1(\omega\times(0,t))}. \quad (1.11)$$

1.2.1 Free boundary and finite speed of propagation

Under suitable conditions, there occurs a phenomenon called localization of solutions, i.e., $\overline{\mathcal{S}}(y_u) \cap \Omega$ is strictly contained in Ω for some $u \in U_{\mathrm{ad}}$. We present first the result for the elliptic system and then for the parabolic one, both of them could be find in [8].

Suppose that $y \geq 0$ is a solution of (1.1) with source term $u \geq 0$. Let us fix the attention on a point $x_0 \in \Omega$ and call $d_0 = \mathrm{dist}\,(x_0, \partial\Omega)$. We add the following structural assumption:

$$C|r|^{\sigma+1} \leq f(r)r, \quad \text{for all } r \in \mathbb{R}, \quad (1.12)$$

for some $\sigma > 0$ and $C > 0$. We define the energy

$$E(d) = \int_{B_d(x_0)} a(y, \nabla y) \cdot \nabla y \,\mathrm{d}x, \quad (1.13)$$

where $B_d(x_0)$ is the open ball of radius d centered in x_0. We have the following result which guarantees the presence of a dead core, i.e., a non empty set of points in Ω where the solution vanishes.

Proposition 1.2.4. *Let $u \in U_{ad}$ be such that $u \equiv 0$ in $B_{d_1}(x_0)$ for some $0 < d_1 < d_0$. Assume that (1.12) holds with $\sigma < p - 1$ and that*

$$||u||^{\frac{(1+\sigma)(1-\gamma)}{\sigma}}_{L^{\frac{1+\sigma}{\sigma}}(B_d(x_0))} \leq \varepsilon(d - d_1)^{\frac{1-\gamma}{\gamma}}, \quad \textit{for all } d \in (d_1, d_2),$$

for some $d_2 \in (d_1, d_0)$,

$$\gamma = \frac{(1-\theta)(p-\sigma-1)}{(p-1)(\sigma+1)} \quad \textit{and} \quad \theta = \frac{N(p-\sigma-1)+\sigma+1}{N(p-\sigma-1)+p(\sigma+1)}.$$

Then, if $y(x)$ is the solution of (1.1) *and E is its corresponding energy (see* (1.13)*), there exist $\bar{E}, \bar{\varepsilon}$ such that, if $E(d_0) \leq \bar{E}$ and $\varepsilon \leq \bar{\varepsilon}$, then*

$$y(x) = 0, \quad \textit{for all } x \in B_{d_1}(x_0).$$

Proof. See [8]. □

Remark 1.2.5. For a small subset ω, $u\chi_\omega$ is mostly zero in Ω and this, combined with Proposition 1.2.4, produces, for any solution with not too big energy, the presence of a dead core (see Example 1).

Example 1. Let us consider the one dimensional quasilinear system

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}x}(|y_x|^{p-2}y_x) + |y|^{q-1}y = u_s & \text{in } (-2,2), \\ y(-2) = y(2) = 0, \end{cases} \tag{1.14}$$

where $q \in (0,1)$, $p \geq 2$ and u_s is given by

$$u_s(x) = \begin{cases} 2(p-1)c_2^{p-1}x^{p-2} + (c_1 - c_2x^2)^q & x \in (-1/2, 1/2), \\ 0 & x \in [-2,-1/2] \cup [1/2,2]. \end{cases}$$

The two positive constant c_1, c_2 appearing in the formula for u_s are actually functions of a parameter $x_s \in (1/2, 2)$ and are given by

$$\begin{aligned} c_1 &= \frac{c_2}{4} + k\left(1 - \frac{1}{2x_s}\right)^{\frac{p}{p-q-1}}, \\ c_2 &= \frac{kp}{x_0(p-q-1)}\left(1 - \frac{1}{2x_s}\right)^{\frac{q+1}{p-q-1}}, \\ k &= \left(x_s^p \frac{(p-q-1)^p}{p^{p-1}(q+1)(p-1)}\right)^{\frac{1}{p-q-1}}. \end{aligned} \tag{1.15}$$

The solution of (1.14) is

$$y_s(x) = \begin{cases} c_1 - c_2x^2 & x \in (-1/2, 1/2), \\ k\left(1 - \dfrac{|x|}{x_s}\right)_+^{\frac{p}{p-q-1}} & x \in [-2,-1/2] \cup [1/2,2]. \end{cases}$$

In fact, $y_s \in C^{1,1}([-2,2])$ and satisfies (1.14) pointwise in $(-2,-1/2) \cup (-1/2,1/2) \cup (1/2,2)$.
From the formula of u_s we see that $u_s \geq 0$ and in particular $u_s = 0$ in $(-2,1/2) \cup (1/2,2)$. If we look at the solution, we see that $\mathcal{N}(y) =$

$(-2, -x_0) \cup (x_0, 2)$. In this specific case, the energy of y_s in a neighbourhood of a point x_0 is

$$E(d) = \int_{B_d(x_0)} |y_{sx}|^p \mathrm{d}x$$
$$= 2c_2^p + \frac{k^p}{x_s^{p-1}} \frac{p^p}{(p-q-1)^{p-1}(p(q+2)-(q+1))} \left(1 - \frac{1}{2x_s}\right)^{\frac{p(q+2)-(q+1)}{p-q-1}}. \tag{1.16}$$

From this last expression we see that in order to increase the energy of y_s in a neighbourhood of a point one has to increase the constants c_2 and k, which is equivalent to take bigger values of x_s. So even if $\mathcal{S}(u_s) = [-1/2, 1/2]$ for all choices of $x_s > 1/2$, $\mathcal{N}(y_s)$ disappears for $x_s \geq 2$.

For the parabolic system, under suitable hypothesis, we are going to show that the solutions have the property of finite speed of propagation. Starting from the zero state at time zero, the solution at time T is not strictly positive whenever $u(x,t)$ is not identically zero. In order to get this result, we introduce a new structural condition:

$$C|r|^{k+1} \leq \psi(r)r - \int_0^r \psi(\tau)\, d\tau \leq \bar{C}|r|^{k+1} \text{ for all } r \in \mathbb{R}, \tag{1.17}$$

with $C, \bar{C}, k > 0$.

Proposition 1.2.6. *Let y_0 be an initial data not necessarily identically zero. Assume that $y_0(x) = 0$ for $x \in B_{d_1}(x_0)$, $0 < d_1 < d_0$, and that (1.17) holds with $1 + k < p$. If $u(x,t) = 0$ in $B_{d_1}(x_0) \times (0, T)$, then there exist a time $0 < T^* \leq T$ and a decreasing function $d(t) : (0, T^*) \to (0, \infty)$ such that the solution y of (1.2) is non negative and*

$$y(x,t) = 0 \quad \textit{for all } x \in B_{d(t)}(x_0), \textit{and any } t \in (0, T^*).$$

An estimation of $d(t)$ and T^* is possible and it depends on different parameters of the model and energies of the solution.

1.3 Existence of a minimum

We start this section by giving the family of admissible controls. Let us consider the cone $K = (L^\infty_+(Q_T); ||\cdot||_{L^1(Q_T)}) \subset L^1(Q_T)$, which is the set of non negative bounded functions with the metrics given by the L^1 norm. We define

$$U_{\mathrm{ad}} := \text{any compact subset of } K. \tag{1.18}$$

Theorem 1.3.1. *Under structural assumptions* (1.5), (1.6), *there exists at least one minimum of* J *in* U_{ad}.

To have a better understanding of J we focuse first on the behaviour of J_1. In the following lemma we show a sort of semicontinuity result for J_1.

Lemma 1.3.2. *Consider a sequence* $\{y_n\}_n \subset L^1_+(\Omega) = \{y \in L^1(\Omega), y \geq 0\}$ *converging to* $y \geq 0$ *in* $L^1(\Omega)$. *Then*

$$\liminf_{n\to\infty} \mathcal{L}^N(\mathcal{S}(y_n) \cap B) \geq \mathcal{L}^N(\mathcal{S}(y) \cap B), \tag{1.19}$$

where $\mathcal{L}^N$ *stands for the Lebesgue measure.*

Proof. We start from a result which can be found in [26]. For $0 < \delta_n \to 0$, let us define

$$\Gamma_n = \{x \in \Omega : y_n(x) \leq \delta_n\}.$$

Then one have

$$\mathcal{L}^N(\Gamma_n \setminus \mathcal{N}(y)) \to 0, \text{ as } n \to \infty. \tag{1.20}$$

We show now that (1.20) implies (1.19). $\mathcal{N}(y_n) \subset \Gamma_n$ and it is immediate to see that (1.20) implies

$$\mathcal{L}^N(\mathcal{N}(y_n) \setminus \mathcal{N}(y)) \to 0, \text{ as } n \to \infty. \tag{1.21}$$

Of course, if we call $E_n := \mathcal{N}(y_n) \setminus \mathcal{N}(y)$, we have that

$$\begin{aligned} &\cdot \mathcal{L}^N(E_n) \to 0, \text{ as } n \to \infty, \\ &\cdot \mathcal{N}(y) \supset \mathcal{N}(y_n) \setminus E_n, \\ &\cdot \mathcal{S}(y_n) \cap E_n = \emptyset. \end{aligned} \tag{1.22}$$

If we look at the second line in (1.22) and we take the complement of both side, we notice that it gives $\mathcal{S}(y) \subset \mathcal{S}(y_n) \cup E_n$, which, due to the subadditivity of the measure, implies

$$\mathcal{L}^N(\mathcal{S}(y) \cap B) \leq \mathcal{L}^N(\mathcal{S}(y_n) \cap B) + \mathcal{L}^n(E_n \cap B).$$

This together with the first expression of (1.22) gives (1.19). □

Now we are ready to give the proof of Theorem 1.3.1.

Proof of Theorem 1.3.1. Let $\{u_n\} \subset U_{\mathrm{ad}}$ be a minimizing sequence for J. As U_{ad} is compact, there exists a subsequence (which we still denote with $\{u_n\}$) which converges to a function $u \in U_{\mathrm{ad}}$ in L^1 norm. As $u_n \to u$, from Proposition 1.2.2 we have $\psi(y_n(\cdot,T)) \to \psi(y(\cdot,T))$ in $L^1(\Omega)$, where y and y_n are the solutions of (1.2) with source term u and u_n respectively. From now on, we will refer to $y_n(\cdot,T)$ and $y(\cdot,T)$ as y_{u_n} and y_u.

We can extract another subsequence such that $\psi(y_{u_n}) \to \psi(y_u)$ a.e. in Ω, and this, for the continuity of ψ, implies that $y_{u_n} \to y_u$ a.e.. Function $\frac{1}{G(y)}$ is uniformly bounded for all $0 \leq y \in L^1(\Omega)$ because $G(0) > 0$ and G is increasing, which means that

$$0 \leq \frac{1}{G(y)} \leq \frac{1}{G(0)}.$$

So, using the Lebesgue dominate convergence theorem, we get

$$\int_\Omega \frac{1}{G(y_{u_n})}\,\mathrm{d}x \to \int_\Omega \frac{1}{G(y_u)}\,\mathrm{d}x.$$

From Lemma 1.3.2 we already know that

$$\int_\Omega \chi_{\mathcal{S}(\psi(y_u))\cap B}(x)\,\mathrm{d}x$$

is lower semicontinuous in $L^1(\Omega)$. Due to the structure assumption on ψ (see (1.6)), we have that $\mathcal{S}(\psi(y_u)) = \mathcal{S}(y_u)$, and hence

$$\int_\Omega \chi_{\mathcal{S}(\psi(y_u))\cap B}(x)\,\mathrm{d}x = \int_\Omega \chi_{\mathcal{S}(y_u)\cap B}(x)\,\mathrm{d}x.$$

To conclude, we obtain that

$$J(u_n) \to \min_{v\in U_{\mathrm{ad}}} J(v) \geq J(u).$$

Hence $J(u) = \min_{v\in U_{\mathrm{ad}}} J(v)$. □

1.3.1 A related problem

One can ask what would happen if, instead of functional J we take a slightly different one, J^*, with similar structure to J,

$$J^* = J_1^* + J_2^* = \int_\Omega \chi_{\mathcal{N}(y_u)\cap B}(x) + \int_\Omega G(y_u). \tag{1.23}$$

This time, J_1^* measures the part of B invaded by the null set of y_u while J_2^* is increasing in y_u. Minimising one of the two terms singularly would increase the value of the other one. So, also in this case the minimisation problem of J^* is not trivial. One can see J^* as specular to J, indeed when u increases $J_1(u)$ increases and J_2 decreases, while in J^* is the opposite.

The difficulty now is that the functional $y \mapsto \int_\Omega \chi_{\mathcal{N}(y)\cap B}(x)$ is not lower semicontinuous in $L^1(\Omega)$. Consider the following example:

Example 2. Let us take for $N = 1$ the semilinear problem

$$\begin{cases} -\varphi''(r) + \varphi^q(r) = u(r) & r \in (-2,2), \\ \varphi(-2) = \varphi(2) = 0, \end{cases}$$

with $q \in (0,1)$, which is (1.1) with $a(r,q) = q$ and $f(r) = r^q$.

We set

$$\varphi_\epsilon(r) = \begin{cases} 0 & r \in (1,2), \\ \mathrm{e}^{-\frac{1}{1-\mathrm{r}}} & r \in (1-\epsilon, 1), \\ C_1 - C_2 r^2 & r \in (0, 1-\epsilon), \end{cases}$$

and define $\varphi_\epsilon(r)$ by reflection on the interval $(-2,0)$. The constants C_1 and C_2 have to be chosen so as to make $\varphi_\epsilon \in C^1(-2,2)$, which means

$$C_1 = \mathrm{e}^{-\frac{1}{\epsilon}} + \mathrm{e}^{-\frac{1}{\epsilon}} \frac{1-\epsilon}{2\epsilon^2}, \quad \mathrm{C}_2 = \mathrm{e}^{-\frac{1}{\epsilon}} \frac{1}{2\epsilon^2(1-\epsilon)}.$$

In fact, the derivative is

$$\varphi'_\epsilon(r) = \begin{cases} 0 & r \in (1,2), \\ -\dfrac{1}{(1-r)^2}\mathrm{e}^{-\frac{1}{1-\mathrm{r}}} & r \in (1-\epsilon, 1), \\ -2C_2 r & r \in (0, 1-\epsilon). \end{cases}$$

To have a continuous derivative it is enough to check that the limit from the left and the limit from the right in $r = (1-\varepsilon)$ coincide. That is to say

$$2C_2(1-\varepsilon) = \frac{\mathrm{e}^{-\frac{1}{\varepsilon}}}{\varepsilon^2},$$

which gives C_2. Now to obtain C_1, we check the continuity of φ_ϵ in $r = (1-\varepsilon)$, i.e.,

$$\mathrm{e}^{-\frac{1}{\varepsilon}} = \mathrm{C}_1 - \frac{1-\varepsilon}{2\varepsilon^2}\mathrm{e}^{-\frac{1}{\varepsilon}},$$

which gives C_1.
We want to check now that these functions satisfy

$$-\varphi_\epsilon'' + \varphi_\epsilon^q \geq 0. \tag{1.24}$$

On the interval $(0, 1-\epsilon)$ the functions are concave and positive and the result follows. On $(1-\epsilon, 1)$ we have that

$$\begin{aligned} -\varphi_\epsilon''(r) + \varphi_\epsilon^q(r) &= \mathrm{e}^{-\frac{1}{1-\mathrm{r}}} \left[\frac{2}{(1-\mathrm{r})^3} - \frac{1}{(1-\mathrm{r})^4} \right] + \mathrm{e}^{-\frac{\mathrm{q}}{1-\mathrm{r}}} \\ &= \mathrm{e}^{-\frac{\mathrm{q}}{1-\mathrm{r}}} \left[\mathrm{e}^{\frac{\mathrm{q}-1}{1-\mathrm{r}}} \frac{1-2\mathrm{r}}{(1-\mathrm{r})^4} + 1 \right] \geq 0, \end{aligned} \tag{1.25}$$

for $1 - r < 1 - r_0$ for some $r_0 > 0$. In fact, if we take $(1-r)$ sufficiently small,

$$\mathrm{e}^{\frac{\mathrm{q}-1}{1-\mathrm{r}}} \frac{1-2\mathrm{r}}{(1-\mathrm{r})^4} > -\frac{\mathrm{e}^{\frac{\mathrm{q}-1}{1-\mathrm{r}}}}{(1-\mathrm{r})^4} > -1.$$

So if we take $\epsilon < r_0$ we obtain (1.24) on the whole interval.

Now it easy to check that $u_\epsilon := -\varphi_\epsilon'' + \varphi_\epsilon^q \to 0$ uniformly as $\epsilon \downarrow 0$. On interval $(1, 2)$ it is evident. On interval $(1-\varepsilon, 1)$, for ε sufficiently small, we deduce from equation (1.25) that

$$|-\varphi_\epsilon''(r) + \varphi_\epsilon^q(r)| \leq \mathrm{e}^{-\frac{\mathrm{q}}{1-\mathrm{r}}} \leq \mathrm{e}^{-\frac{\mathrm{q}}{\varepsilon}}.$$

On $(0, 1-\varepsilon)$, for ε sufficiently small

$$\begin{aligned} |-\varphi_\epsilon''(r) + \varphi_\epsilon^q(r)| &= |2C_2 + (C_1 - C_2 r^2)^q| \leq |2C_2 + C_1^q| \\ &\leq 2\frac{\mathrm{e}^{-\frac{1}{\varepsilon}}}{\varepsilon^2} + \left(\frac{2\mathrm{e}^{-\frac{1}{\varepsilon}}}{\varepsilon^2}\right)^q \leq 2\left(\frac{2\mathrm{e}^{-\frac{1}{\varepsilon}}}{\varepsilon^2}\right)^q. \end{aligned}$$

Hence we can conclude that

$$|-\varphi_\epsilon''(r) + \varphi_\epsilon^q(r)| \leq 2^{q+1} \left(\frac{\mathrm{e}^{-\frac{1}{\varepsilon}}}{\varepsilon^2}\right)^q, \quad \text{for all } r \in (0, 2),$$

which goes to zero as $\varepsilon \to 0$.

In addition we see that $\varphi_\epsilon \to 0$. Nevertheless $\mathcal{N}(\varphi_\epsilon) = \emptyset$, for any $\epsilon < r_0$, while in the limit it is the whole interval $(-2, 2)$.

Here we want to show a condition that, if satisfied by a family of function $\{y_n\} \subset L^1(\Omega)$ which converges to $y \in L^1(\Omega)$, guarantees not only the semicontinuity but also the continuity of the null set. To be more precise, we want to show that if $y_n \to y$ in $L^1(\Omega)$ then

$$\int_\Omega |\chi_{\mathcal{N}(y_n)}(x) - \chi_{\mathcal{N}(y)}(x)|\mathrm{d}x \to 0, \quad \text{as } n \to \infty,$$

or, with a measure notation,

$$\mathcal{L}^N(\mathcal{N}(y) \setminus \mathcal{N}(y_n)) + \mathcal{L}^N(\mathcal{N}(y_n) \setminus \mathcal{N}(y)) \to 0, \quad \text{as } n \to \infty. \tag{1.26}$$

We have already seen in the proof of Lemma 1.3.2 that

$$\mathcal{L}^N(\mathcal{N}(y_n) \setminus \mathcal{N}(y)) \to 0,$$

so it remains to show that also the first term in (1.26) goes to zero. The idea is to provide the family of functions with a "not-arbitrarily-flat" condition near the zero level set, an instrument which comes from the works of Caffarelli [24], [25], Brezzi and Caffarelli [23], Phillips [67], Rodrigues [69] and Nochetto [65], [64], among other authors.

Definition 1.3.3 (No-flat condition)**.** *Let us take* $L^1_+(\Omega) \supset Y = \{y_n\}_{n\in\mathbb{N}}$. *We say that* Y *has the no-flat condition if there exist* $\epsilon_0 > 0$ *and* $h : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\lim_{t\to 0} h(t) = 0$ *such that*

$$\mathcal{L}^N(\{x \in \Omega \,:\, 0 < y_n(x) \le \epsilon\}) \le h(\epsilon) \quad \forall \epsilon < \epsilon_0. \tag{1.27}$$

Example 3. We give here a very simple family of functions which does not satisfy the *no-flat condition* and a family which instead satisfy the *no-flat condition.* Let us consider the domain $\Omega = (0,1) \subset \mathbb{R}$ and define

$$y_n(x) = \frac{1}{n} \quad \text{for all } x \in (0,1).$$

For every $\varepsilon_0 > 0$, we can find n_0 sufficiently big such that $1/n < \varepsilon_0$ for all $n > n_0$. If we call

$$m_n(\varepsilon) := \mathcal{L}^N(\{x \in \Omega \,:\, 0 < y_n(x) \le \epsilon\}),$$

we see that, for all $n > n_0$,

$$m_n(\varepsilon) = \begin{cases} 0 & \varepsilon \in (0, 1/n), \\ 1 & \varepsilon \in [1/n, \varepsilon_0] \end{cases}$$

If $n \to \infty$, $m_n(\varepsilon) \to 1$ for all $\varepsilon \in (0, \varepsilon_0)$. Hence (1.27) does not hold.

On the same domain consider instead, for $n \geq 1$,

$$w_n(x) = \max(1/n - x, 0) \quad \text{for all } x \in (0,1).$$

Defining in the same way the function $m_n(\varepsilon)$ we see that, once fixed any $\varepsilon_0 < 1$, if $1/n > \varepsilon_0$ then $m_n(\varepsilon) = \varepsilon$, if on the other hand $1/n < \varepsilon_0$,

$$m_n(\varepsilon) = \min(1/n, \varepsilon).$$

It comes naturally to set the bound $h(\varepsilon) = \varepsilon$, which satisfies the assumption of Definition 1.3.3.

Lemma 1.3.4. *Let Y satisfy the no-flat condition 1.3.3 and $y_n \to y$ in $L^1(\Omega)$. Then*

$$\mathcal{L}^n(\mathcal{N}(y) \setminus \mathcal{N}(y_n)) \to 0, \ \text{as } n \to \infty. \tag{1.28}$$

Proof. We take a sequence $\{\varepsilon_n\}_{n \in N}$ with $\varepsilon_n > 0$, $\varepsilon_n \to 0$ as $n \to \infty$ and such that

$$\lim_{n\to\infty} \frac{||y - y_n||_{L^1(\Omega)}}{\varepsilon_n} = 0.$$

We write

$$\mathcal{N}(y) \setminus \mathcal{N}(y_n) = A_n^< \cup A_n^>,$$

where

$$\begin{aligned} A_n^< &= \{x \in \Omega \,:\, y(x) = 0 \text{ and } 0 < y_n(x) \leq \varepsilon_n\}, \\ A_n^> &= \{x \in \Omega \,:\, y(x) = 0 \text{ and } y_n(x) > \varepsilon_n\}. \end{aligned} \tag{1.29}$$

Let us consider first $A_n^>$. Since $A_n^> \subset \{x \in \Omega \,:\, |y(x) - y_n(x)| > \varepsilon_n\}$, from the Chebychev inequality we see

$$\mathcal{L}^N(A_n^>) \leq \frac{1}{\varepsilon_n} \int_\Omega |y - y_n| \, dx.$$

Hence $\mathcal{L}^N(A_n^>) \to 0$ as $n \to \infty$. Now comes the part where we need the no-flat condition.
For n sufficiently big, $\mathcal{L}^N(A_n^<) \leq h(\varepsilon_n)$, because Y satisfies the no-flat condition, and for $n \to \infty$, $h(\varepsilon_n) \to 0$. So we easily have

$$\lim_{n\to\infty} \mathcal{L}^N(A_n^<) = 0.$$

Combining both results we obtain

$$\lim_{n\to\infty} \mathcal{L}^n(\mathcal{N}(y) \setminus \mathcal{N}(y_n)) = \lim_{n\to\infty} A_n^< + \lim_{n\to\infty} A_n^> = 0.$$

□

In order to state a result similar to Theorem 1.3.1 also for a functional J^* with J_1^* given by (1.23), one should find an appropriate set of admissible controls U_{ad}^* such that the family of all solutions of (1.1) or (1.2) with control term $u \in U_{\mathrm{ad}}^*$ satisfies the no-flat condition.

1.4 Numerical experiments

In this part we develop a numerical implementation of the minimization problem of functional (1.4). We will deal with the one and two dimensional case.

1.4.1 One dimensional case

The domain of the problem will be an interval. In particular, without loss of generality, we take $\Omega = (0, I)$ where $I > 0$.

We apply first a finite element scheme to solve the boundary value problem

$$\begin{cases} -y_{xx} + f(y) = u\chi_\omega(x) & \text{in } (0, I), \\ y(x) = 0 & x = 0, x = I. \end{cases} \tag{1.30}$$

where $f(y) = |y|^{q-1}y$, $q \in (0, 1)$.

In the discretization procedure of the domain in subintervals we consider subintervals of the the same length and denote with n the number of internal nodes. The procedure to arrive to a matrix representation of the problem is standard (see for instance [68]). Starting by the definition of weak solution, y, solution of (1.30), has to verify

$$\int_0^I y_x \varphi \,\mathrm{d}x + \int_0^I f(y)\varphi \,\mathrm{d}x = \int_\omega u\phi \,\mathrm{d}x \quad \text{for all } \varphi \in H_0^1(0, I). \tag{1.31}$$

This variational formulation requires to test equation (1.30) against an infinite dimensional vector space of test function, i.e., $H_0^1(0, I)$. One replaces the infinite dimensional problem in $H_0^1(0, I)$ by a finite dimensional one on the n-dimensional subspace $V_n \subset H_0^1(0, I)$,

$$\begin{aligned} V_n = &\{v : (0, I) \to \mathbb{R} : v \in C[0, I], \\ &\quad v_{|[x_k, x_{k+1}]} \text{ linear}, k = 0, \ldots, n, v(0) = v(I) = 0\}, \end{aligned}$$

where $x_k = kI/(n + 1)$, $k = 0, \ldots, n + 1$ is a uniform grid of points in $(0, I)$. This means that we approximate y with $y_n \in V_n$ and check that y_n satisfies (1.31) with $\varphi \in V_n$ arbitrary.

The commonly used basis of V_n is given by φ_k, $k = 1, \ldots, n$ with

$$\varphi_k = \begin{cases} \frac{x - x_{k-1}}{x_k - x_{k-1}} & x \in [x_{k-1}, x_k], \\ \frac{x_{k+1} - x}{x_{k+1} - x_k} & x \in [x_k, x_{k+1}], \\ 0 & \text{otherwise.} \end{cases}$$

The solution will be then a linear combination of the basis, $y_n = \sum y_k \varphi_k$. Plugging y_n into (1.31) and choosing as test functions the elements of the basis φ_k we obtain

$$\sum_i \int_0^I y_i \varphi_{i_x} \varphi_{k_x} + \int_0^I f\left(\sum_i y_i \varphi_i\right) \varphi_k = \int_0^I u \varphi_k, \quad k = 1, \ldots, n.$$

If we call the step length $h = x_k - x_{k-1}$, due to the small support of the φ_k, the expression can be simplified to

$$\frac{1}{h}(2y_k - y_{k-1} - y_{k+1}) + \int_{x_{k-1}}^{x_{k+1}} f(\sum_{i=k-1}^{k+1} y_i \varphi_i) \varphi_k = \int_{x_{k-1}}^{x_{k+1}} u \varphi_k,$$

and using the method of the trapezoids for the last two integrals we can approximate that equation by

$$\frac{1}{h}(2y_k - y_{k-1} - y_{k+1}) + h f(y_k) = h u(x_k),$$

for $k = 1, \ldots, n$. We point out that this scheme can be also obtained in this simple 1-D case by using of finite differences instead of finite elements. In a compact way, if we put $Y = [y_1, \cdots, y_n]^T$, $U = [u(x_1), \cdots, u(x_n)]^T$,

$$AY + f(Y) = U,$$

where

$$A = \frac{1}{h^2} \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & \cdots \\ -1 & 2 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 2 & 1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \quad \text{and} \quad f(Y) = \begin{pmatrix} f(y_1) \\ \vdots \\ f(y_n) \end{pmatrix}.$$

To solve the nonlinear system so obtained we apply the Newton's method. We remind that if we want to find a zero of the function $F(y)$, we take an initial guess y_0 and we look for y_1, such that $F(y_1) = 0$, with the first order approximated formula $F(y_1) \sim F(y_0) + \nabla F(y_0)(y_1 - y_0)$. This formula and $F(y_1) = 0$ gives the iterative scheme

$$y_{n+1} = y_n - (\nabla F(y_n))^{-1} F(y_n).$$

In our case $F(Y) = AY + f(Y) - U$.

For computational purposes the non linear term $|y|^{q-1}y$ is here approximated close to the origin by $\arctan(1000y)$, which grows very fast near the origin and gives a similar qualitative behaviour. The protected region is given by the subinterval $B = (a, b)$ with $0 < a < b < I$.

In order to compute J we use the following functions:

$$J_1 = \int_\Omega \chi_{S(y(x;u))\cap B}(x)\,\mathrm{d}x, \quad \text{and} \quad J_2 = \lambda \int_\Omega \frac{1}{G(y(x;u))}\,\mathrm{d}x,$$

each of which is computed separately. The parameter $\lambda > 0$ appearing in J_2 is a weight which is used to give more relevance to J_2 ($\lambda > 1$) or to J_1 ($\lambda < 1$).

For the value of J_1, a simple counting method is applied. This method consists in counting the number of interval of the form (x_k, x_{k+1}) which are contained in $\mathcal{S}(y_n)$ multiplied by the length h. We approximate the value of J_2 by using the method of trapezoids. Function G is chosen to be $G(y) = (1 + ky)$, with $k = 10^6$.

As we are dealing with an approximation at different levels, the set $\{x \in (0, I) : y(x) > 0\}$ is approximated by $\{x \in (0, I) : y(x) > \delta\}$, with $\delta = 10^{-6}$.
For the global minimization problem we make use of the software *GOP* (Global Optimization Platform) developed by MOMAT group and which can be found at *www.mat.ucm.es/momat/software.htm*. The need for a global optimization method arises from the possible existence of local minima. We also point out that the theoretical result contained in this work does not say anything about uniqueness of solution of the minimization problem corresponding to (1.30). We can see in our experiments that this problem can have multiple solutions: in experiment 4 (see also Figure 1.2) we show two of these possible solutions, while in all the others we just put one.

1D numerical experiments

In these numerical experiments we denote with M the upper bound of the control vector, $\omega = (w_1, w_2)$ is the control region and u_{opt} the non zero part of the optimal control vector that we have found with our algorithm. The dimension of u_{opt} depends mainly on the dimension of ω and on n.

Experiment 1.1 (see Figure 1.1)
$I = 10$; $n = 100$; $M = 1$; $w1 = 1.5$; $w2 = 2$; $a = 2.5$; $b = 6$; $\lambda = 0.1$;

$u_{\text{opt}} = 0.9970 \quad 0.3486 \quad 0.0002 \quad 0.0006 \quad 0.0001$;

$J(u_{\text{opt}}) = 0.7678$;

Experiment 1.2 (see Figure 1.1)
$I = 10$; $n = 100$; $M = 1$; $w1 = 1.5$; $w2 = 2$; $a = 8.5$; $b = 9$; $\lambda = 0.1$;

$u_{\text{opt}} = 1 \quad 1 \quad 1 \quad 1 \quad 1$;

$J(u_{\text{opt}}) = 0.5930$;

Experiment 1.3
$I = 10$; $n = 100$; $M = 1$; $w1 = 1.5$; $w2 = 2$; $a = 5$; $b = 9$; $\lambda = 0.1$;

$u_{\text{opt}} = 1 \quad 1 \quad 1 \quad 1 \quad 1$;

$J(u_{\text{opt}}) = 0.5930$;

Experiment 1.4 (see Figure 1.2)
$I = 10$; $n = 100$; $M = 7$; $w1 = 1.5$; $w2 = 2$; $a = 5$; $b = 9$; $\lambda = 0.1$;

First solution : $u^1_{\text{opt}} = 2.7032 \quad 4.3910 \quad 0.0022 \quad 2.2641 \quad 0.0025$;

Second solution : $u^2_{\text{opt}} = 4.7708 \quad 4.9370 \quad 0.0125 \quad 0.0131 \quad 0.0138$;

$J(u_{\text{opt}}) = 0.4802$;

Experiment 1.5 (see Figure 1.3)
$I = 10$; $n = 100$; $M = 7$; $w1 = 1.5$; $w2 = 2$; $a = 8$; $b = 8.5$; $\lambda = 0.1$;

$u_{\text{opt}} = 7.0000 \quad 7.0000 \quad 4.4883 \quad 2.4839 \quad 4.1998$;

$J(u_{\text{opt}}) = 0.1832$;

Experiment 1.6 (see Figure 1.3)
$I = 10$; $n = 100$; $M = 1$; $w1 = 1.5$; $w2 = 4$; $a = 8$; $b = 8.5$; $\lambda = 0.1$;

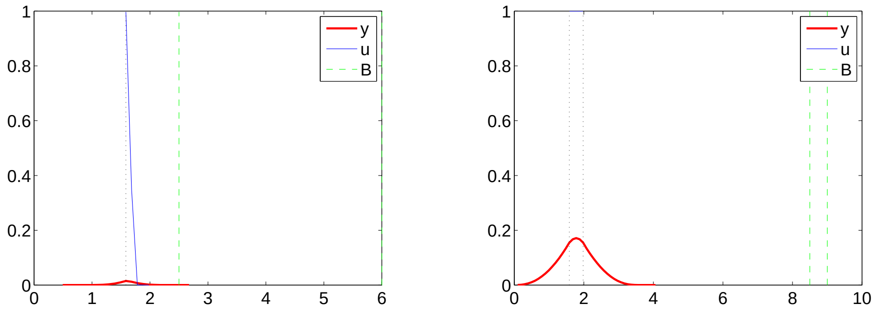

Figure 1.1: Experiments 1.1 and 1.2

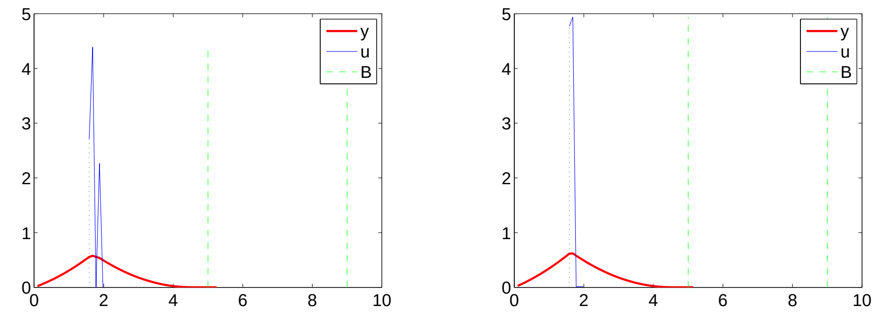

Figure 1.2: Experiments 1.4

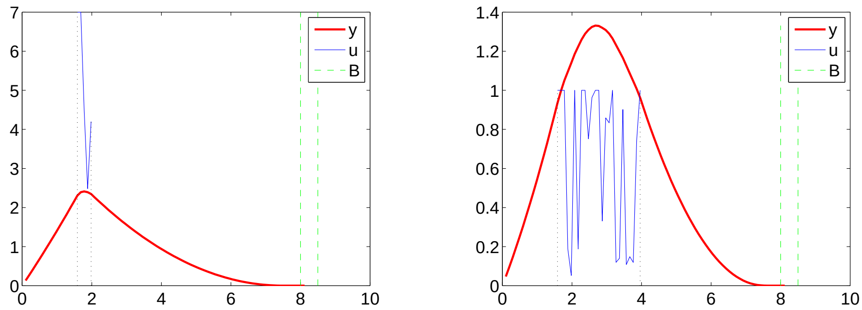

Figure 1.3: Experiments 1.5 and 1.6

$u_{opt} =$

Columns 1 through 9

1.0000 1.0000 1.0000 0.1881 0.0527 1.0000 0.1878 1.00001.0000

Columns 10 through 18

0.7518 0.9629 1.0000 1.0000 0.3309 0.8586 0.8335 1.0000 0.1195

Columns 19 through 25

0.1408 0.9015 0.1087 0.1484 0.1193 0.7423 1.0000

$J(u_{\text{opt}}) = 0.1832$;

From the pictures we can see the qualitative behaviour of the solutions depending on the different parameter values. Actually, once we set λ, it is mostly the relation between M, a, w_2 which imposes a specific profile to the solutions. If, for example, the distance $a - w_2$ is big compared to M, then no invasion of the protected region occurs (see Figure 1.1, experiment 1.2). On the other hand, if these two quantity are somehow balanced then a partial invasion appears (Figure 1.1, experiment 1.1 or Figure 1.2 and Figure 1.3).

1.4.2 Two dimensional case

In two dimensions, we set the domain to be the easiest possible for the moment, $\Omega = (0, I) \times (0, I)$. The boundary value problem is now

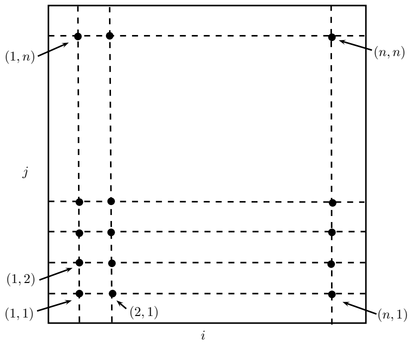

Figure 1.4: Discretization grif of y

$$\begin{cases} -\Delta y(x) + f(y(x)) = u(x)\chi_\omega & \text{in } \Omega = (0, I) \times (0, I), \\ y = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.32}$$

Also in this case $f(y) = |y|^{q-1}y$, $q \in (0, 1)$.

Using the finite difference method we solve the state equation (with non complex domains we could have used again finite elements). We discretize the square with a grid of points labelled with two indices, $x_{i,j}$, as shown in Figure 1.4. We look for approximate values of y in those points, $y_{i,j} \approx y(x_{i,j})$. The following approximations for the derivatives are used

- $\dfrac{\partial y}{\partial x_1}(x_{i,j}) \simeq (y_{i+1,j} - y_{i,j})/h$,
- $\dfrac{\partial^2 y}{\partial x_1^2}(x_{i,j}) \simeq (y_{i+1,j} - 2y_{i,j} + y_{i-1,j})/h^2$,
- $-\Delta y(x_{i,j}) \simeq [4y_{i,j} - (y_{i+1,j} + y_{i,j+1} + y_{i-1,j} + y_{i,j-1})]/h^2$.

Function $u\chi_\omega$ evaluated in the internal nodes of Ω is arranged in a square matrix which we call U. We also define the matrix $Y = (y_{i,j})_{ji}$.

Using the finite difference scheme we are reduced to solve the nonlinear system

$$A\overline{Y} + f(\overline{Y}) = \overline{U}, \tag{1.33}$$

with $\overline{Y}$ and $\overline{U}$ being the column vectors

$$\overline{Y} = \begin{pmatrix} Y_{j,1} \\ Y_{j,2} \\ \vdots \\ Y_{j,n} \end{pmatrix} \quad \text{and} \quad \overline{U} = \begin{pmatrix} U_{j,1} \\ U_{j,2} \\ \vdots \\ U_{j,n} \end{pmatrix}.$$

This time A is a $(n^2 \times n^2)$ square matrix with the value 4 on the main diagonal, the $(n+1)(n-1)$-vector P

$$Q = (\underbrace{1, \ldots, 1}_{n-1}, 0),$$
$$P = -(\underbrace{Q, \ldots, Q}_{n-1}, \underbrace{1, \ldots, 1}_{n-1})$$

on the first sub and super diagonal and the value -1 on the n^{th} sub and super diagonal. $f(\overline{Y})$ is as in Section 1.4.1.

The optimal $\overline{U}$ found (a minimizer of J) and the corresponding solution of (1.33), reshaped in matrix form, are denoted with U_{opt} and Y_{opt}.

The main difference with respect to the 1-D case is that now the protected region may be very general. Once again, we restrict to the simple case of a rectangle (for general 2-D domain we can apply finite elements approximations, see [68]) which is characterized by two points: BL, the lower-left corner and BR, the upper-right corner. The same idea is applied to the control region, whose lower-left and upper-right corners will be written as wL and wR respectively. In this case M is again the upper bound for the control matrix.

We apply also in this case the approximations we have applied in section 1.4.1 for what concern f and $\mathcal{S}(y)$. The evaluation of J_1 and J_2 is also similar to the one performed in section 1.4.1. For solving (1.33), Newton's method is used and for solving the minimization problem we use again the software *GOP*.

2-D numerical experiments

We perform here three different numerical experiments with different values of the parameter $\lambda = \{0.1,\ 1,\ 10\}$, but with the same settings for the other parameters.

General settings for experiments:

$$I = 3;\ n = 15;\ BL = [1,1];\ BR = [2.5, 2.5];$$
$$wL = [0.2, 0.2];\ wR = [1,1];\ M = 2.$$

For each case a graph of Y_{opt} and U_{opt} are shown in the next figures. As we can see in figures 1.5,1.7, 1.9 the profile of Y_{opt} is in light grey on those points where the support does not meet the protected region and in a darker tone of grey where it does. Figures 1.6, 1.8, 1.10 represents the control graphs.

We can notice that, according to different values of the parameter λ, which changes the relative weight of the two parts of the functional J, the best strategy may consist in invading or not the protected region. For example we can see from Figure 1.5 and 1.6 that in case $\lambda = 10$ the best strategy, as we could have expected, is to take the maximal value of the control everywhere, i.e. $U_{\text{opt}} = 2$.

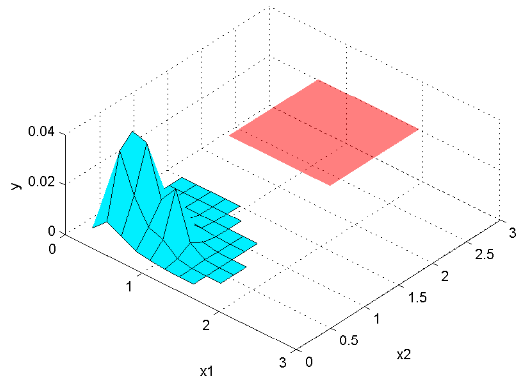

Figure 1.5: Graph of Y_{opt} when $\lambda = 0.1$

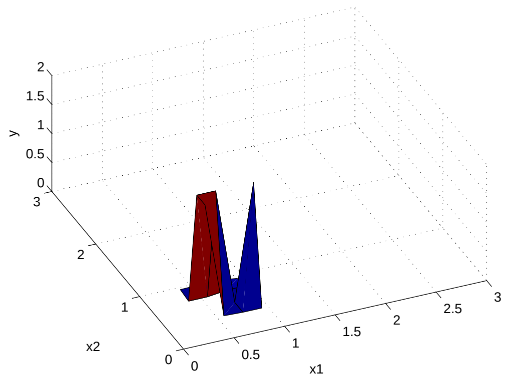

Figure 1.6: Graph of U_{opt} when $\lambda = 0.1$

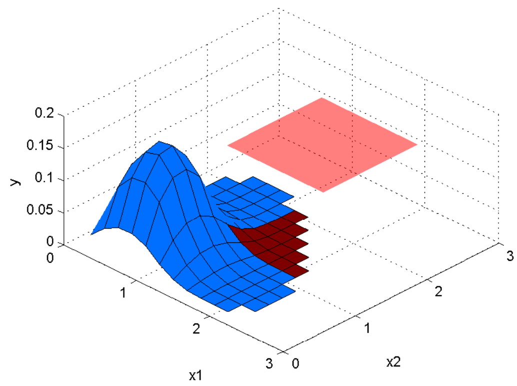

Figure 1.7: Graph of Y_{opt} when $\lambda = 1$

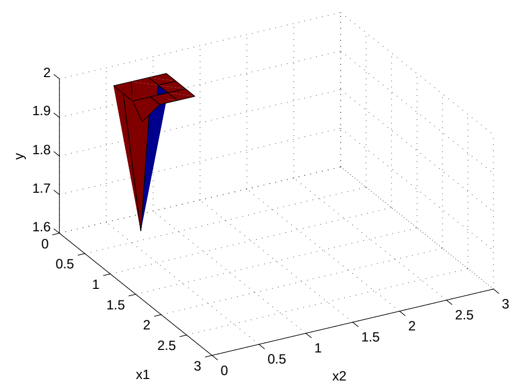

Figure 1.8: Graph of U_{opt} when $\lambda = 1$

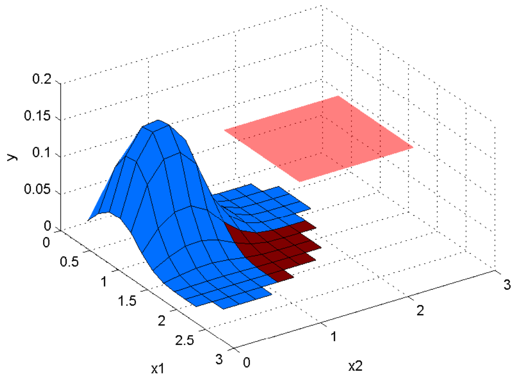

Figure 1.9: Graph of Y_{opt} when $\lambda = 10$

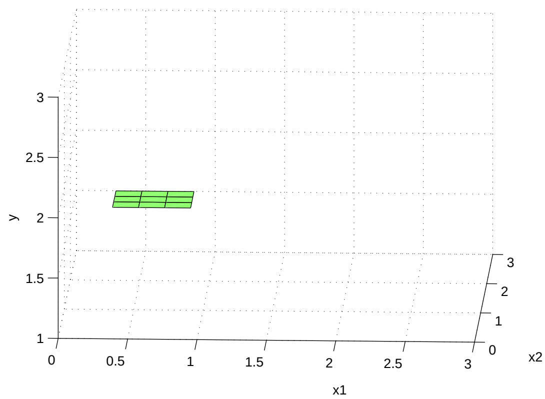

Figure 1.10: Graph of U_{opt} when $\lambda = 10$

Chapter 2

Porous Media Equation

This chapter has been written starting from the papers:

- J.-M. Coron, J. I. Díaz, A. Drici, T. Mingazzini, Global Null Controllability of the 1-Dimensional Nonlinear Slow Diffusion Equation, Chinese Annals of Mathematics, 34B(3), 2013, 333-344.

- A. Drici, T. Mingazzini, Feedback Stabilization of the 1-Dimensional Porous Medium Equation. (Submitted).

In this chapter we deal with other important issues of Control Theory, which are the null controllability and the stabilization of a particular kind of nonlinear, infinite-dimensional dynamical system. While the theory for finite dimensional system is well understood at least in a linear framework, the control in infinite dimension, which is the abstract way of treating PDEs of evolution type, is quite recent and full of open questions even for the linear cases. In the nonlinear case the situation is even more complicated and there is not one complete theory which can describe the controllability properties of all systems. During the last decades different tecniques have been developed and we refer to [34] for a survey which includes both the linear and nonlinear part.

In our specific case the dynamical system consists of nonlinear slow diffusion equations: the first two sections are devoted to the Porous Media Equation (shortly PME), while the last one focuses shortly on the p-Laplacian type equations.

Setting a parameter $m \in R$, $m \geq 1$, the PME has the form

$$y_t - \Delta \left(|y|^{m-1} y \right) = f. \tag{2.1}$$

It belongs to the more general family of non linear diffusion equations

$$y_t - \Delta\phi(y) = f, \tag{2.2}$$

where ϕ is a continuous nondecreasing function with $\phi(0) = 0$.

This family of equations arises in many different frameworks and, depending on the nature of ϕ, it models different diffusion processes, mainly grouped into three categories: "slow diffusion", "fast diffusion"and linear processes.

The "slow diffusion"case is characterized by a *finite speed of propagation* and the formation of free boundaries, while the "fast diffusion"one is characterized by a *finite extinction time*, which means that the solution becomes identically zero after a finite time.

If one neglects the source term, i.e. $f \equiv 0$, and imposes the constraint of nonnegativeness to the solutions (which is fundamental in all the applications where y represents for example a density), then one can precisely characterize these phenomena. In fact, it was shown in [38] that the homogeneous Dirichlet problem associated to (2.2) on a bounded open set Ω of $\mathbb{R}^N$ satisfies a finite extinction time if and only if

$$\int_0^1 \frac{\mathrm{d}s}{\phi(s)} < +\infty,$$

which for constitutive laws given by (2.1) corresponds to the case $m \in (0, 1)$. On the contrary, if

$$\int_0^1 \frac{\mathrm{d}s}{\phi(s)} = +\infty, \tag{2.3}$$

(which is the case for $m \geq 1$) then, for any initial datum $y_0 \in H^{-1}(\Omega) \cap L^1(\Omega)$ with $(-\Delta)^{-1} y_0 \in L^\infty(\Omega)$, there is a kind of "retention property". This means that, if $y_0(x) > 0$ on a positively measured subset $\Omega' \subset \Omega$, then $y(\cdot, t) > 0$ on Ω' for any $t > 0$. In addition to (2.3), if ϕ satisfies

$$\int_0^1 \frac{\phi'(s)\mathrm{d}s}{s} < +\infty,$$

(i.e. $m > 1$ in the case of (2.1)) then the solution enjoys a finite speed of propagation and generates a free boundary given by that of its support ($\partial\{y > 0\}$).

Most typical applications of "slow diffusion"are: nonlinear heat propagation, groundwater filtration and the flow of an ideal gas in a homogeneous porous medium. With regard to the "fast diffusion", it rather finds a paradigmatic application to the flow in plasma physics. Many results and references can by found in the monographs [8] and [70].

2.1 Global null controllability

As already said, the aim of section is to show how a combined action of boundary controls and a spatially homogeneous internal control may allow the global extinction of the solution (the so-called *global null controllability*) in any prescribed temporal horizon $T > 0$. We shall prove the global null controllability for the following two control problems

$$P_{DD}\begin{cases} y_t - (y^m)_{xx} = u(t)\chi_I(t) & \text{in } (0,1)\times(0,T), \\ y(0,t) = v_0(t)\chi_I(t) & t \in (0,T), \\ y(1,t) = v_1(t)\chi_I(t) & t \in (0,T), \\ y(x,0) = y_0(x) & x \in (0,1), \end{cases} \tag{2.4}$$

and

$$P_{DN}\begin{cases} y_t - (y^m)_{xx} = u(t)\chi_I(t) & \text{in } (0,1)\times(0,T), \\ (y^m)_x(0,t) = 0 & t \in (0,T), \\ y(1,t) = v_1(t)\chi_I(t) & t \in (0,T), \\ y(x,0) = y_0(x) & x \in (0,1), \end{cases} \tag{2.5}$$

where $I := (t_1, T)$ with $t_1 \in (0,T)$, $m \geq 1$ and χ_I is the characteristic function of I. In both problems, y represents the state variable and $U_{DN} := (u\chi_I, 0, v_1\chi_I)$, respectively $U_{DD} := (u\chi_I, v_0\chi_I, v_1\chi_I)$, is the control variable. The function y^m should be more properly written in form (2.1), but as we shall impose the constraint $y \geq 0$ it makes no real difference.

We emphasize the fact that the internal control $u(t)$ has the property to be independent of the space variable x and that all the controls are active only on a part of the time interval. Moreover, as we shall show later, the systems are null controllable in arbitrarily fixed time, and then the localized form of the control $u(t)\chi_I(t)$ (the same for the boundary controls) on a subinterval of $[0,T]$ is more an emphatic than a real difficulty. It serves mostly to underline that the controls are not active in the first time lapse. In the same way, it could be possible to take a control interval $(\underline{t}, \overline{t})$ with $\underline{t}, \overline{t} \in (0,T)$ or even more generally three different intervals, one for each control v_0, v_1, u, such that the intersection of the three is not empty.

The main results of this paper are contained in the following statement.

Theorem 2.1.1. *Let $m \in [1, +\infty)$.*
i) For any initial data $y_0 \in H^{-1}(0,1)$ such that $y_0 \geq 0$ and any time $T > 0$, there exist controls $v_0(t), v_1(t)$ and $u(t)$ with $v_0(t)\chi_I(t), v_1(t)\chi_I(t) \in H^1(0,T)$, $v_0, v_1 \geq 0$ and $u \in L^\infty(0,T)$ such that the solution y of P_{DD} satisfies $y \geq 0$ on $(0,1)\times(0,T)$, and $y(\cdot,T) \equiv 0$ on $(0,1)$.
ii) For any initial data $y_0 \in H^{-1}(0,1)$ such that $y_0 \geq 0$ and any time $T > 0$,

there exist controls $v_1(t)$ *and* $u(t)$ *with* $v_1(t)\chi_I(t) \in H^1(0,T)$, $v_1 \geq 0$ *and* $u \in L^\infty(0,T)$ *such that the solution* y *of* P_{DN} *satisfies* $y \geq 0$ *on* $(0,1)\times(0,T)$, *and* $y(\cdot,T) \equiv 0$ *on* $(0,1)$.

Notice that since $H^{-1}(0,1) = (H_0^1(0,1))'$ and $H_0^1(0,1) \subset C([0,1])$, we have $H^{-1}(0,1) \supset \mathcal{M}(0,1)$, where $\mathcal{M}(0,1)$ is the set of bounded Borel measures on $(0,1)$; for instance, the initial datum can be a Dirac mass distribution at a point in $(0,1)$. As said before in the case of "slow diffusion" ($m > 1$), the solution may present a free boundary given by that of its support (as soon as the support of y_0 is strictly smaller than $[0,1]$). Nevertheless, our strategy is built in order to prevent such a situation. Indeed, on the set of points (x,t) where y vanishes (i.e. on the points $(x,t) \in (0,1) \times (0,T) \setminus \operatorname{supp}(y)$), the diffusion operator is not differentiable at $y \equiv 0$ and so, some linearization methods which works quite well for second order semilinear parabolic problems (see, e.g., [45, 51, 53, 55]) can not be applied directly. Moreover, the evanescent viscosity perturbation with some higher order terms only gives some controllability results for suitable functions ϕ, as the ones of the Stefan problem ([45], [43] and [44]).

Here we follow a different approach which is mainly based on the so-called *return method* introduced in [32, 33] (see [34, Chapter 6] for information on this method). More precisely, we shall prove first the null controllability of problem (2.4) by applying an idea appeared in [28] (for the controllability of the Burgers equation). In a second step, we shall show, using some symmetry arguments, that the same result holds for (2.5).

Our version of the return method consists in choosing a suitable one parameter family of trajectories $a(t)/\varepsilon$, which is independent of the space variable, going from the initial state $y \equiv 0$ to the final state $y \equiv 0$. We shall use the controls to reach one of such trajectories, no matter which one, in some large positive time smaller than the final T. Once we fix a partition of the form $0 < t_1 < t_2 < t_3 < T$, we shall choose a function $a(t)$ satisfying the following properties:

i) $a \in C^2([0,T])$;

ii) $a(t) = 0$, $0 \leq t \leq t_1$ and $t = T$;

iii) $a(t) > 0$, $t \in (t_1, T)$;

iv) $a(t) = 1$, $t_2 \leq t \leq t_3$.

Then, we can write the decomposition of the solution y of problem P_{DD} as a perturbation of the explicit solution $a(t)/\varepsilon$ of the same equation with

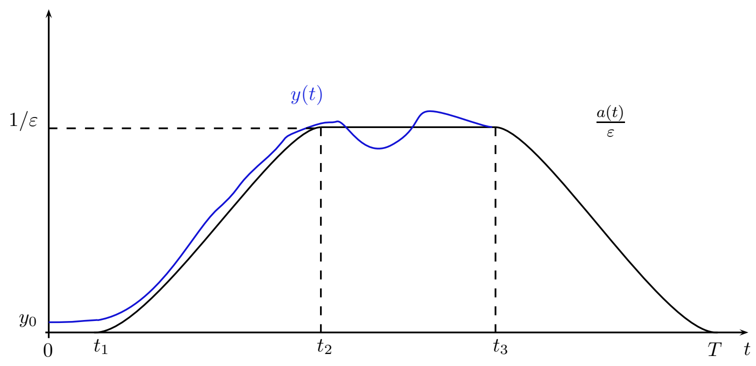

Figure 2.1: Solution profile.

the control $U := (a(t)/\varepsilon, a(t)/\varepsilon, a(t)/\varepsilon)$ in the following way:

$$y(x,t) = \left(\frac{a(t)}{\varepsilon} + z(x,t)\right). \tag{2.6}$$

Now, our aim is to find controls such that $z(\cdot, t_3) \equiv 0$, which means that we have controlled our solution $y(\cdot, t)$ to the state $1/\varepsilon$ at time $t = t_3$; this will be done by using a slight modification of a result in [11]. On the final time interval (t_3, T), we shall use the same trajectory $y(\cdot, t) \equiv a(t)/\varepsilon$ to reach the final state $y(\cdot, T) \equiv 0$. An ideal representation of the trajectory can be seen in Figure 2.1.

One can see that the central core of our procedure is to drive the initial state to a constant state in a finite time thanks to the use of a boundary and internal control which only depends on the time variable.

On the first interval $(0, t_1)$ we shall not make any use of the controls. So we let the solution $y(t) := y(\cdot, t)$ regularize itself from an initial state in $H^{-1}(0,1)$ to a smoother one in $H_0^1(0,1)$ for $t = t_1$. Then, as the degenerate character of the diffusion operator neglects the diffusion effects outside the support of the state, we move $y(t)$ away from the zero state by asking $z(t) := z(\cdot, t)$ to be nonnegative at least on the interval (t_1, t_2). With this trick, the solution $y(t)$ will be far enough from zero. On the interval (t_2, t_3) the states $y(t)$ will be kept strictly positive even if the internal control $u(t)$ will be allowed to take negative values.

As already mentioned concerning the local retention property, we point out that the presence of the control $u(t)$ is fundamental for the global null controllability. To be more precise, notice that if we assume $u(t) \equiv 0$ then we can find initial states which cannot be steered to zero at time T just with some nonnegative boundary controls. As a matter of fact, one can use the well-known family of Barenblatt solutions [10, 70] (also known as ZKB solutions) to show it. Indeed, if we introduce the parameters

$$\alpha = \frac{1}{m+1}, \quad k = \frac{m-1}{2m(m+1)}, \quad \tau << 1,$$

and choose C such that $(C/k)^{1/2}(T+\tau)^{\alpha} < 1/2$, then the function

$$y_m(x,t) = (t+\tau)^{-\alpha} \left(C - k\,|x-1/2|^2\,(t+\tau)^{-2\alpha} \right)_+^{1/(m-1)}$$

is a solution of system (2.4) with $u = 0, v_0 = v_1 = 0$ and $y_m(\cdot,T) \neq 0$. Any other solution of system (2.4) with the same initial datum and $v_0, v_1 \geq 0$ would be a supersolution of y_m which implies that $y_m(\cdot,0)$ cannot be connected with $y(\cdot,T) \equiv 0$.

2.1.1 Well-posedness of the Cauchy problem

For the existence theory of problem (2.4) we refer to [4, 18, 21, 19, 29, 30, 57, 70]; in particular, we shall use a frame similar to the ones in [4] and [18]. More precisely, we adopt the following definition.

Definition 2.1.2. *Let $(v_0, v_1) \in L^\infty(0,T)^2$ and $v_D = (1-x)v_0(t) + xv_1(t)$ and let $u \in L^\infty(0,T)$. Assume that $y_0 \in H^{-1}(0,1)$. We say that y is a weak solution of*

$$P_{DD} \begin{cases} y_t - (|y|^{m-1}\,y)_{xx} = u(t) & \text{in } (0,1)\times(0,T), \\ y(0,t) = v_0(t) & t \in (0,T), \\ y(1,t) = v_1(t) & t \in (0,T), \\ y(x,0) = y_0(x) & x \in (0,1), \end{cases} \tag{2.7}$$

if

$$y \in C^0([0,T];H^{-1}(0,1)) \text{ and } y(0) = y_0 \text{ in } H^{-1}(0,1), \tag{2.8}$$

$$y \in L^\infty(\tau,T;L^1(0,1)),\ \forall \tau \in (0,T], \tag{2.9}$$

$$\partial_t y \in L^2(\tau,T;H^{-1}(0,1)),\ \forall \tau \in (0,T], \tag{2.10}$$

$$|y|^{m-1}\,y \in |v_D|^{m-1}\,v_D + L^2(\tau,T;H_0^1(0,1)),\ \forall \tau \in (0,T], \tag{2.11}$$

and for every $\tau \in (0,T]$, for every $\xi \in L^2(0,T;H_0^1(0,1))$,

$$\int_\tau^T \langle \partial_t y, \xi \rangle + \int_\tau^T \int_0^1 (|y|^{m-1}\,y)_x \xi_x = \int_\tau^T \int_0^1 u\xi, \tag{2.12}$$

where the symbol $\langle\cdot,\cdot\rangle$ stands for the dual pairing between $H^{-1}(0,1)$ and $H_0^1(0,1)$.

Remark 2.1.3. We have changed the definition of weak solution given in [4] in order to handle the case where y_0 is only in $H^{-1}(0,1)$, instead of $y_0 \in L^{m+1}(0,1)$ as assumed in [4].

The modifications to extend the previous definition to the case of problem (P_{ND}) are straightforward (see [4]). For instance, the extension to the interior of the boundary datum can be taken now as $v_D = (c_1 + c_2 x^2)v_1(t)$.

With this definition, one has the following proposition.

Proposition 2.1.4. *The boundary-value problem* (2.4) *has at most one weak solution.*

Proof. The proof is almost the same as the one in [4, Theorem 2.4]. We repeat the principal steps and underline the differences.

Suppose y_1, y_2 are two solutions. We call $y = y_1 - y_2 \in L^2(0,T,H^{-1})$ and by duality representation there exists a function $v \in L^2(0,T,H_0^1)$ such that

$$\int_0^T \int_0^1 v_x \xi_x \, dx \, dt = \int_0^T < y, \xi > \, dt \tag{2.13}$$

for all $\xi \in L^2(0,T;H_0^1)$. We can also state that $v \in L^\infty(0,T,H_0^1)$. In fact as $y \in C(0,T,H^{-1}(0,1))$ and thanks to the suriectivity of $-\Delta : H_0^1 \to H^{-1}$ we have that for every $t \in [0,T]$

$$\int_0^1 v_x(t)\xi_x \, dx =< y(t), \xi >$$

for every $\xi \in H_0^1(0,1)$. This implies that $v_x(t)$ is weakly continuous in the $L^2(0,1)$ topology on $[0,T]$, which gives the boundedness.

So we pick $\xi = v$. With the same computations as in [4]

$$\int_\tau^t < \partial_t y, v > + < y(\tau), v(\tau) >= \frac{1}{2}\int_0^1 (v_x)^2(t). \tag{2.14}$$

On the other hand we have

$$\int_\tau^t \int_0^1 v_x (y_1^{1/m} - y_2^{1/m})_x = \int_\tau^t < y, y_1^{1/m} - y_2^{1/m} >= \int_\tau^t \int_0^1 (y_1 - y_2)(y_1^{1/m} - y_2^{1/m}). \tag{2.15}$$

Hence using v as test function in the definition of solution and (2.14) and (2.15)

$$\frac{1}{2}\int_0^1 (v_x)^2(t) + \int_\tau^t \int_0^1 (y_1 - y_2)(y_1^{1/m} - y_2^{1/m}) =< y(\tau), v(\tau) >,$$

and as $(y_1 - y_2)(y_1^{1/m} - y_2^{1/m}) \geq 0$,

$$\frac{1}{2}\int_0^1 (v_x)^2(t) \leq < y(\tau), v(\tau) > .$$

Sending $\tau \to 0$, the right hand side goes to zero because $y(\tau) \to 0$ in $H^{-1}(0,1)$ and $v(\tau)$ is bounded in $H_0^1(0,1)$. Hence we can conclude that $\int_0^1 (v_x)^2(t) = 0$ for a.e. $t \in (0,T)$. Hence $y = 0$. □

Next proposition is taken from [4, Theorem 1.7, Theorem 2.4].

Proposition 2.1.5. *Suppose that* $(v_0, v_1) \in H^1(0,T)^2$ *and that* $y_0 \in L^{m+1}$, *then there exists one and only one weak solution* y *of problem* (2.4). *Moreover this solution satisfies*

$$y \in L^\infty(0,T; L^1(0,1)), \tag{2.16}$$
$$\partial_t y \in L^2(0,T; H^{-1}(0,1)), \tag{2.17}$$
$$|y|^{m-1} y \in |v_D|^{m-1} v_D + L^2(0,T; H_0^1(0,1)). \tag{2.18}$$

With 2.1.5 and [21] we get the following result.

Proposition 2.1.6. *Suppose that* $(v_0, v_1) \in H^1(0,T)^2$ *and vanishes in a neighbourhood of* $t = 0$, *then there exists one and only one weak solution of problem* (2.4).

Proof. □

Now, we emphasize that the solution of problem (P_{DD}) enjoys an additional semigroup property (we will need it to construct the final trajectory), which directly follows from Definition 2.1.2, Proposition 2.1.6 and Proposition 2.1.5.

Lemma 2.1.7 (Matching)**.** *Suppose that* y_1, *respectively* y_2, *is a weak solution of* (2.4) *on the interval* $(0,T_1)$, *respectively* (T_1,T), *with* $y_2(T_1) = y_1(T_1) \in L^2(0,1)$. *If we denote*

$$y(t) = \begin{cases} y_1(t) & t \in (0,T_1), \\ y_2(t) & t \in (T_1,T), \end{cases}$$

then y *is a weak solution of* (2.4) *on the interval* $(0,T)$.

2.1.2 Proof of the main theorem: first step

In the interval $(0,t_1]$ the solution with no control evolves as in [21], hence $0 \le y^m(t) \in H_0^1(0,1)$ for all $t \in (0,t_1]$. Due to the inclusion $H_0^1(0,1) \subset L^\infty(0,1)$ we get that $y_1(x) := y(x,t_1)$ is a bounded function. We call the solution on this first interval y^0, i.e.

$$y_{|(0,t_1)} = y^0. \tag{2.19}$$

In order to be able to apply the null controllability result in [11] to the function $z(x,t)$, given in the decomposition (2.6), on the interval (t_2,t_3) we need the H^1-norm of $z(t_2)$ to be small enough. We want to find some estimates of the solution z of

$$\begin{cases} z_t - \left(m \left(\dfrac{a(t)}{\varepsilon} + z \right)^{m-1} z_x \right)_x = 0, & \text{in } (0,1)\times(t_1,t_2), \\ z_x(t,0) = z_x(t,1) = 0, & t \in (t_1,t_2), \\ z(x,0) = y_1(x), & x \in (0,1). \end{cases} \tag{2.20}$$

For the existence, regularity and comparison results for this problem we refer to [52], where the equation is recast in the form $(|Y|^{1/m}\operatorname{sign}(Y))_t - Y_{xx} = a'/\varepsilon$.
From the maximum principle, we deduce that $y_1 \in L^\infty(0,1)$ and $y_1 \ge 0$ imply that $z \in L^\infty((0,1)\times(t_1,t_2))$ and $z \ge 0$. In fact, we have $0 \le z \le M$, where $M := \|y_1\|_{L^\infty(0,1)}$ is a solution of the state equation of (2.20), and in particular a super solution of (2.20).

To study the behaviour of z, we will actually make use of rescaling.

2.1.3 Small initial data and a priori estimates

For $\delta > 0$, we define $\tilde z := \delta z$. Then $\tilde z$ satisfies

$$\begin{cases} \tilde z_t - \left(m \left(\dfrac{a(t)}{\varepsilon} + \dfrac{1}{\delta}\tilde z \right)^{m-1} \tilde z_x \right)_x = 0, & \text{in } (0,1)\times(t_1,t_2), \\ \tilde z_x(t,0) = \tilde z_x(t,1) = 0, & t \in (t_1,t_2), \\ \tilde z(x,0) = \delta y_1, & x \in (0,1). \end{cases} \tag{2.21}$$

After collecting the factor $\dfrac{1}{\varepsilon}$ and rescaling the time $\tau := \dfrac{t}{\varepsilon^{m-1}}$, we get

$$\tilde z_t - \left(m \left(a(\tau) + \frac{\varepsilon}{\delta}\tilde z \right)^{m-1} \tilde z_x \right)_x = 0.$$

Choosing $\delta := \varepsilon^{1-\alpha}$ with $0 < \alpha < 1$, the system can be written in the following form

$$\begin{cases} \tilde{z}_\tau - \left(m\left(a(\tau) + \varepsilon^\alpha \tilde{z}\right)^{m-1} \tilde{z}_x\right)_x = 0, & \text{in } (0,1)\times(\tau_1,\tau_2), \\ \tilde{z}_x(\tau,0) = \tilde{z}_x(\tau,1) = 0, & \tau \in (\tau_1,\tau_2), \\ \tilde{z}(x,0) = \varepsilon^{1-\alpha} y_1, & x \in (0,1), \end{cases} \tag{2.22}$$

where $\tau := \dfrac{t}{\varepsilon^{m-1}}$. For simplicity, we take $\alpha = 1/2$.

Thus, the null controllability of system (2.20) is reduced to the null controllability of system (2.22). As we can see, the initial data in (2.22) is now depending on ε and tends to 0 as $\varepsilon \to 0$.

2.1.4 H^1-estimate

We recall that, according to regularity theory for linear parabolic equations with bounded coefficients, $\tilde{z}(t) \in H^2(0,1)$ for $t > 0$, see e.g. [50, pp. 360-364]. Multiplying by $\tilde{z}_{xx}$ the first equation of (2.22) and integrating on $x \in (0,1)$, we get

$$\int_0^1 \tilde{z}_\tau \tilde{z}_{xx}\,\mathrm{d}x = \int_0^1 \left(m\left(a(\tau) + \sqrt{\varepsilon}\tilde{z}\right)^{m-1} \tilde{z}_x\right)_x \tilde{z}_{xx}\,\mathrm{d}x.$$

Then, integrating by parts and using the boundary condition in (2.22), we are led to

$$\begin{aligned} \frac{1}{2m}\frac{d}{d\tau}\int_0^1 \tilde{z}_x^2\,\mathrm{d}x &= -\int_0^1 \left(a(\tau) + \sqrt{\varepsilon}\tilde{z}\right)^{m-1} \tilde{z}_{xx}^2\,\mathrm{d}x \\ &\quad - \frac{(m-1)}{3}\sqrt{\varepsilon}\int_0^1 \left(a(\tau) + \sqrt{\varepsilon}\tilde{z}\right)^{m-2} \left(\tilde{z}_x^3\right)_x\,\mathrm{d}x \\ &= -\int_0^1 \left(a(\tau) + \sqrt{\varepsilon}\tilde{z}\right)^{m-1} \tilde{z}_{xx}^2\,\mathrm{d}x \\ &\quad + \frac{(m-1)(m-2)}{3}\varepsilon\int_0^1 \left(a(\tau) + \sqrt{\varepsilon}\tilde{z}\right)^{m-3} \tilde{z}_x^4\,\mathrm{d}x. \end{aligned}$$

We denote by

$$\begin{aligned} IT_1 &:= -\int_0^1 \left(a(\tau) + \sqrt{\varepsilon}\tilde{z}\right)^{m-1} \tilde{z}_{xx}^2\,\mathrm{d}x, \\ IT_2 &:= \frac{(m-1)(m-2)}{3}\varepsilon\int_0^1 \left(a(\tau) + \sqrt{\varepsilon}\tilde{z}\right)^{m-3} \tilde{z}_x^4\,\mathrm{d}x. \end{aligned}$$

We observe that $IT_1 \leq 0$. Let us look at the term IT_2. For $m \in (1,2)$, we have that $IT_2 \leq 0$. Otherwise,

$$IT_2 \leq \frac{(m-1)(m-2)}{3} \left(a(\tau) + \sqrt{\varepsilon} \left\|\tilde{z}\right\|_{\infty}\right)^{m-3} \varepsilon \int_0^1 \tilde{z}_x^4 \, \mathrm{d}x.$$

The fact that the L^∞-norm of $\tilde{z}$ is finite comes from that $\tilde{z} = \delta z$ and that the supremum of z is bounded, as already pointed out. We use now a well-known Gagliardo-Nirenberg's inequality in the case of a bounded interval:

Lemma 2.1.8. *Suppose $z \in L^\infty(0,1)$ with $z_{xx} \in L^2(0,1)$ and either $z(0) = z(1) = 0$ or $z_x(0) = z_x(1) = 0$, then*

$$\left\|z_x\right\|_{L^4} \leq \sqrt{3} \left\|z_{xx}\right\|_{L^2}^{\frac{1}{2}} \left\|z\right\|_{L^\infty}^{\frac{1}{2}}.$$

Proof of lemma 2.1.8. Integrating by parts and using the boundary conditions, we obtain

$$\int_0^1 z_x^4 \, \mathrm{d}x = \int_0^1 z_x^3 z_x \, \mathrm{d}x = -3 \int_0^1 z_x^2 z_{xx} z \, \mathrm{d}x.$$

Then, using Cauchy–Schwarz's inequality, we get

$$\left\|z_x\right\|_{L^4}^4 \leq 3 \left\|z_x\right\|_{L^4}^2 \left\|z\right\|_{L^\infty} \left\|z_{xx}\right\|_{L^2},$$

and the result follows immediately. □

Setting $C' := C \left\|\tilde{z}\right\|_{L^\infty}^2$ and considering that $\left\|\tilde{z}_x\right\|_{L^4}^4 \leq C' \left\|\tilde{z}_{xx}\right\|_{L^2}^2$, we have

$$\begin{aligned}
\frac{1}{2m}\frac{d}{d\tau}\int_0^1 \tilde{z}_x^2 \, \mathrm{d}x \leq & -\int_0^1 \left(a(\tau) + \sqrt{\varepsilon}\tilde{z}\right)^{m-1} \tilde{z}_{xx}^2 \, \mathrm{d}x \\
& + \frac{(m-1)(m-2)}{3} \left(a(\tau) + \sqrt{\varepsilon} \left\|\tilde{z}\right\|_{\infty}\right)^{m-3} \varepsilon \int_0^1 \tilde{z}_x^4 \, \mathrm{d}x, \\
\leq & -\left(a(\tau)\right)^{m-1} \int_0^1 \tilde{z}_{xx}^2 \, \mathrm{d}x \\
& + C' \frac{(m-1)(m-2)}{3} \left(a(\tau) + \sqrt{\varepsilon} \left\|\tilde{z}\right\|_{\infty}\right)^{m-3} \varepsilon \int_0^1 \tilde{z}_{xx}^2 \, \mathrm{d}x, \\
= & \, C''(m,\tau,\varepsilon) \int_0^1 \tilde{z}_{xx}^2 \, \mathrm{d}x,
\end{aligned}$$

where

$$C''(m,\tau,\varepsilon) := \left(C' \frac{(m-1)(m-2)}{3} \left(a(\tau) + \sqrt{\varepsilon} \left\|\tilde{z}\right\|_{\infty}\right)^{m-3} \varepsilon - \left(a(\tau)\right)^{m-1} \right).$$

For $\tau > 0$, we have

$$C''(m,\tau,\varepsilon) < 0,$$

if ε is small enough.

From these estimates, we deduce that the H^1-norm is non increasing in the interval (τ_1, τ_2). Hence, for all $\rho \geq 0$, we can choose ε small enough to get $\|\tilde{z}(\tau_2)\|_{H^1(0,1)} \leq \varepsilon \|y_1\|_{H^1(0,1)} \leq \rho$.

2.1.5 End of the proof of the main theorem

Now, we go back to problem (2.22) but with Dirichlet boundary conditions and initial data $\tilde{z}(\tau_2)$. We apply an extension method that can be found for instance in [53, Chapter 2]. It consists in extending the space domain from $(0,1)$ to $E := (-d, 1+d)$ and inserting a sparse control in ω, a nonempty open interval whose closure in $\mathbb{R}$ is included in $(-d, 0)$. We look at the following system

$$\begin{cases} w_t - \left(m(1+\sqrt{\varepsilon}w)^{m-1}w_x\right)_x = \chi_\omega \tilde{u}, & (x,\tau) \in Q', \\ w(-d,\tau) = w(1+d,\tau) = 0, & \tau \in (\tau_2, \tau_3), \\ w(x,\tau_2) = w_2(x), & x \in E, \end{cases} \tag{2.23}$$

where $Q' := E \times (\tau_2, \tau_3)$ and $\tau_3 := t_3/\varepsilon^{m-1}$. The function $w_2 \in H_0^1(E) \cap H^2(E)$ is an extension of $\tilde{z}(\tau_2)$ to E which does not increase the H^1-norm, i.e. $\|w_2\|_{H^1(E)} \leq k \|\tilde{z}(\tau_2)\|_{H^1(0,1)} \leq \sqrt{\varepsilon}k \|y_1\|_{H^1(0,1)}$, for some $k > 0$ independent of $\tilde{z}(\tau_2)$.

Proposition 2.1.9. *There exists $\rho > 0$ such that, for any initial data w_2 with $\|w_2\|_{H^1} \leq \rho$ and for any ε sufficiently small, system* (2.23) *is null controllable, i.e. there exists $\tilde{u} \in L^2(Q')$ such that $w(\tau_3) = 0$.*

Proof. The proof is substantially the same as in [11]. We just have to choose ε sufficiently small such that the solution of the control problem satisfies, $||w||_{L^\infty} < \frac{1}{\sqrt{\varepsilon}}$. In [11] the proof is split in two main part: Theorem 4.1 and Theorem 4.2. We will only show how to adapt the proof of Theorem 4.1. The modification in the proof of Theorem 4.2 can be done similarly.

To begin with we change the time interval in the new one $(0, t_\varepsilon)$, with $t_\varepsilon = \varepsilon^{1-m}(\tau_3 - \tau_2)$ and always for simplicity we denote $Q = E \times (0, t_\varepsilon)$. Also, we rewrite the initial data as $w(0) = \sqrt{\varepsilon}w_0$.

$$\begin{cases} w_t - (a_\varepsilon(w)w_x)_x = \chi_\omega \tilde{u}, & (x,t) \in Q, \\ w(-d,t) = 0, \quad w(1+d,t) = 0, & \tau \in (0, t_\varepsilon), \\ w(0) = \sqrt{\varepsilon}w_0(x), & x \in E, \end{cases} \tag{2.24}$$

with $a_\varepsilon(w) = m(1+\sqrt{\varepsilon}w)^{m-1}$ We also define the primitive A_ε of a_ε. We take w_0 such that $(A_\varepsilon(w_0))_{xx}, (w_0)_x, w_0 \in L^2(E)$.

Now let's fix $\mu << 1$ and $\varepsilon > 0$ sufficiently small. Consider the set of functions

$$K := \{w : \|w_x\|_{L^\infty(Q_T)}, \left\|\sqrt{t}w_t\right\|_{L^\infty(Q_T)}, \|w_{xt}\|_{L^2(Q_T)} \le \rho, \qquad w(0) = \sqrt{\varepsilon}w_0, w(\partial E) = 0\}$$

with

$$\rho \le \left|\frac{-1+\mu^{1/(m-1)}}{\sqrt{\varepsilon}}\right| \tag{2.25}$$

to be specified later. This bound for ρ is chosen to obtain an elliptic behaviour for $a_\varepsilon(w)$. In fact, see that as $w(-d,t) = w(1+d,t) = 0$, and taking $d < 1/2$

$$|w(x,t)| \le \min\left(w(-d,t) + \int_{-d}^{x} \rho \,\mathrm{d}s\ ,\ w(1+d,t) + \int_{1+d}^{x} -\rho \,\mathrm{d}s\right) \le \rho,$$

which implies $\|w\|_{L^\infty(Q_T)} \le \rho$ and $a_\varepsilon(w) \ge \mu$. Then with this general setting of parameters we follow [11] theorem 4.1 to stretch the slight modifications one needs.

Define $b_\varepsilon = a_\varepsilon(\tilde{w})$ for a fixed $\tilde{w} \in K$. Consider the linear equation

$$w_t - (b_\varepsilon w_x)_x = u\chi_\omega \tag{2.26}$$

and the optimal control problem: minimize

$$\int e^{-2s\alpha}\phi^{-3}u^2 + \sigma^{-1}\int_E w(t_\varepsilon)^2 \tag{2.27}$$

subject to (2.26). The solution w_σ of (2.27) (we have omitted in the notation the dependence of w_σ on ε), satisfies the inequality

$$\|(w_\sigma)_x\|_{L^\infty(Q_T)} + \left\|\sqrt{(t)}(w_\sigma)_t\right\|_{L^\infty(Q_T)} + \|(w_\sigma)_{xt}\|_{L^2(Q_T)} \le C^*(1+\rho^6)\int_E (A_\varepsilon(\sqrt{\varepsilon}w_0))^2_{xx} + (\sqrt{\varepsilon}w_0)^2_x + (\sqrt{\varepsilon}w_0)^2 \,\mathrm{d}x.$$

In order to be able to continue with the proof we need $w_\sigma \in K$. So we want to find ε and ρ such that

$$\int_E (A_\varepsilon(\sqrt{\varepsilon}w_0))^2_{xx} + (\sqrt{\varepsilon}w_0)^2_x + (\sqrt{\varepsilon}w_0)^2 \mathrm{d}x < \frac{\rho}{C^*(1+\rho^6)}. \tag{2.28}$$

Here we have pay a bit of attention. In fact, modifying ε we are also modifying a_ε, A_ε, t_ε. So we now choose ρ such that (2.25) holds and that estimate 3.24 in [11] holds for a given ε_0 sufficiently small. Now we just pick up one $\varepsilon < \varepsilon_0$ such that (2.28) holds (notice that (2.25) is till valid). This is possible because for every $x \in \mathbb{R}$, $|a_\varepsilon(x)|$ is decreasing for $\varepsilon \to 0$ and the same is for $|A_\varepsilon(x)|$. So the left hand side of (2.28) is going to zero. For the right hand side the only unknown is C^* which may depend on t_ε, ρ and μ. Without trying to compute exactly this dependence we argue in the following way: ρ and μ are fixed and decreasing ε does not affect them. t_ε instead is increasing. So if $C^* \downarrow$ for $t_\varepsilon \uparrow$ then the right hand side is increasing making for ε sufficiently small (2.28) true. On the contrary, if $C^* \uparrow$ for $t_\varepsilon \uparrow$ then we can just fix a time $t^* < t_\varepsilon$ and prove the null controllability of system (2.24) at time t^*.

From now on, having found such an ε which makes (2.28) true, the proof follows exactly as in [11]. □

Remark 2.1.10. Note that, combining the results in [11] and [50, pp. 360-364], the solution of (2.23) satisfies $w(0,\cdot), w(1,\cdot) \in H^1(\tau_2, \tau_3)$.

Proof of Theorem 2.1.1. We consider the function

$$y(\cdot,t) = \begin{cases} y^0(\cdot,t), & t \in (0,t_1), \\ \dfrac{a(t)}{\varepsilon} + z(\cdot,t) = \dfrac{a(t)}{\varepsilon} + \dfrac{\tilde{z}(\cdot,t)}{\sqrt{\varepsilon}}, & t \in (t_1,t_2), \\ \dfrac{a(t)}{\varepsilon} + \dfrac{w(\cdot,t)}{\sqrt{\varepsilon}}, & t \in (t_2,t_3), \\ \dfrac{a(t)}{\varepsilon}, & t \in (t_3,T), \end{cases} \tag{2.29}$$

which is a solution of system (2.4) with controls given by

$$u(t) := \frac{a'(t)}{\varepsilon}, \quad t \in (0,T), \tag{2.30}$$

$$v_0(t) := \begin{cases} 0, & t \in (0,t_1), \\ \dfrac{a(t)}{\varepsilon} + \dfrac{\tilde{z}(0,t)}{\sqrt{\varepsilon}}, & t \in (t_1,t_2), \\ \dfrac{a(t)}{\varepsilon} + \dfrac{w(0,t)}{\sqrt{\varepsilon}}, & t \in (t_2,t_3), \\ \dfrac{a(t)}{\varepsilon}, & t \in (t_3,T), \end{cases} \tag{2.31}$$

and

$$v_1(t) := \begin{cases} 0, & t \in (0, t_1), \\ \dfrac{a(t)}{\varepsilon} + \dfrac{\tilde{z}(1,t)}{\sqrt{\varepsilon}}, & t \in (t_1, t_2), \\ \dfrac{a(t)}{\varepsilon} + \dfrac{w(1,t)}{\sqrt{\varepsilon}}, & t \in (t_2, t_3), \\ \dfrac{a(t)}{\varepsilon}, & t \in (t_3, T). \end{cases} \tag{2.32}$$

The function $y \in C([0,T]; H^{-1}(0,1))$ and, as one can check using the improved regularity of the solution when it is strictly positive, $(v_1, v_2) \in H^1(0,T)^2$. Combining Proposition 2.1.6, Proposition 2.1.5 and Lemma 2.1.7, it is easy to see that the function given by (2.29) is the solution on the interval $(0,T)$ of problem (2.4) with nonhomogeneous term (2.30) and boundary conditions given by (2.31)-(2.32).

To conclude, we have from construction that $y(\cdot, T) \equiv 0$. □

The proof of part ii) follows the common argument of extension by symmetry. First, one notices that using the smoothing property of (2.5) when $u \equiv 0$ and $v_1 \equiv 0$, we may assume that y_0 is in $L^2(0,1)$. Then, we consider the auxiliary problem

$$P^s_{DD} \begin{cases} y_t - (y^m)_{xx} = \widetilde{u}(t)\chi_I(t) & \text{in } (-1,1) \times (0,T), \\ y(-1,t) = v_0(t)\chi_I(t) & t \in (0,T), \\ y(1,t) = v_1(t)\chi_I(t) & t \in (0,T), \\ y(x,0) = \widetilde{y}_0(x) & x \in (-1,1), \end{cases} \tag{2.33}$$

with $\widetilde{y}_0 \in L^2(-1,1)$ defined by

$$\widetilde{y}_0(x) = y_0(x) \text{ and } \widetilde{y}_0(-x) = y_0(x), \; \forall x \in (0,1). \tag{2.34}$$

and with $v_0(t) = v_1(t)$. We apply the arguments of part i) to P^s_{DD} with $(0,1)$ replaced by $(-1,1)$ and adjusting the formulation of (2.23) in such a way that the control region ω is now symmetric with respect to $x = 0$. Then, as we show later, the restriction of the solution of P^s_{DD} to the space interval $(0,1)$ is the sought trajectory for system P_{DN}.

Lemma 2.1.11. *Let ω be a nonempty open subset of $[-1-d, 1+d] \setminus [-1,1]$ which is symmetric with respect to (w.r.t.) $x = 0$. Then, if w_2 is symmetric w.r.t. $x = 0$, we can find a control u_s, symmetric w.r.t. $x = 0$, such that the solution w of system* (2.23) *satisfies*

1. *w is symmetric w.r.t. $x = 0$,*

2. $w(\cdot, \tau_3) = 0$.

Proof. The proof follows almost straightforward from [11, Theorems 4.1 and 4.2]. We just have to minimize the functional which appear in [11, Theorems 4.1] in the space of L^2 functions which are symmetric w.r.t. $x = 0$.

The symmetry of the initial value implies as a consequence the symmetry of the solution w. □

To conclude the proof of part ii) of Theorem 2.1.1, we note that as the solution $y(\cdot, t)$ of (2.33) belongs to $H^2(-1, 1)$ for all $t \in (0, T)$, we see that $y_x(0, t) = 0$ for all $t \in (0, T)$ and so, the conclusion is a direct consequence of part i).

2.2 Stabilization

In this section, we introduce a simple procedure to stabilize the zero state for the porous medium equation with homogeneous Neumann boundary condition. We consider the following control system

$$\begin{cases} y_t - (y^m)_{xx} = u, & (x,t) \in Q_\infty, \\ (y^m)_x(0,t) = 0, & t \in (0,\infty), \\ (y^m)_x(1,t) = 0, & t \in (0,\infty), \\ y(x,0) = y_0, & x \in (0,1), \end{cases} \tag{2.35}$$

where $Q_\infty := (0, 1) \times (0, \infty)$ and $m > 1$. The initial datum y_0 can be of changing sign, so in order to have the well-posedness of the problem (2.35), the nonlinear term y^m must be intended as $|y|^{m-1} y$.

For $u \equiv 0$, the behavior of the solutions of (2.35) is well described in [3] where it is shown that if $y_0 \in L^\infty(\Omega)$, with $\Omega \subset \mathbb{R}^{\mathbb{N}}, N \geq 1$ then

$$y(\cdot, t) \xrightarrow{L^p} \frac{1}{|\Omega|} \int_\Omega y_0 \,\mathrm{d}x, \quad \text{for } t \to \infty.$$

This convergence can be even in the L^∞-norm if $N = 1$ (for $N > 1$, one needs the initial data to be strictly positive in $\overline{\Omega}$). Also, different rates of homogenization are proved depending on the mean of the initial data. For the case of zero mean, they showed the existence of $C > 0$ such that for every $y_0 \in L^\infty(\Omega)$ with $\int_\Omega y_0 \, dx = 0$

$$\|y(\cdot, t)\|_{L^\infty(\Omega)} \leq \frac{C \|y_0\|_{L^\infty(\Omega)} \|y_0\|^{-1}_{L^{m+1}(\Omega)}}{\left(c(m-1)t + \|y_0\|^{1-m}_{L^{m+1}(\Omega)}\right)^{\frac{1}{m-1}}}.$$

Our purpose is to stabilize the system to zero independently of the initial data, which will be chosen in a proper energy space. To do that, we will use an internal feedback control $u : y \mapsto u(y) \in \mathbb{R}$, $u(t) := u(y(\cdot, t))$ constant in space. We focus specifically on a control of the form

$$u(t) := -\int_0^1 y^m(x,t)\,\mathrm{d}x. \tag{2.36}$$

Remark 2.2.1. We could have chosen a more general function u as control, multiplying the integral by a parameter $\lambda > 0$ that could be settled according to needs. The point is that adding λ is not really effective in improving the stabilization rate. This is due to the fact that the nonlinear term $(y^m)_{xx}$ plays an important role in all inequalities, and in order to get an arbitrarily small estimate of the type

$$\|y(\cdot,t)\|_{L^{m+1}(0,1)} \leq \frac{C}{\lambda} t^{-\frac{1}{m-1}},$$

one should modify the equation into the different one

$$y_t - \lambda(y^m)_{xx} = -\lambda \int_0^1 y^m\,\mathrm{d}x.$$

In the next section, we shall prove the existence for small time and the uniqueness of solutions of system (2.35) with control law given by (2.36). Because of the changing sign of the solutions, we could not adopt the same technique of monotonicity that was applied in [6, 37] to prove the existence. What we do instead is to make use of a fixed point theorem. Then, we shall show that the existence can be extended for all time and give an uniform decay rate.

2.2.1 Well-posedness for small time

We start recalling what we mean by a weak solution of the system (2.35) on a finite cylinder $Q_T = (0,1) \times (0,T)$ for $T < \infty$. Given $u \in L^1(0,T)$ and $y_0 \in L^1(0,1)$, a measurable function y defined in Q_T is said to be a weak solution of the system (2.35) if:

1. $y \in L^1(Q_T)$ and $y^m \in L^1(0,T;W^{1,1}(0,1))$;
2. y satisfies

$$\iint_{Q_T} (y^m)_x\eta_x - y\eta_t\,\mathrm{d}x\mathrm{d}t = \int_0^1 y_0\eta(x,0)\,\mathrm{d}x + \int_0^T u\left(\int_0^1 \eta\,\mathrm{d}x\right)\mathrm{d}t, \tag{2.37}$$

for every $\eta \in C^1(\overline{Q}_T)$ with $\eta(\cdot,T) = 0$.

Proposition 2.2.2 (see [70]). *Assume that* $y_0 \in L^{m+1}(0,1)$ *and* $u \in L^\infty(0,T)$. *Problem* (2.35) *has a unique weak solution. Moreover, if* y *is the solution then* $y \in L^\infty(0,T;L^{m+1}(0,1))$, $y^m \in L^2(0,T;H^1(0,1))$ *and the following energy inequality holds*

$$\frac{1}{m+1}\int_0^1 |y|^{m+1}\,\mathrm{d}x + \iint_{Q_T} |(y^m)_x|^2\,\mathrm{d}x\mathrm{d}t \\ \leq \frac{1}{m+1}\int_0^1 |y_0|^{m+1}\,\mathrm{d}x + \int_0^T u\left(\int_0^1 y^m\,\mathrm{d}t\right)\mathrm{d}x. \quad (2.38)$$

Lemma 2.2.3. *Let us assume that* $y \in L^2(Q_T)$ *is a solution of* (2.35)-(2.36) *with* $y^m \in L^2(0,T;H^1(0,1))$, *then* y *is the unique solution.*

Proof. The proof is the same as in [70, p. 79], we just repeat it to show that the nonlinear term does not alter its validity. Let y_1, y_2 be two solutions with the same initial datum. Let $w_i := y_i^m$ for $i = 1,2$ and, according to (2.37), we have

$$\iint_{Q_T} (w_1 - w_2)_x\eta_x - (y_1 - y_2)\eta_t\,\mathrm{d}x\mathrm{d}t = -\int_0^T\int_0^1 (w_1 - w_2)\,\mathrm{d}x\int_0^1 \eta\,\mathrm{d}x\mathrm{d}t,$$

for all η in the space of test functions. To get the result, we use a function introduced by Oleĭnik

$$\eta(x,t) := \int_t^T (w_1(x,s) - w_2(x,s))\,\mathrm{d}s,$$

even if it does not possess the regularity required. In fact, it can be approximated by a sequence of test functions, and (2.37) remains true in the limit. Thus,

$$\iint_{Q_T} (w_1 - w_2)(y_1 - y_2)\,\mathrm{d}x\mathrm{d}t \\ + \iint_{Q_T} (w_1 - w_2)_x \int_t^T (w_1 - w_2)_x\,\mathrm{d}s\mathrm{d}x\mathrm{d}t \\ = -\int_0^T\int_0^1 (w_1 - w_2)\,dx\int_0^1\int_t^T (w_1 - w_2)\,\mathrm{d}s\mathrm{d}x\mathrm{d}t. \quad (2.39)$$

Observing that

$$(w_1 - w_2)_x\int_t^T (w_1 - w_2)_x\mathrm{d}s = -\frac{1}{2}\frac{d}{dt}\left(\int_t^T (w_1 - w_2)_x\,\mathrm{d}s\right)^2,$$

and

$$\int_0^1 (w_1 - w_2)\,\mathrm{d}x \int_t^T \int_0^1 (w_1 - w_2)\,\mathrm{d}x\mathrm{d}s = -\frac{1}{2}\frac{d}{dt}\left(\int_t^T \int_0^1 (w_1 - w_2)\,\mathrm{d}x\mathrm{d}s\right)^2,$$

we can rewrite (2.39) as

$$\iint_{Q_T} (w_1 - w_2)(y_1 - y_2)\,\mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_0^1 \left(\int_0^T (w_1 - w_2)_x\,\mathrm{d}s\right)^2 \mathrm{d}x + \frac{1}{2}\left(\int_0^T \int_0^1 (w_1 - w_2)\,\mathrm{d}x\mathrm{d}s\right)^2 = 0.$$

Since all terms are nonnegative, we have $y_1 \equiv y_2$. □

To prove the well-posedness for small time of the nonlocal problem (2.35)-(2.36), we use the Banach fixed point theorem. We first restrict our attention to initial data that belong to $L^\infty(0,1)$.

Theorem 2.2.4. *For any $y_0 \in L^\infty(0,1)$, there exists a time T (depending on the L^∞-norm of y_0) such that the boundary value problem* (2.35)-(2.36) *has a unique solution.*

We introduce the following map

$$K : z \mapsto y := K(z),$$

where y is the solution of

$$\begin{cases} y_t - (y^m)_{xx} = -\int_0^1 z^m\,\mathrm{d}x, & (x,t) \in Q_T, \\ (y^m)_x(0,t) = 0, & t \in (0,T), \\ (y^m)_x(1,t) = 0, & t \in (0,T), \\ y(x,0) = y_0, & x \in (0,1), \end{cases} \tag{2.40}$$

and the domain of K is the closed subset

$$B := \left\{z \in L^1(Q_T) : ||z||_{L^\infty(Q_T)} \le c\right\} \subset L^1(Q_T).$$

From [70, ps. 29-31], we know that given any two solutions y, $\tilde{y}$ of problem (2.35) (respectively with forcing terms u, $\tilde{u}$ and initial data y_0, $\tilde{y}_0$), we have the following estimates

$$\|y\|_{L^\infty(Q_T)} \le \|y_0\|_{L^\infty(0,1)} + T\,\|u\|_{L^\infty(0,T)}\,, \tag{2.41}$$

$$\|y(\cdot,t)-\tilde{y}(\cdot,t)\|_{L^1(0,1)} \leq \|y_0-\tilde{y}_0\|_{L^1(0,1)} + \int_0^t |u(s)-\tilde{u}(s)|\,\mathrm{d}s. \qquad (2.42)$$

In (2.40), $u(t) := -\int_0^1 z^m(x,t)dx$. If we take $T < \frac{1}{mc^{m-1}}$ and $\|y_0\|_{L^\infty(0,1)} \leq \frac{m-1}{m}c$, we see from (2.41) that

$$\|y\|_{L^\infty(Q_T)} \leq \|y_0\|_{L^\infty(0,1)} + T\,\|z\|^m_{L^\infty(Q_T)} < c,$$

hence $K : B \to B$. Before passing to show the contraction property of K, we give an estimate of $|a^m - b^m|$ with $a, b \in \mathbb{R}$ (remind $r^m := |r|^{m-1}\,r$). In the case where $|a|\,, |b| \leq c$, the above inequalities can be summarized as $|a^m - b^m| \leq m\,|a-b|\,c^{m-1}$. From (2.42), we obtain for any couple of solutions of (2.40) with same initial datum,

$$\begin{aligned}\|y-\tilde{y}\|_{L^1(Q_T)} &\leq T\int_0^T\int_0^1 |z^m - \tilde{z}^m|\,\mathrm{d}x\mathrm{d}t \\ &\leq T\int_0^T\int_0^1 mc^{m-1}\,|z-\tilde{z}|\,\mathrm{d}x\mathrm{d}t \\ &\leq Tmc^{m-1}\,\|z-\tilde{z}\|_{L^1(Q_T)}\,.\end{aligned}$$

For any c in the definition of the Banach space B, choosing $T < \frac{1}{mc^{m-1}}$, we see that K is a contraction. We have thus proved that it admits a unique fixed point in B, which is the solution for small time of system (2.35)-(2.36).

2.2.2 Decay rate and initial data in L^{m+1}

We start by proving a decay estimate of an energy functional. Suppose that y is a solution of (2.35)-(2.36), then inequality (2.38) gives

$$\begin{aligned}\int_0^1 |y|^{m+1}(x,t)\,\mathrm{d}x &\leq \int_0^1 |y|^{m+1}(x,0)\,\mathrm{d}x \\ &\quad - (m+1)\left(\int_0^T \left(\int_0^1 y^m\,\mathrm{d}x\right)^2 + \int_0^1 |(y^m)_x|^2\,\mathrm{d}x\mathrm{d}t\right). \qquad (2.43)\end{aligned}$$

Let $w := y^m$, we see that

$$
\begin{aligned}
|w|^{\frac{m+1}{m}} &\leq \left(\left|\int_0^1 w\,\mathrm{d}x\right| + \left(\int_0^1 |w_x|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\right)^{\frac{m+1}{m}} \\
&= \left(\left(\int_0^1 w\,\mathrm{d}x\right)^2 + \int_0^1 |w_x|^2\,\mathrm{d}x + 2\left|\int_0^1 w\,\mathrm{d}x\right|\left(\int_0^1 |w_x|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\right)^{\frac{m+1}{2m}} \\
&\leq 2^{\frac{m+1}{2m}}\left(\left(\int_0^1 w\,\mathrm{d}x\right)^2 + \int_0^1 |w_x|^2\,\mathrm{d}x\right)^{\frac{m+1}{2m}}.
\end{aligned}
$$

Integrating in space yields

$$
\frac{1}{2}\left(\int_0^1 |w|^{\frac{m+1}{m}}\,\mathrm{d}x\right)^{\frac{2m}{m+1}} \leq \left(\int_0^1 w\,\mathrm{d}x\right)^2 + \int_0^1 |w_x|^2\,\mathrm{d}x. \tag{2.44}
$$

Going back to y and using (2.43) and (2.44), we have

$$
\int_0^1 |y|^{m+1}\,\mathrm{d}x \leq \int_0^1 |y_0|^{m+1}\,\mathrm{d}x - \frac{m+1}{2}\int_0^t \left(\int_0^1 |y|^{m+1}\,\mathrm{d}x\right)^{\frac{2m}{m+1}}\,\mathrm{d}t.
$$

Hence, applying the comparison principle and elevating afterwards to the power $1/(m+1)$, one can show with a direct computation that

$$
\begin{aligned}
\|y(\cdot,t)\|_{L^{m+1}(0,1)} &\leq \left(\frac{m-1}{2}t + \|y_0\|_{L^{m+1}(0,1)}^{1-m}\right)^{-\frac{1}{m-1}} \\
&\leq Ct^{-\frac{1}{m-1}},
\end{aligned} \tag{2.45}
$$

where $C := \left(\frac{m-1}{2}\right)^{-\frac{1}{m-1}}$. We emphasizes the fact that the decay rate is independent of the initial datum.

We now extend the set of solutions to admit as initial datum a function which is not bounded but merely in the energy space $L^{m+1}(0,1)$.

To prove the existence of at least one solution, we pick a sequence of initial data $(y_{0,n})_{n\in\mathbb{N}} \subset L^\infty(0,1)$ such that $y_{0,n} \to y_0$ in $L^{m+1}(0,1)$ and, without loss of generality, we may assume that $|y_{0,n}| \leq |y_0|$ a.e. $x \in (0,1)$. In what follows, C will denote a general constant independent of n but whose value may change from line to line. Owing to (2.45), the solutions y_n to (2.35)-(2.36) with initial datum $y_{0,n}$ satisfies

$$
\|y_n\|_{L^\infty(0,T;L^{m+1}(0,1))} \leq C,
$$

where C depends on $\|y_0\|_{L^{m+1}(0,1)}$. From (2.38)

$$\iint_{Q_T} |(y_n^m)_x|^2 \,\mathrm{d}x\mathrm{d}t \leq \frac{1}{m+1}\int_0^1 |y_0|^{m+1} \,\mathrm{d}x \\ + \|y_n\|^m_{L^\infty(0,T;L^m(0,1))} \|y_n^m\|_{L^1(Q_T)},$$

which means that $\|(y_n^m)_x\|_{L^2(Q_T)} \leq C$. As

$$\|y_n\|^m_{L^\infty(0,T;L^{m+1}(0,1))} = \|y_n^m\|_{L^\infty(0,T;L^{\frac{m+1}{m}}(0,1))},$$

it follows that $\|y_n^m\|_{L^{\frac{m+1}{m}}(Q_T)} \leq C$. Since $\frac{m+1}{m} > 1$, the space $L^{\frac{m+1}{m}}(Q_T)$ is reflexive. Therefore, up to a subsequence, we obtain the following weak convergences

$$\begin{aligned} y_n &\rightharpoonup y, \text{ in } L^{m+1}(Q_T), \\ y_n^m &\rightharpoonup \tilde{y}, \text{ in } L^{\frac{m+1}{m}}(Q_T), \\ (y_n^m)_x &\rightharpoonup v, \text{ in } L^2(Q_T). \end{aligned}$$

It is clear that $v \equiv (\tilde{y})_x$. To establish that we actually have $\tilde{y} \equiv y^m$, it suffices to show that $y_n^m \to \tilde{y}$ a.e. in Q_T. Indeed, from this we infer $y_n \to \tilde{y}^{1/m}$ a.e. in Q_T which, combining with the weak convergence to y, implies $y \equiv \tilde{y}^{1/m}$. Concerning the a.e. convergence, we need a time derivative estimate. From [70, ps. 34-35], we know that if we call $\phi(y_n) := y_n^m$,

$$\iint_{Q_T} t\phi'(y_n)\,|y_{nt}|^2 \,\mathrm{d}x\mathrm{d}t + T\int_0^1 |\phi(y_n)_x(T)|^2 \,\mathrm{d}x \leq \iint_{Q_T} |\phi(y_n)_x|^2 \,\mathrm{d}x\mathrm{d}t \\ + \iint_{Q_T} t\phi'(y_n)u_n^2 \,\mathrm{d}x\mathrm{d}t.$$

Hence, using Young's inequality, we get

$$m\iint_{Q_T} t\,|y_n|^{m-1}\,|y_{nt}|^2 \,\mathrm{d}x\mathrm{d}t + T\int_0^1 |(y_n^m)_x(T)|^2 \,\mathrm{d}x \leq \iint_{Q_T} |(y_n^m)_x|^2 \,\mathrm{d}x\mathrm{d}t \\ + mT\frac{m-1}{m+1}\iint_{Q_T} |y_n|^{m+1} \,\mathrm{d}x\mathrm{d}t + mT\frac{2}{m+1}\int_0^T |u_n|^{m+1} \,\mathrm{d}t.$$

Going back to our nonlocal problem, Holder's inequality yields

$$\|u_n\|_{L^{m+1}(0,T)} \leq C\,\|y_n\|^m_{L^\infty(0,T;L^{m+1}(0,1))}.$$

Define $Q_T^\tau := (0,1) \times (\tau, T)$, we have

$$\|y_n^m\|_{L^\infty(\tau,T;H^1(0,1))} \leq C,$$

where in this case the constant depends on τ, i.e., $C = C(\tau)$. Then, by Rellich–Kondrachov's theorem,

$$\|y_n\|_{L^\infty(Q_T^\tau)} \leq C.$$

This leads to

$$m \int_{Q_T^\tau} t\, |y_n|^{2(m-1)}\, |y_{nt}|^2 \, \mathrm{d}x\mathrm{d}t \leq mC^{m-1} \int_{Q_T^\tau} t\, |y_n|^{m-1}\, |y_{nt}|^2 \, \mathrm{d}x\mathrm{d}t,$$

that is

$$\|(y_n^m)_t\|_{L^2(Q_T^\tau)} \leq C.$$

Thus, we have shown $\|y_n^m\|_{H^1(Q_T^\tau)} \leq C$. Consequently, we can extract a subsequence still labelled $(y_n)_{n\in\mathbb{N}}$ such that $y_n^m \to y^m$ in $L^2(Q_T^\tau)$. By the diagonal argument, passing to another subsequence, we may assume that $y_n \to y$ a.e. in Q_T . It remains to check that

$$\int_0^T \left(\int_0^1 y_n^m \, \mathrm{d}x \right) \left(\int_0^1 \eta \, \mathrm{d}x \right) \mathrm{d}t \to \int_0^T \left(\int_0^1 y^m \, \mathrm{d}x \right) \left(\int_0^1 \eta \, \mathrm{d}x \right) \mathrm{d}t,$$

which follows from Lebesgue's theorem. Indeed, $\|y_n^m(\cdot,t)\|_{L^1(0,1)}$ is bounded in $L^\infty(0,T)$ and, passing to a subsequence, $\|y_n^m(\cdot,t)\|_{L^1(0,1)} \to \|y^m(\cdot,t)\|_{L^1(0,1)}$ a.e. $t \in (0,T)$. Therefore, we can pass to the limit in (2.37) as $n \to \infty$, which gives that y is a solution of (2.35)-(2.36).

2.2.3 Existence for all time

In this part, we shall prove the existence of a solution for all time $t \geq 0$. The argument is standard and we repeat it just for completeness. Let $y_0 \in L^{m+1}(0,1)$ be fixed. If y_1, y_2 are solutions on $[0,T_1)$, $[0,T_2)$ respectively, then $y_1 \equiv y_2$ on $[0, \min(T_1,T_2))$.

Consider the set of solutions $\{y_s : [0,T_s) \to L^{m+1}(0,1)\}$ and let $\widetilde{T} := \sup T_s$. We define $y : [0,\widetilde{T}) \to L^{m+1}(0,1)$ in the following way:

$$y(t) := y_s(t), \quad \text{if } t < T_s,$$

which, thanks to the uniqueness, is well-defined and is also a solution on the interval $[0,T_s)$. Now, choose a sequence $(t_k)_{k\in\mathbb{N}}$ with $t_k \to \widetilde{T}$. We know from (2.45) that $\|y(t_k)\|_{L^{m+1}(0,1)} \leq C$ for all $k \in \mathbb{N}$. Due to our assumptions, there exists $T > 0$ independent of k such that the problem (2.35)-(2.36) with initial datum $y(t_k)$ possesses a unique solution $y_k : [0, T] \to L^{m+1}(0, 1)$. By uniqueness, $y_k(t) = y(t + t_k)$ for small t. Fix k such that $t_k \in (\widetilde{T} - T, \widetilde{T})$ and set

$$\tilde{y}(t) := \begin{cases} y(t), & \text{if } t \in [0, t_k], \\ y_k(t - t_k), & \text{if } t \in [t_k, t_k + T]. \end{cases}$$

It follows that $\tilde{y}$ is a solution of (2.35)-(2.36) on $[0, t_k + T]$ and $t_k + \widetilde{T} > \widetilde{T}$, which contradicts the definition of $\widetilde{T}$.

2.3 Derivative dependent nonlinearities

In this part we deal with a different type of nonlinearity, whose ellipticity value does not depend any more on the value of the solution but of its first derivative. To begin with we treat the case of non degenerate equations and in a second moment we address the problem of the p-laplacian equation. In both cases we are interested in the null controllability of these equations.

2.3.1 Non degenerate system

The control system in this case has the form

$$\begin{cases} y_t - a(y_x)_x = u(t) & (x, t) \in Q, \\ y_x(0, t) = 0 & t \in (0, T), \\ y_x(1, t) = v(t) & t \in (0, T), \\ y(x, 0) = y_0 & x \in (0, 1), \end{cases} \tag{2.46}$$

with $Q = (0, 1) \times (0, T)$ and $a : \mathbb{R} \to \mathbb{R}$ which satisfies the uniform ellipticity condition $0 < \mu \leq a'(x)$, and the bounds $|a'(x)|, |a''(x)|, |a'''(x)| \leq M < \infty$ for all $x \in \mathbb{R}$. As in section 1 we make use of an internal constant control u, but this time the boundary control is of Neumann type. Our main result is the following one:

Theorem 2.3.1. *For any sufficiently small δ and for any initial data with $||y_{0x}||_{H^1} \leq \delta$ there exists controls u, v_1, v_2 such that the solution of system (2.46) verifies $y(\cdot, T) = 0$.*

We start differentiating formally the state equation of (2.46) with respect to x and we call $w = y_x$. We look at the equation for w in the new domain $Q' = (0, 1) \times (0, T')$ with $T' < T$,

$$\begin{cases} w_t - a(w)_{xx} = 0 & (x,t) \in Q', \\ w(0,t) = 0 & t \in (0,T'), \\ w(1,t) = v_1(t) & t \in (0,T'), \\ w(x,0) = w_0 & x \in (0,1), \end{cases} \tag{2.47}$$

where $w_0 = y_{0x}$. In a first step we would like to steer w to zero at time T' with the only dirichlet boundary condition in $x = 1$. To prove such a result we extend the domain Q' to $Q'_2 = (0,2) \times (0,T')$, and the initial datum w_0 to $z_0 \in H^1(0,2)$ with $||z_0||_{H^1(0,2)} \leq c||w_0||_{H^1(0,1)}$. We build the new problem as in section 1

$$\begin{cases} z_t - a(z)_{xx} = u(x,t)\chi_\omega & (x,t) \in Q', \\ z(0,t) = 0 & t \in (0,T'), \\ z(2,t) = 0 & t \in (0,T'), \\ z(x,0) = z_0 & x \in (0,1), \end{cases} \tag{2.48}$$

where $\omega \subset (1,2)$. We can state the following.

Proposition 2.3.2. *For any $\delta > 0$ there exists $\eta > 0$ such that for every $z_0 \in H^1(0,2)$ with $||z_0||_{H^1} \leq \eta$, (2.48) is exactly null controllable.*

At this point we define the function

$$y(x,t) = \int_{1/2}^{x} z(s,t)\,\mathrm{d}s + C \quad x \in (0,1), t \in (0,T'),$$

where C is a constant to be chosen later. One can check that y solve the following problem

$$\begin{cases} \dfrac{d}{dt}y(x,t) - a(y_x)_x(x,t) = -a(z)_x(1/2,t) & x \in (0,1) \times (0,T'), \\ y_x(0,t) = 0 & t \in (0,T'), \\ y_x(1,t) = z(1,t) & t \in (0,T'), \\ y(x,0) = \displaystyle\int_{1/2}^{x} z_0(s)\,\mathrm{d}s + C & x \in (0,1). \end{cases} \tag{2.49}$$

We remark that since $z(\cdot,t) \in H^1(0,2)$, the state equation of (2.49) makes sense. To conclude we define $\bar{y}$ as

$$\bar{y} = \begin{cases} y(x,t) & t \in (0,T'), \\ k(t) & t \in (T',T), \end{cases} \tag{2.50}$$

where $k \in C^1(T', T)$ is such that $k(T) = 0$ and $\bar{y}$ is sufficiently regular. In the same spirit we define

$$\bar{v}(t) = \begin{cases} z(1,t) & t \in (0,T'), \\ 0 & t \in (T',T). \end{cases} \tag{2.51}$$

The vector $(\bar{y}, \bar{v})$ is a trajectory of system (2.46) with $\bar{y}(T') = 0$. Hence Theorem 2.3.1 holds.

2.3.2 p-Laplacian case

With the p-laplacian operator in the diffusion the situation is more complicated. In this case the degeneracy occurs at the points where $y_x = 0$. For the null control of the p-laplacian equation the set of control we use is more sophisticated and the hypothesises on the initial condition are more restrictive.

$$\begin{cases} y_t - (|y_x|^{p-2}y_x)_x = u_1(t)x + u_2(t) & (x,t) \in Q, \\ y_x(0,t) = v_0(t) & t \in (0,T), \\ y_x(1,t) = v_1(t) & t \in (0,T), \\ y(x,0) = y_0 & x \in (0,1), \end{cases} \tag{2.52}$$

The meaning of the symbols is the same as in section 2.3.1. We can see that the internal control is composed by two parts. The first one , u_1 has the function of steering the initial data to a constant state while u_2 is mainly needed in a second time to go from a generic constant state to zero, even though it plays a role also in the first part

Proposition 2.3.3. *Let $m \in [2, +\infty)$. For any initial data $y_0 \in W^{1,p}(0,1)$ with $y_{0x} \geq 0$ and any time $T > 0$, there exist controls $v_0(t), v_1(t)$ and $u_1(t), u_2(t)$ such that the solution y of (2.52) satisfies $y(\cdot, T) \equiv 0$ on $(0,1)$.*

The proof is a combination of the ideas used in sections 2.1 and 2.3.1. In particular, as in section 2.3.1, we make use of the existing relation between the PME and the p-laplacian equation.

the first step we formally derive with respect to x system (2.52) as in section 2.3.1 and we focus on the cylinder $Q' = (0,1) \times (0,T')$. We set $w = y_x$ and we have a new system

$$\begin{cases} w_t - (|w|^{p-2}w)_{xx} = u_1(t) & (x,t) \in Q' \\ w(0,t) = v_0(t) & t \in (0,T'), \\ w(1,t) = v_1(t) & t \in (0,T'), \\ w(x,0) = y_{0x} := w_0 & x \in (0,1), \end{cases} \tag{2.53}$$

with $w_0 \in L^p(0,1)$. We apply Theorem 2.1.1 to obtain controls $\bar{u}_1, \bar{v}_0, \bar{v}_1$ and a solution $\bar{w}$ such that $\bar{w}(T') = 0$. Again we define on $(0, T')$ the function

$$y(x,t) = \int_{\frac{1}{2}}^{x} w(s,t)\,\mathrm{d}s + C \quad x \in (0,1), t \in (0,T') \tag{2.54}$$

with $C = y_0(1/2)$. We can see that (2.54) is a solution of (2.52) with

$$u_2(t) = -(|w(x,t)|^{p-2} w(x,t))_x(1/2,t) - \frac{1}{2}u_1(t).$$

The choice of 1/2 as integration point is due to the higher regularity that w enjoys in the interior, giving sense to u_2.

If we check at y we see that $y(T') \equiv C$. So on the interval (T', T) we switch off u_1 and we follow a constant in space trajectory up to zero. This means we choose a smooth function $c : [T', T] \to \mathbb{R}$ such that $c(T') = C$, $c(T) = 0$: it is a solution of (2.52) on $(0,1) \times (T', T)$ with controls $u_1 = 0$, $u_2 = c'$, $v_0 = v_1 = c$.
The function obtained as

$$\bar{y}(t) = \begin{cases} y & t \in (0,T'), \\ c & t \in (T',T), \end{cases} \tag{2.55}$$

is the sought trajectory.

Remark 2.3.4. The hypothesis $y_{0x} \geq 0$ is quite restrictive and it would be nice to find a method to extend proposition 2.3.3 to the general case of $y_0 \in W^{1,p}(0,1)$.

Chapter 3

Expansion on the boundary of the support

This chapter has been written starting from the paper:

· J.I. Díaz, T.Mingazzini, A criterion on the boundary non-diffusion or expansion of the support for some reaction-diffusion free boundary problems, or how the free boundary approaches to the boundary. (In preparation).

In this chapter we shall study the way the free boundary of solutions to some partial differential equations behaves depending on the trace of the solutions. The free boundary problems we consider are of two different types: i) *Elliptic reaction-diffusion type problems,* as

$$\begin{cases} -Lu + \lambda u^q = 0 & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

under the fundamental assumption

$$q \in (0,1), \tag{3.2}$$

which guaranties the formation of the free boundary (at least for $\lambda > 0$ large enough, if Ω is bounded, or for any $\lambda > 0$, if Ω is unbounded). Such problem arises, for instance, in Chemical Engineering when a catalytic chemical reactor occupying a domain Ω has a reactant feed channel (entrance boundary) which is represented by the part $\Gamma_+ \subset \partial\Omega$, where the reactant concentration is $h(x) > 0$ and the rest of walls of the chemical reactant are isolated in such a way that, if we denote by $\Gamma_0 := \partial\Omega \setminus \Gamma_+$, then $h(x) = 0$ on Γ_0. Here we

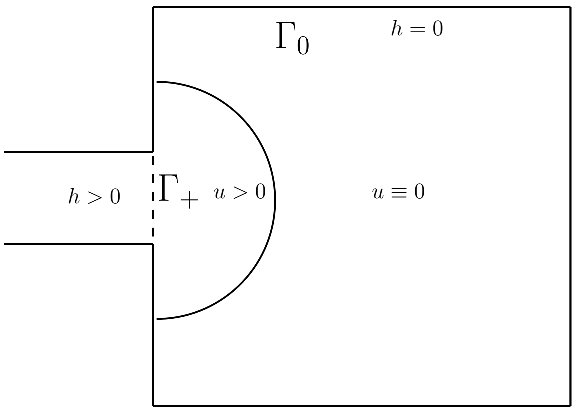

Figure 3.1: Chemical reactor scheme

assume that there is no exit boundary (see Figure 3.1). The exponent q is called the order of the reaction.
ii) *The obstacle problem*

$$\begin{cases} -Lu \geq f(x), \quad u \geq 0 \text{ and } (-Lu - f(x))u = 0 & \text{in } \Omega, \\ u(x) = h(x) & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Here the free boundary is given by the boundary of the *coincidence set* (the set of points where $u = 0$); according for instance to [39] a sufficient condition for the existence of the free boundary is that $f(x) \leq -\mu$ for some $\mu > 0$ on a large enough open subset of Ω (see, for instance, [69] for a full treatment of the obstacle problem). Among the many frameworks in which the obstacle problem arises we could mention, for instance, the unilateral problem of the stationary shape of a membrane which is forced downwards by a constant force f, is fixed on the boundary to a hight $h(x)$ and constrained to lie over the hyperplane $u = 0$. Actually, here we shall consider the special case in which (3.3) can be formulated in terms of

$$\begin{cases} -Lu + \lambda\beta(u) \ni \varepsilon & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

for some constant $\varepsilon \in [0, \lambda)$, where $\beta(u)$ is the maximal monotone graph of $\mathbb{R}^2$ given by

$$\beta(u) = \begin{cases} 0 & \text{for } u < 0, \\ [0, 1] & \text{for } u = 0, \\ 1 & \text{for } u > 0. \end{cases} \tag{3.5}$$

If u "solve problem" (3.4) (the rigorous definition of solution will be given later) then u is also a solution of the obstacle problem (3.3) with $f = -\lambda + \varepsilon$: indeed, we will see that $\varepsilon \geq 0$ and $h \geq 0$ imply that $u \geq 0$. Then, if $u > 0$, $-Lu + \lambda = \varepsilon$ which is the same as $-Lu - f = 0$. Finally, since there is uniqueness of solution for both formulations we get that the solutions must be the same.

Another interesting application of problem (3.4) arises also in the context of Chemical Engineering (as problem (3.1) with $q = 0$: see, e.g., [9]).

For some general purposes, such as the existence, uniqueness and regularity of the solutions, the domain Ω will be assumed to be an open regular set of $\mathbb{R}^N$. Nevertheless, when studying the qualitative properties of the solutions we focus on the bi-dimensional case, and we adopt as domain Ω the upper half plain in $\mathbb{R}^2$, $\Omega = \mathbb{R} \times [0, \infty)$. In this setting we use the following notation: $x := (x_1, x_2)$ with $x_1 \in \mathbb{R}$ and $x_2 \in [0, \infty)$. The unbounded boundary of the domain is then $\partial\Omega = \mathbb{R} \times \{0\}$ and so the boundary function h will depend only on the variable x_1. In addition, L denotes a second order elliptic operator of the form

$$Lu = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}u\right) = \operatorname{div}(\mathbf{A}(\mathbf{x})\nabla u), \tag{3.6}$$

with $a_{ij} \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$, such that the corresponding matrix $\mathbf{A}(\mathbf{x})$ is symmetric and positive definite. Actually, in the parts concerning the behaviour of the support and free boundary of the solutions we shall restrict to the case of constant coefficients. This restriction serves merely to simplify the calculations and does not affect the local behaviour. For what concerns the boundary datum h, we assume that

$$h \in L^\infty(\partial\Omega) \text{ and } h \geq 0 \text{ on } \partial\Omega,$$

even though the existence and uniqueness results on a bounded domain hold for $h \in L^1(\partial\Omega)$ (and even for signed boundary measures).

A general exposition containing many references on both problems can be found in the monograph [39]. One can see that both problems are special cases of the wider formulation

$$\begin{cases} -Lu + \lambda\beta(u) \ni f & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega, \end{cases} \tag{3.7}$$

where $\beta(u)$ is a maximal monotone graph of $\mathbb{R}^2$ such that $0 \in \beta(0)$: β is given by

$$\beta(u) = \lambda\,|u|^{q-1}\,u \tag{3.8}$$

in case of problem (3.1) and by (3.5) in case of problem (3.3). We define, as usual, the domain of β as $D(\beta) = \{r \in \mathbb{R} : \beta(r) \neq \emptyset\}$ where $\emptyset$ stands for the empty set.

We also consider the associated parabolic problem

$$\begin{cases} u_t - Lu + \lambda\beta(u) \ni f(x,t) & \text{in } Q_\infty, \\ u = h(t,x) & \text{on } \Sigma_\infty, \\ u(x,0) = u_0(x) & \text{on } \Omega, \end{cases} \tag{3.9}$$

where $Q_\infty = \Omega \times (0,\infty)$, $\Sigma_\infty = \partial\Omega \times (0,\infty)$ and for some $f \in L^\infty(Q_\infty) \cap L^1_{\text{loc}}((0,\infty); L^1_{\text{loc}}(\Omega))$, $h \in L^\infty(\Sigma_\infty) \cap L^1_{\text{loc}}((0,+\infty); L^1(\partial\Omega))$, with f, $h \geq 0$ respectively on Q_∞ and on Σ_∞, and $u_0 \in L^\infty(\Omega)$ with $u_0 \geq 0$ on Ω.

As mentioned before, the above problems, both elliptic and parabolic, give rise to a free boundary defined as the boundary of the support of the solution. If we denote the positivity set of a non-negative function u by $\mathcal{S}(u) := \{x \in \Omega : u(x) > 0\}$, then the free boundary is defined as $\mathcal{F}(u) = \partial\mathcal{S} \cap \Omega$ (we also introduce the null set of u as $\mathcal{N}(u) := \{x \in \Omega : u(x) = 0\}$ and the support of u as $\overline{\mathcal{S}}(u)$). Similar notations can be introduced also for the parabolic problem, applying the definitions to $u(t,\cdot)$. Our main goal in this paper is to study the behaviour of the free boundary near the support of the boundary datum h (respectively $h(t,\cdot)$). We shall assume always that

$$\mathcal{S}(h) \subsetneq \partial\Omega,$$

respectively

$$\mathcal{S}(h(t,\cdot)) \subsetneq \partial\Omega, \text{ for } a.e.\ t > 0.$$

The main question we investigate in this paper is whether the free boundary $\mathcal{F}(u)$ is connected or not with the boundary of the support of the boundary datum h (and similar question for the parabolic formulation). In some sense, this research can be considered as a natural continuation of the study of the so called *non-diffusion of the support property* (see [39] and [5]) in the case where $h \equiv 0$; under a suitable behaviour of f near the boundary of its support $\mathcal{S}(u) = \mathcal{S}(f)$. In the case of parabolic free boundary problems this question is related with the behaviour of the free boundary for small times (the so called *waiting time property*) and received a great attention in the last 40 years (see, e.g., the monographs [70], [8] and their many references). Another study, not too far from our interest is the paper by Martel and Souplet [62] regarding the behaviour of solutions of linear parabolic problems with incompatible initial data.

To be more precise, our main goal is to find some sufficient criterion on the behaviour of h near the boundary of its support ensuring that the free boundary $\mathcal{F}(u)$ is in contact with $\partial\mathcal{S}(h)$. In this way the support of the datum is not diffused on the boundary of the domain and we would have

$$\partial\mathcal{S}(u) \cap \partial\Omega = \mathcal{S}(h). \tag{3.10}$$

It is what we can call the *non-diffusion on the boundary of the support* property. In addition, we want to give some sufficient conditions ensuring the opposite qualitative behaviour, i.e., to find conditions on h implying that there is a strict expansion of the support $\mathcal{S}(h)$ on the boundary $\partial\Omega$. In other words, we want to know cases in which $\mathcal{F}(u)$ has no contact with $\partial\mathcal{S}(h)$ and so

$$\mathcal{S}(h) \subsetneq \partial\mathcal{S}(u) \cap \partial\Omega.$$

We call this phenomenon the *expansion on the boundary of the support* property. The only paper in the previous literature about such boundary qualitative behaviour we are aware of is [48] in which they proved the *expansion on the boundary of the support* property for problem (3.1) in the special case of $Lu = \Delta u$, h given by the Heaviside function and Ω the half plane $\mathbb{R} \times \mathbb{R}^+$. As we shall see later, this property also holds even for suitable continuous boundary data h.

Before stating our main results we need to make precise the notion of solution. The delicate point in our study is that we want to allow the boundary datum to be discontinuous and so the notion of the trace of the solution must be taken in a very general framework (something which, in our opinion, is not discussed enough in [48]).

We recall that the notion of boundary trace of a function u in Ω depends on the regularity properties of such function u. For instance, when $u \in C(\overline{\Omega})$ the boundary trace $u|_{\partial\Omega}$ is clearly well defined and belongs to $C(\partial\Omega)$. If u is in some Sobolev space $W^{1,p}(\Omega)$, for some $p > 1$, then the boundary trace can be defined and is a function in the space $L^p(\partial\Omega)$ (more precisely in the Sobolev space $W^{1-\frac{1}{p},p}(\partial\Omega)$: see, e.g., [58] and [2]) . Nevertheless, the identification of the elements of the trace space $W^{1-\frac{1}{p},p}(\partial\Omega)$ is not always easy and leads to some pathological results against intuition. For instance in the book by Mikhailov [63] one can see that already when $\Omega = B$, the unit ball of $\mathbb{R}^2$, there are continuous functions $h \in C(\partial B)$ which are not the trace of any function in $H^1(\Omega)$ (i.e., $C(\partial B) \not\subseteq H^{\frac{1}{2}}(\partial B)$).

A different approach was proposed by Haïm Brezis, in an unpublished paper (1972) profusingly mentioned in the literature (see [71], [61] and [42]), which holds for semilinear second order boundary value problems with boundary data in $L^1(\partial\Omega)$ (later extended to measures on $\partial\Omega$). The main idea is to multiply by a "regular" test function ($\varphi \in W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega)$) and to integrate twice by parts. We introduce the adjoint operator

$$L^*u = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial}{\partial x_i}u\right) = \mathrm{div}(\mathbf{A}^*\nabla u)$$

($\mathbf{A}^*$ the transposed matrix of $\mathbf{A}$) and for $x \in \partial\Omega$ we define

$$\partial_A u := (A^*\nabla u)\cdot \mathbf{n},$$

where $\mathbf{n}(x)$ is the outward normal vector to $\partial\Omega$ in x. A solution is then a function u which satisfies

$$-\int_\Omega uL^*\varphi\,\mathrm{d}x + \lambda\int_\Omega b\varphi\,\mathrm{d}x = \int_\Omega f\varphi\,\mathrm{d}x - \int_{\partial\Omega} h\partial_A\varphi\,\mathrm{d}\sigma, \tag{3.11}$$

for all $\varphi \in W^{2,\infty}(\Omega)\cap W_0^{1,\infty}(\Omega)$ and for some $b \in L^1_{loc}(\Omega)$ such that $b(x) \in \beta(u(x))$ for a.e. $x \in \Omega$ (in case of problem (3.1), it is $b = u^q$). In order to give a meaning to all the above integrals it is useful to recall that since $\varphi \in W_0^{1,\infty}(\Omega)$, we know that $c_1\rho(x) \leqslant \varphi_1(x) \leqslant c_2\rho(x)$ $\forall x \in \Omega$, where

$$\rho(x) := \mathrm{dist}\,(x, \partial\Omega),$$

and c_1, c_2 are positive constants. Thus we must require at least that

$$f \in L^1(\Omega;\rho),$$

where

$$L^1(\Omega;\rho) := \{f \in L^1_{loc}(\Omega) : \int_\Omega |f(x)|\rho(x)\,\mathrm{d}x < +\infty\}.$$

Definition 3.0.5. *Given $f \in L^1(\Omega;\rho)$ and $h \in L^1(\partial\Omega)$, we say that u is a very weak solution of problem (3.7) if $u \in L^1(\Omega)$ and there exists $b \in L^1(\Omega;\rho)$ such that $b(x) \in \beta(u(x))$ for a.e. $x \in \Omega$, and for any test function $\varphi \in W^{2,\infty}(\Omega)\cap W_0^{1,\infty}(\Omega)$ identity (3.11) holds.*

It is not too difficult to adapt to our framework some existence and uniqueness results in the literature (see [20], [71] and [61]).

Theorem 3.0.6. *Let Ω be a bounded regular open set of R^N, let β be a maximal monotone graph of R^2 such that $0 \in \beta(0)$ and let $f \in L^1(\Omega;d)$ and $h \in L^1(\partial\Omega)$. Then there exists a unique very weak solution u of problem (3.7). Moreover, there exists a constant C, only dependent of Ω, such that if $\widehat{u}$ is the very weak solution corresponding to the data $\widehat{f} \in L^1(\Omega;d)$ and*

$\widehat{h} \in L^1(\partial\Omega)$, with $\widehat{b}(x) \in \beta(\widehat{u}(x))$ for a.e. $x \in \Omega$ as in Definition 3.0.5, then we have

$$\begin{aligned} &\left\| [u-\widehat{u}]_+ \right\|_{L^1(\Omega)} + \lambda \left\| \left[b-\widehat{b} \right]_+ \right\|_{L^1(\Omega;d)} \\ &\qquad \le C \left(\left\| \left[f-\widehat{f} \right]_+ \right\|_{L^1(\Omega;d)} + \left\| \left[h-\widehat{h} \right]_+ \right\|_{L^1(\partial\Omega)} \right) \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} &\left\| u-\widehat{u} \right\|_{L^1(\Omega)} + \lambda \left\| b-\widehat{b} \right\|_{L^1(\Omega;d)} \\ &\qquad \le C \left(\left\| f-\widehat{f} \right\|_{L^1(\Omega;d)} + \left\| h-\widehat{h} \right\|_{L^1(\partial\Omega)} \right). \end{aligned} \tag{3.13}$$

In particular, $f \le \widehat{f}$ and $h \le \widehat{h}$ implies that $u \le \widehat{u}$ on Ω.

To study the behaviour of the solution close to the boundary of $\mathcal{S}(h)$ we consider the particular case where $\Omega = \mathbb{R} \times [0,\infty)$, a_{ij} is constant for $i,j \in \{1,2\}$, $f \equiv 0$, and $\beta(u)$ is given by (3.8) or (3.5). For what concern the boundary datum, we are interested in the case of h satisfying, $h \in L^\infty(\partial\Omega), h(x_1) = 0$ on $(-\infty,0)$ and $h(x_1) > 0$ on $(0,+\infty)$.

The reason why we consider boundary data in $L^\infty(\partial\Omega)$ instead of in $L^1(\partial\Omega)$ (remember that now $\partial\Omega$ is unbounded, so $L^\infty(\partial\Omega) \nsubseteq L^1(\partial\Omega)$) is that we know the explicit solution in the unperturbed linear case ($\lambda = 0$, $L = \Delta$, $f \equiv 0$) with boundary data given by the Heaviside function (3.50). Such solution is given by

$$u(x_1,x_2) = 1 - \frac{1}{\pi} \arctan\left(\frac{x_1}{x_2}\right) \tag{3.14}$$

(the result can be found in [48] formula (2.6)). Having at disposal an explicit solution like (3.14) is really useful in the study of the behaviour of general solutions close to the point $x = (0,0)$. In addition, since our main interest, as already said, is specifically the behaviour near the boundary of the support $\partial\mathcal{S}(h)$ and not in the whole Ω, we shall assume also that h is non decreasing and that $h(x_1) = c_+ > 0$ for $x_1 \ge \delta > 0$. We can resume this set of hypothesis in

$$\begin{cases} \Omega = \mathbb{R} \times [0,\infty),\ a_{ij} \text{ constant for } i,j \in \{1,2\},\ f \equiv 0, \\ \beta(u) \text{ is given by (3.8) or (3.5)}, \\ h \in L^\infty(\partial\Omega), h \text{ non decreasing }, \\ h(x_1) = 0 \text{ on } (-\infty,0) \text{ and } h(x_1) = c_+ > 0 \text{ on } (\delta,+\infty). \end{cases} \tag{H_{hp}}$$

To give a definition of solution for the new setting H_{hp}, we first introduce the family of rectangles

$$R_n = \{(x_1, x_2) : |x_1| < n, 0 < x_2 < \bar{x}\},$$

where $\bar{x}$ is a constant which will be make explicit later on. We define now the bounded domain Ω_n, which is any sufficiently smooth regularization of the rectangular R_n. We call Γ_n^1 the horizontal boundary of Ω_n, i.e., $\Gamma_n^1 := \{(x_1, x_2) \in \partial\Omega_n \,:\, x_2 = 0 \text{ or } x_2 = \bar{x}\}$, and $\Gamma_n^2 = \partial\Omega_n \setminus \Gamma_n^1$. Then we consider the problem

$$\begin{cases} -Lu + \beta(u) \ni 0 & \text{in } \Omega_n, \\ u(x_1, 0) = h(x_1), & u(x_1, \bar{x}) = 0 \quad |x_1| \leq n, \\ u(x_1, x_2) = 0 & (x_1, x_2) \in \Gamma_n^2, x_1 < 0, \\ u(x_1, x_2) = z(x_2) & (x_1, x_2) \in \Gamma_n^2, x_1 > 0, \end{cases} \tag{3.15}$$

where z is given by (3.48). We define the class of "limit very weak solutions".

Definition 3.0.7. *In the framework of H_{hp}, we say that u is a limit very weak solution of problem* (3.7) *if $u = \lim_{n\to\infty} u_n$, where u_n is the solution of the truncated problem* (3.15)*.*

Theorem 3.0.8. *Assume* (H_{hp})*. There exists a unique limit very weak solution of problem* (3.7) *on the hyperplane $\Omega = \mathbb{R} \times [0, \infty)$. Moreover that solution satisfies the comparison principle with respect to the boundary data h: if $h \leq \widehat{h}$ then the corresponding limit very weak solutions satisfy $u \leq \widehat{u}$ on Ω.*

Remark 3.0.9. It is easy to see that any "limit very weak solution" is a very weak solution in the sense that $u \in L^\infty(\Omega)$, $u \geq 0$ and there exists $b \in L^\infty(\Omega)$ such that $b(x) \in \beta(u(x))$ for a.e. $x \in \Omega$ and, for any test function $\varphi \in W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega)$ with compact support, identity (3.11) holds. Actually, it also satisfies that $u \in H^1_{loc}(\Omega) \cap C(\Omega)$ (indeed, it is enough to apply the local regularity result by Zhang and Bao [72] and to argue by bootstrap for other Sobolev spaces; see their Theorem 1.6, so as to arrive to continuity on open subsets).

In the specific setting of $L = \Delta$, boundary datum the Heaviside function and $\beta(u) = |u|^{q-1} u$, $q \in (0, 1)$, it is possible to show (see [48]) that the uniqueness holds for local $H^1_{loc}(\Omega)-$weak solutions which are continuous up to the boundary except the origin (i.e. in $C(\overline{\Omega} - (0, 0))$. Note that, although they prove uniqueness of a solution without asking the condition of being limit solution, their formulation is very restrictive (in particular it does not include the possibility of h just in $L^\infty(\partial\Omega)$).

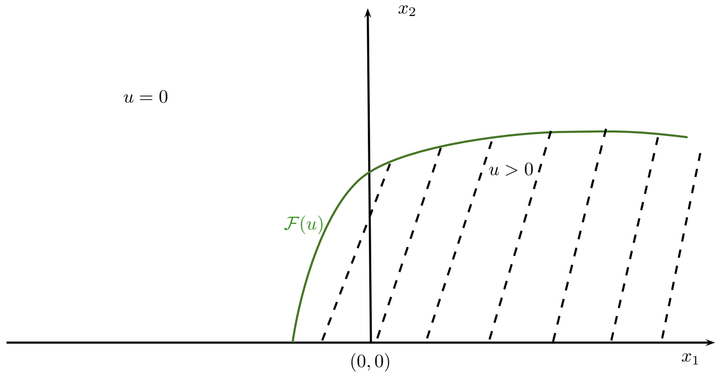

Figure 3.2: Expansion on the boundary of the support

Our technique allows to extend the uniqueness result in [48] in several senses ($L \neq \Delta$, case of β multivalued, and, for instance, cases in which the set of discontinuity of the boundary datum h is a countable set of points) that we shall not develop here.

Our main result concerns the qualitative behaviour of the solution of *(3.7) under the assumption* (H_{hp}).

Theorem 3.0.10. *Assume* (H_{hp}). *Then there exist four positive constants* $\underline{C} < \overline{C}$, $\underline{\varepsilon} < \overline{\varepsilon}$ *and two boundary points* $\underline{x}_{1,\varepsilon}, \overline{x}_{1,\varepsilon} > 0$, *such that :*

i) If $h(x_1) \geq \overline{C} x_1^{\frac{2}{1-q}}$ *for a.e.* $x_1 \in (0, \overline{x}_{1,\varepsilon})$ *and* $h(x) \geq \overline{\varepsilon}$ *for a.e.* $x_1 \in (\overline{x}_{1,\varepsilon}, +\infty)$ *then the expansion on the boundary of the support property holds.*

ii) If $h(x_1) \leq \underline{C} x_1^{\frac{2}{1-q}}$ *for a.e.* $x_1 \in (0, \underline{x}_{1,\varepsilon})$ *and* $h(x_1) \leq \underline{\varepsilon}$ *for a.e.* $x_1 \in (\underline{x}_{1,\varepsilon}, +\infty)$ *then the non-diffusion on the boundary of the support property holds.*

In both cases, $q \in (0, 1)$ *when* β *is given by* (3.8) *and* $q = 0$ *when* β *is chosen as* (3.5).

The pictures of an indicative qualitative behaviour of solutions illustrated in Theorem 3.0.10 is resumed is Figure 3.2 (i) and 3.3 (ii).

Corollary 3.0.11. *In the same framework of Theorem 3.0.10 we have the additional consequences:*

1) Under condition (i), $u = \frac{\partial}{\partial n} u = 0$ *on* $(-\infty, 0) \times \{0\}$.

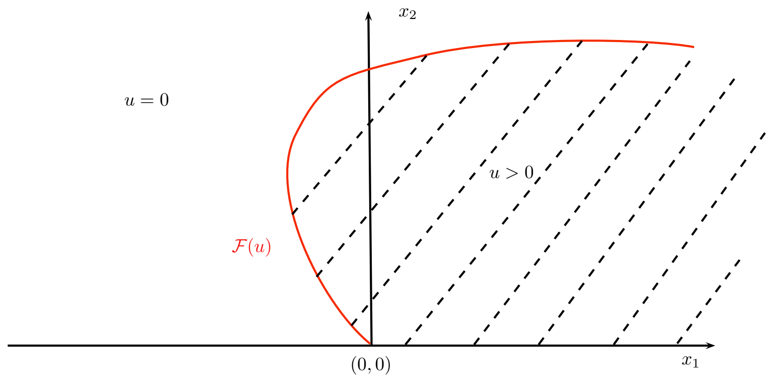

Figure 3.3: Non-diffusion on the boundary of the support

2) *Under condition (ii),* $\frac{\partial}{\partial n}u > 0$ *on* $(-\zeta, 0) \times \{0\}$ *for some* $\zeta > 0$. *If in particular, h is the a multiple of the Heaviside function then* $\frac{\partial}{\partial n}u \notin C(\partial\Omega)$ *and* $u \notin C(\overline{\Omega})$.

In both situations, $n = (0, 1)$ *is the normal vector to* $\partial\Omega$.

Concerning the parabolic problem our main interest consists in analysing the stabilization of the solution to the solution of stationary problem in order to well understand the *expansion on the boundary of the support* property. When Ω is a general open bounded set the notion of very weak solution is quite similar to the elliptic case.

Definition 3.0.12. *Let us take any* $T > 0$, $f \in L^1(0, T; L^1(\Omega; d))$, $h \in L^1(0, T; L^1(\partial\Omega))$ *and* $u_0 \in L^1(\Omega; d)$ *with* $u_0(x) \in \overline{D(\beta)}$. *We say that u is a very weak solution of problem* (3.9) *if* $u \in L^1(0, T; L^1(\Omega))$ *and there exists* $b \in L^1(0, T; L^1(\Omega; d))$ *such that* $b(t, x) \in \beta(u(t, x))$ *for a.e.* $(t, x) \in (0, T) \times \Omega$, *and for every test function* $\varphi \in W^{1,\infty}([0, T]; L^\infty(\Omega)) \cap L^\infty(0, T; W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega))$ *with* $\varphi(T, \cdot) = 0$ *the following identity holds*

$$\begin{aligned} &-\int_0^T \int_\Omega u \frac{\partial \varphi}{\partial t} \, \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega u L^* \varphi \, \mathrm{d}x + \lambda \int_0^T \int_\Omega b\varphi \, \mathrm{d}x\mathrm{d}t \\ &\quad = \int_\Omega u_0(x)\varphi(0, x) \, \mathrm{d}x + \int_0^T \int_\Omega f\varphi \, \mathrm{d}x\mathrm{d}t - \int_0^T \int_{\partial\Omega} h \, \partial_A \varphi \, \mathrm{d}\sigma\mathrm{d}t. \end{aligned} \tag{3.16}$$

Once again, it is not too difficult to adapt to our framework some existence and uniqueness results in the literature (see [59]).

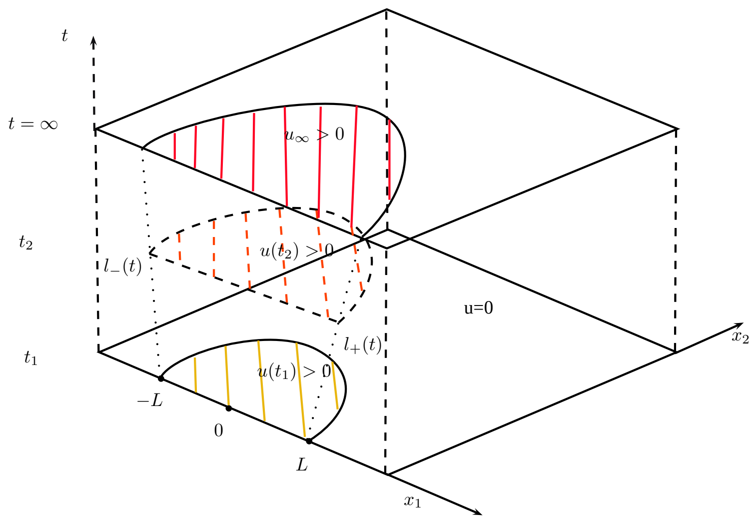

Figure 3.4: Convergence of $\mathcal{S}(u(t,\cdot))$ to $\mathcal{S}(u_\infty)$

Theorem 3.0.13. *i) For data f, h and u_0 as in Definition 3.0.12 there exists a unique very weak solution of* (3.9)*. Moreover we have a smoothing effect ([59]).*

ii) If in addition $h \in W^{1,1}(0,T;L^1(\partial\Omega))$, then the very weak solution satisfies $u \in C([0,T];L^1(\Omega;d))$.

Our result on the asymptotic behaviour, for t$\to +\infty$, seems to be new in the context of very weak solutions (check [41] and [56] for similar results on more regular solutions).

Remark 3.0.14. Whenever we are dealing at the same time with the parabolic and the elliptic problem, as we are going to do, we use the symbols u_∞, h_∞, f_∞ to denote the solution and the data of the elliptic boundary value problem.

For the next result we will add the following hypothesis:

$$\begin{aligned} &\exists\, q \in [0,1) \text{ such that } |b| \leq C\,|r|^q \\ &\text{for any } b \in \beta(r) \text{ and for any } r \in \mathbb{R}. \end{aligned} \tag{3.17}$$

In the above condition the case $q = 0$ means that $R(\beta)$ (the range of β, i.e., $r \in \mathbb{R}$ such that there exists $x \in \mathbb{R}$ for which $r \in \beta(x)$) is bounded.

Theorem 3.0.15. *Consider the case of condition* (3.17). *Assume* $h \in W^{1,1}(0,T;L^1(\partial\Omega))$ *for any* $T>0$ *and that there exists a sequence* $t_n \to +\infty$, *as* $n \to +\infty$, *such that*

$$\int_{t_n-1}^{t_n+1}\int_{\Omega} |f(s,x)-f_\infty(x)|\,\rho(x)\,\mathrm{d}x\mathrm{d}s \to 0 \text{ as } n\to+\infty \tag{3.18}$$

and

$$\int_{t_n-1}^{t_n+1}\int_{\partial\Omega} |h(s,x)-h_\infty(x)|\;\mathrm{d}\sigma\mathrm{d}s \to 0 \text{ as } n\to+\infty. \tag{3.19}$$

Assume in addition that

$$u \in L^\infty(0,\infty;L^1(\Omega;\rho)). \tag{3.20}$$

Then $u(t_n,\cdot)\to u_\infty$ *in* $L^1(\Omega;\rho)$ *with* u_∞ *the very weak solution of* (3.7) *with data* f_∞ *and* h_∞.

For the qualitative behaviour of the solutions, we consider now the half plane case under the assumptions

$$\begin{cases} \Omega=\mathbb{R}\times[0,\infty), a_{ij} \text{ are constants, } f\equiv 0, u_0\geq 0,\\ \beta(u) \text{ is given by (3.8) or (3.5),}\\ h\in W^{1,1}_{loc}(0,+\infty;L^1(\partial\Omega))\cap L^\infty((0,+\infty)\times\partial\Omega),\\ h(t,\cdot)=0 \text{ on } (-\infty,0) \text{ and } h(t,\cdot)>0 \text{ on } (0,+\infty),\\ h(t,x_1)=c_+\forall\, x_1\in[\delta,\infty), t>0. \end{cases} \tag{$\widehat{H}_{hp}$}$$

In this setting we adapt the definition of limit very weak solution from the elliptic case using truncated-in-space solutions. Existence and uniqueness can be obtain in similar way.

Corollary 3.0.16. *Assume* $(\widehat{H}_{hp})$. *Then there exist four positive constants* $\underline{C}<\overline{C}, \underline{\varepsilon}\leq\overline{\varepsilon}$ *and two boundary points* $\underline{x}_{1,\varepsilon}, \overline{x}_{1,\varepsilon}>0$ *such that*

i) if $u_0=0$, $h(t,x_1)\leq \underline{C}x_1^{\frac{2}{1-q}}$ *for a.e.* $x_1\in(0,\underline{x}_{1,\varepsilon})$ *and* $h(t,x_1)\leq\underline{\varepsilon}$ *for a.e.* $x_1\in(\underline{x}_{1,\varepsilon},+\infty)$, *for any* $t\geq 0$, *then the nondiffusion on the boundary of the support property holds for any* $t\geq0$, *i.e.,* $\partial\mathcal{S}(u(t,.))\cap\partial\Omega=\mathcal{S}(h(t,.))$ *for any* $t\geq 0$ *(infinite waiting time property).*

ii) Assume that $u_0(x_1,x_2)\geq u_1(x_1,x_2)+u_2(x_1,x_2)$ *with* u_1 *and* u_2 *solutions of the problems* (3.56) *and* (3.57) *and that* $h(t,x_1)\geq\overline{C}x_1^{\frac{2}{1-q}}$ *for a.e.* $x_1\in(0,\overline{x}_{1,\varepsilon})$ *and* $h(t,x_1)\geq\overline{\varepsilon}$ *for a.e.* $x_1\in(\overline{x}_{1,\varepsilon},+\infty)$, *for any* $t\in(0,T)$.

Then the expansion on the boundary of the support property holds for any $t \in (0,T]$, *i.e.,* $\mathcal{S}(h(t,.)) \subsetneq \partial\mathcal{S}(u(t,.)) \cap \partial\Omega := [-\delta_0, \infty] \times \{0\}$, *for some* $\delta_0 > 0$.

Remember that $q \in (0,1)$ *when* $\beta(u)$ *is given by* (3.8) *and* $q = 0$ *for the case* (3.5).

Corollary 3.0.17. *The conclusions of Corollary 3.0.11 remain valid for* $u(t,\cdot)$ *under the corresponding assumptions.*

We point out that the analysis of the free boundary studied in this chapter can be of great interest in the study of some optimal control problems. Consider, for example, a functional J as the one given in Chapter 1 but with B being a neighbourhood in $\mathbb{R} \times [0,\infty)$ of the interval $(-k,0) \times \{0\}$, and assume that the control variable this time is the boundary value h, with h as in H_{hp}. Then, in the minimization problem we will not consider those controls h whose growth near $x_1 = 0$ is too fast while we will prefer those with bigger mass at infinity and low growth close to the origin.

The organization of the rest of this chapter is the following. Section 3.1 is devoted to the proof of the general existence and uniqueness results, Theorems 3.0.6 and Theorem 3.0.13. The stabilization of very weak solutions, when $t \to +\infty$, is considered in Section 3.2 and, in particular Theorem 3.0.15 is proved there. Finally the special case of the half plane is considered in Section 3.3. After proving Theorem 3.0.8 we present the proof of Theorem 3.0.10 in Subsections 3.3.3 and 3.3.4. The special case of discontinuous boundary data plays an important role in such proof and so it is previously discussed there.

3.1 On the existence and uniqueness of very weak solutions

3.1.1 Proof of Theorem 3.0.6

We need to introduce first a result on the corresponding linear problem

$$\begin{cases} -Lu = f & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega. \end{cases} \tag{3.21}$$

Definition 3.1.1. *Assume $f \in L^1(\Omega;\rho)$ and $h \in L^1(\partial\Omega)$. A function $u \in L^1(\Omega)$ is a weak solution of (3.21) if it satisfies*

$$\int_\Omega u L^*\phi \,\mathrm{d}x = \int_{\partial\Omega} h\,\partial_A\phi \,\mathrm{d}\sigma - \int_\Omega f\phi\,\mathrm{d}x$$

for every function $\phi \in C_0^2(\bar{\Omega})$.

This lemma is a known result and we cite it for further needs.

Lemma 3.1.2. *Assume u solves (3.21) with $h = 0$ and $f \in L^2(\Omega)$. Then $u \in H_0^1(\Omega)$ and*

$$\int_\Omega -Lu \cdot u\,dx \geq C||u||^2_{L^2(\Omega)}$$

Next proposition is a consequence of a generalization of the estimates of Brezis [20] applied to the operator L. The proof when L is the Laplacian can be found in [61]. The case of an even more general second order linear operator of the form

$$Lu = -\operatorname{div}(\mathbf{A}\nabla u) + \mathbf{b}\cdot\nabla u - \operatorname{div}(\mathbf{c}u) + du$$

under appropriate structural and regularity assumptions on the coefficients $\mathbf{A} \in M^{n\times n}(\mathbb{R}), \mathbf{b}, \mathbf{c} \in \mathbb{R}^n, d \in \mathbb{R}$ (essentially the maximum principle should hold) is contained in [71].

Proposition 3.1.3. *Let $f \in L^1(\Omega;\rho)$ and $h \in L^1(\partial\Omega)$. Then there exists a unique solution $u \in L^1(\Omega)$ of problem (3.21) in the sense of Definition 3.1.1. Moreover there exists $C = C(\Omega, L) > 0$ such that*

$$||u||_{L^1(\Omega)} \leq C(||f||_{L^1(\Omega;\rho)} + ||h||_{L^1(\partial\Omega)}) \tag{3.22}$$

and u satisfies

$$-\int_\Omega u_+ L^*\phi\,\mathrm{d}x \leq \int_\Omega f(\operatorname{sgn}_+ u)\phi\,\mathrm{d}x - \int_{\partial\Omega}\partial_A\phi\, h_+\,\mathrm{d}\sigma, \tag{3.23}$$

and

$$-\int_\Omega |u| L^*\phi\,\mathrm{d}x \leq \int_\Omega f(\operatorname{sgn} u)\phi\,\mathrm{d}x - \int_{\partial\Omega}\partial_A\phi\,|h|\,\mathrm{d}\sigma, \tag{3.24}$$

for every non negative $\phi \in C_0^2(\bar{\Omega})$. We have used the notation

$$\operatorname{sgn} r = \begin{cases} 1 & \text{if r} > 0, \\ 0 & \text{if r} = 0, \\ -1 & \text{if r} < 0, \end{cases} \qquad \operatorname{sgn}_+ r = \begin{cases} 1 & \text{if r} \geq 0, \\ 0 & \text{if r} < 0. \end{cases}$$

Proof of Theorem 3.0.6. Uniqueness, monotonicity and estimate (3.12) follow from Proposition 3.1.3. Indeed, assume that u_1 and u_2 are solutions of (3.7) with data f_1, h_1 and f_2, h_2 respectively. It means that there exist $b_1(x) \in \beta(u_1(x))$ and $b_2(x) \in \beta(u_2(x))$ such that (3.11) holds. This implies that $w = u_1 - w_2$ is a solution of

$$\begin{cases} -Lw = f^* = f_1 - f_2 - b_1 + b_2 & \text{in } \Omega, \\ w = h^* = h_1 - h_2 & \text{on } \partial\Omega. \end{cases} \tag{3.25}$$

Then, estimate (3.12) follows from (3.23) when applied to w with test function ϕ_0 solution of

$$\begin{cases} -L^*\phi_0 = 1 & \text{in } \Omega, \\ \phi_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.26}$$

Also monotonicity follows from estimate (3.23) when applied to w with test function ϕ_0. Uniqueness can be derived with the same procedure from (3.24).

Existence. We consider the Yosida approximation β_μ of β, maximal monotone graph of $\mathbb{R}^2$, which we know to be a Lipschitz increasing function (see [21]). We look for a solution of the problem

$$\begin{cases} -Lu + \beta_\mu(u) = f & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega. \end{cases} \tag{3.27}$$

The solution of such a problem is a straightforward generalisation of Proposition 2.1.2 in [61].

Let us call u_μ the solution of (3.27), and let consider $f \in L^\infty(\Omega)$ and $h \in L^\infty(\partial\Omega)$. Then, by the monotonicity of solution one have the upper bound $u_\mu \leq M = \max(\sup_\Omega f, \sup_{\partial\Omega} h)$ for all $\mu > 0$. Also $\{\beta_\mu(u_\mu)\}$ is uniformly bounded in $L^\infty(\Omega)$ (see [22]).

We now show that $\{u_\mu\}$ and $\{\beta_\mu\}$ are a Cauchy sequences in $L^2(\Omega)$. Given $\lambda, \mu > 0$, we subtract the equation for u_λ and u_μ and multiply for $u_\lambda - u_\mu$ and integrate to obtain with the use of Lemma 3.1.2

$$\begin{aligned} 0 &= \int_\Omega -L(u_\lambda - u_\mu)(u_\lambda - u_\mu) + \int_\Omega (\beta_\lambda(u_\lambda) - \beta_\mu(u_\mu))((u_\lambda - u_\mu)) \\ &\geq C||u_\lambda - u_\mu||^2_{L^2(\Omega)} + (\beta_\lambda(u_\lambda) - \beta_\mu(u_\mu), (u_\lambda - u_\mu))_{L^2(\Omega)}, \end{aligned}$$

which following [22] gives

$$C||u_\lambda - u_\mu||^2_{L^2(\Omega)} + (\beta_\lambda(u_\lambda) - \beta_\mu(u_\mu), \lambda\beta_\lambda(u_\lambda) - \mu\beta_\mu(u_\mu))_{L^2(\Omega)} \leq 0.$$

Sending $\lambda, \mu \to 0$ and remembering that $\{\beta_\mu(u_\mu)\}$ is uniformly bounded in $L^\infty(\Omega)$, we get that $||u_\lambda - u_\mu||_{L^2(\Omega)} \to 0$. We set $u := \lim_{\mu\to 0} u_\mu$. By Lemma

2.4 of [36] also $\beta_\mu(u_\mu)$ is a Cauchy sequence in $L^2(\Omega)$ and its limit $b \in L^\infty(\Omega)$ satisfies that $b(x) \in \beta(u(x))$ since β is maximal. Passing to the limit in the definition of solution we have

$$
\begin{aligned}
0 = -\int_\Omega u_\mu L^* \varphi \,\mathrm{d}x + \lambda \int_\Omega \beta_\mu(u_\mu)\,\varphi \,\mathrm{d}x - \int_\Omega f\varphi \,\mathrm{d}x + \int_{\partial\Omega} h\partial_A\varphi \,\mathrm{d}\sigma \to \\
- \int_\Omega u L^* \varphi \,\mathrm{d}x + \lambda \int_\Omega b\,\varphi \,\mathrm{d}x - \int_\Omega f\varphi \,\mathrm{d}x + \int_{\partial\Omega} h\partial_A\varphi \,\mathrm{d}\sigma
\end{aligned}
\tag{3.28}
$$

for any $\varphi \in C_0^2(\bar{\Omega})$. Hence u is a solution of (3.7).

If $(f,h) \in L^1(\Omega;\rho) \times L^1(\partial\Omega)$ we consider $\{(f_n,h_n)\} \subset L^\infty(\Omega) \times L^\infty(\partial\Omega)$ which converges to (f,h) in $L^1(\Omega;\rho) \times L^1(\partial\Omega)$. Call u_n the solution of (3.7) with data f_n and h_n. Thanks to (3.13), u_n and b_n are Cauchy sequences in $L^1(\Omega)$ and hence converges to functions u, b respectively. Since β is maximal $b(x) \in \beta(u(x))$ and passing to the limit in the definition of solution we find that u solves (3.7). □

3.1.2 Proof of Theorem 3.0.13

We start this part by giving a result on the corresponding linear problem:

$$
\begin{cases}
u_t - Lu = f(t,x) & \text{in } Q_T, \\
u = h(t,x) & \text{on } \Sigma_T, \\
u(0,x) = u_0(x) & \text{on } \Omega,
\end{cases}
\tag{3.29}
$$

Proposition 3.1.4. *Assume $f \in L^1(Q_T;\rho)$, $h \in L^1(\Sigma)$ and $u_0 \in L^1(\Omega;\rho)$. Problem (3.29) possesses a unique weak solution $u \in L^1(Q_T)$, in the sense that*

$$
\int_{Q_T} -(\zeta_t + L\zeta)u - f\zeta \,\mathrm{d}x\mathrm{d}t = -\int_{\Sigma_T} h\,\partial_A\zeta \,\mathrm{d}\sigma\mathrm{d}t + \int_\Omega \zeta(x,0)u_0 \,\mathrm{d}x
$$

for every $\zeta \in C^{2,1}(\overline{Q}_T)$. There hold also

$$
||u||_{L^1(Q_T)} \le C(||f||_{L^1(Q_T;\rho)} + ||h||_{L^1(\Sigma_T)} + ||u_0||_{L^1(\Omega;\rho)}) \tag{3.30}
$$

with $C > 0$ and

$$
\begin{aligned}
&\int_{Q_T} -(\zeta_t + L\zeta)|u| - f\zeta \operatorname{sgn}(u) \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \le -\int_{\Sigma_T} |h|\,\partial_A\zeta \,\mathrm{d}\sigma\,\mathrm{d}t + \int_\Omega \zeta(x,0)|u_0| \,\mathrm{d}x
\end{aligned}
\tag{3.31}
$$

for every non-negative $\zeta \in C_0^{2,1}(Q_T)$.

Proof. The proof is exactly the same of [60] with Δ replaced by L. □

Lemma 3.1.5. *Let $h \in L^1(\Sigma)$ and $u_0 \in L^1(\Omega;\rho)$. Then problem* (3.9) *has at most one solution. If u_1, u_2 are solutions with data h_1, u_{01} and h_2, u_{02} respectively, then*

$$\begin{aligned} &||u_1 - u_2||_{L^1(Q_T)} + ||g(u_1) - g(u_2)||_{L^1(Q_T;\rho)} \\ &\qquad \le C\left(||h_1 - h_2||_{L^1(\Sigma)} + ||u_{01} - u_{02}||_{L^1(\Omega;\rho)}\right), \end{aligned} \tag{3.32}$$

with $C > 0$. Furthermore, if $h_1 \le h_2$ and $u_{01} \le u_{02}$ then $u_1 \le u_2$.

Proof. Suppose u_1, u_2 two solutions of (3.9) with same data. Then $w = u_1 - u_2$ satisfies

$$\int_{Q_T} -(\zeta_t + L\zeta)w + (g(u_1) - g(u_2))\zeta \,\mathrm{d}x\,\mathrm{d}t = 0$$

for every $\zeta \in C_0^{2,1}(\overline{Q}_T)$. Take $\zeta = \psi_T$ solution of

$$\begin{cases} -\psi_t - L\psi = 1 & \text{in } \mathrm{Q_T}, \\ \psi = 0 & \text{on } \Sigma, \\ \psi(x,T) = 0 & \text{in } \Omega. \end{cases}$$

Since g is non decreasing, $(g(u_1) - g(u_2))w \ge 0$. We deduce that $w = 0$. Now if u_1 and u_2 are solutions with data h_1, u_{01} and h_2, u_{02} then $u = u_1 - u_2$ is solution of (3.29) with $f = g(u_1) - g(u_2)$, $h = h_1 - h_2$ and $u_0 = u_{01} - u_{02}$. Hence from (3.31) we deduce (3.32). □

Proof of Theorem 3.0.13. The proof of existence for the case of β a continuous function is again an easy adaptation of previous results in the literature concerning the special case of $L = \Delta$ (see, e.g., [1] Lemma 2.7 and [60] Lemma 1.3 and Lemma 1.7: notice that the assumption of β Lipschitz assumed at the beginning of the paper is not needed in both lemmas). The adaptation to the case of L given by (3.6) and β multivalued is completely similar to the one presented in the stationary case.

The continuity in t of the very weak solution will be used in our study of the asymptotic behaviour of solutions and can be obtained by reformulating the parabolic semilinear problem as an abstract Cauchy problem on the Banach space $X = L^1(\Omega;\rho)$

$$(AP)\begin{cases} \frac{du}{dt}(t) + A(t)u(t) \ni f(t) & \text{in } X \\ u(0) = u_0 \end{cases}$$

where $A(t) : D(A(t)) \to \mathcal{P}(X)$ is the operator defined by $(w,z) \in A(t) \subset X \times X$ iff $w \in L^1(\Omega)$ is the very weak solution of

$$\begin{cases} -Lw + \lambda\beta(w) \ni z & \text{in } \Omega, \\ w(x) = h(t,x) & \text{on } \partial\Omega. \end{cases}$$

Here, in the definition of the operator $A(t)$, $t \in (0,T)$ is a parameter (remember that $h \in W^{1,1}(0,T : L^1(\partial\Omega)$, hence $h(t,\cdot)$ makes sense). For a.e. $t \in (0,T)$, this operator is $T - \omega-$accretive on X (see, e.g., [12] or [14]) for some $\omega \geq 0$ large enough. Indeed, we must show that $(I + \mu(A(t) + \omega I))^{-1}$ is a contraction on X and this is equivalent to show that if w^i, with $i = 1,2$, are the very weak solutions of

$$\begin{cases} -\mu L w^i + \lambda\mu\beta(w^i) + \mu\omega w^i \ni z^i(x) & \text{in } \Omega, \\ w^i(x) = h(t,x) & \text{on } \partial\Omega, \end{cases} \tag{3.33}$$

for some $z^i \in L^1(\Omega;\rho)$, then

$$\left\| \left[w^1 - w^2\right]_+ \right\|_{L^1(\Omega;\rho)} \leq \left\| \left[z^1 - z^2\right]_+ \right\|_{L^1(\Omega;\rho)}.$$

But this is a trivial consequence of the estimates proven in Theorem 1 once that ω is taken large enough (in particular $\omega > C$, the constant appearing in estimate (14) which was only dependent of Ω). In addition, for a.e. $t \in (0,T)$ this operator is m-accretive (see [12]) in the sense that $R(I + \mu A(t)) = X$. Indeed we must prove that problem (3.33) (i.e., (3.7)) for a given right hand side in $z^i \in L^1(\Omega;\rho)$ has a unique solution, which, again, is consequence of Theorem 3.0.6. Finally, since $h \in W^{1,1}(0,T : L^1(\partial\Omega))$ we get that the $t-$dependence of the solution has the same regularity: $w \in W^{1,1}(0,T : L^1(\Omega))$ and so the Crandall-Evans theorem ([35],[49]) can be applied ensuring the existence and uniqueness of a mild solution $u \in C([0,T] : L^1(\Omega;\rho))$ of the abstract problem (AP). Finally, since we have uniqueness of the very weak solution of the parabolic problem, it is easy to see (as, for instance, in [15]) that both solutions must coincide and thus we get the desired time regularity result. □

3.2 On the stabilization when $t \to +\infty$

Remember that the solutions of the parabolic problem (3.9) and the elliptic problem (3.7) will be indicated with u and u_∞ respectively.

Proof of Theorem 3.0.15. We follow some of the ideas contained in [41] (see also references therein). We define

$$U_n(s,x) = u(t_n + s, x), \quad F_n(s,x) = f(t_n + s, x), \quad H_n(s,x) = h(t_n + s, x).$$

where $t_n \to \infty$ when $n \to \infty$. By Theorem 3.0.13 we know that there exists $b \in L^1(0,T;L^1(\Omega;\rho))$ such that $b(t,x) \in \beta(u(t,x))$ *a.e.* in Q_T. Thus we also define $B_n(s,x) = b(t_n + s, x)$. Then it is clear that

$$\begin{cases} \dfrac{\partial U_n}{\partial s} - LU_n + \lambda B_n(s,x) = F_n(s,x) & \text{in } (-1,1)\times\Omega, \\ U_n = H_n(s,x) & \text{on } (-1,1)\times\partial\Omega. \end{cases} \tag{3.34}$$

for all $n > 1$. Then from the estimate of Theorem 3.0.13 (which coincides with (1.26) of [60] easily adapted to the case in which L is given by (3.6)), for any $\varepsilon \in (0,2)$

$$\begin{aligned} &\|U_n\|_{L^1((-1+\varepsilon,1)\times\Omega)} + \lambda\,\|B_n\|_{L^1((-1+\varepsilon,1);L^1(\Omega;\rho))} \leq \\ &\qquad C\Big(\|F_n\|_{L^1((-1+\varepsilon,1);L^1(\Omega;\rho))} + \|H_n\|_{L^1((-1+\varepsilon,1);L^1(\partial\Omega))} \\ &\qquad\quad + \|U_n(-1+\varepsilon,\cdot)\|_{L^1(\Omega;\rho))}\Big). \end{aligned}$$

Assumptions (3.18) and (3.19) imply that $F_n \to F_\infty$, and $H_n \to H_\infty$ strongly in $L^1((-1,1);L^1(\Omega;\rho))$ and $L^1((-1,1);L^1(\partial\Omega))$ respectively, where F_∞ and H_∞ are defined as $F_\infty(s,x) = f_\infty(x)$ and $H_\infty(s,x) = H_\infty(x)$. Moreover, from assumption (3.20), $\|U_n(-1,\cdot)\|_{L^1(\Omega;\rho))}$ is bounded independently of n. Consequently, due to assumption (3.17), $\{\rho B_n\}$ is a bounded sequence in $L^p((-1,1);L^p(\Omega))$ with $p = 1/q$ if $q \in (0,1)$ and for every $p > 1$ if $q = 0$. Thus,

$$\rho B_n \rightharpoonup \rho B_\infty$$

weakly in $L^p((-1,1);L^p(\Omega))$ (after passing to a subsequence), for some B_∞ in $L^p((-1,1);L^p(\Omega))$. Then, by [60] (Lemma 1.6 (ii) easily adapted to the case in which $L \neq \Delta$), we get that $U_n \to U_\infty$ (strongly) in $C([-1+\varepsilon,1];L^1(\Omega;\rho))$ for some $U_\infty \in C([-1+\varepsilon,1];L^1(\Omega;\rho))$ for any $\varepsilon \in (0,2)$. Indeed, since problem (3.34) is linear we can use the decomposition

$$U_n = \mathbb{P}(F_n - \lambda B_n, 0, 0) + \mathbb{P}(0, H_n, 0) + \mathbb{P}(0,0,U_n(-1))$$

where $\mathbb{P}$ is the solution mapping (see [60] Lemma 1.6). The operator $\mathbb{P}(F_n - \lambda B_n, 0, 0)$ is compact from $L^p((-1+\varepsilon,1)\times\Omega)\times\{0\}\times\{0\} \to C([-1+\varepsilon,1]\times\Omega)$ since, in general $w = \mathbb{P}(q,0,0)$ is given by

$$w(s,x) = \int_{-1}^{1}\int_\Omega G_L(x,y,s,\tau)\,q(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau$$

(remember that the Green function $G_L(x,\cdot,s,\cdot) \in C_0([-1+\varepsilon,1]\times\overline{\Omega})$. The compactness of the other terms $\mathbb{P}(0,H_n,0)$ and $\mathbb{P}(0,0,U_n(-1))$ was shown in Lemma 1.6 (ii) of the mentioned reference. In our case we know the continuity in time of the functions (Proposition 3.0.13). In particular, since $U_n \to U_\infty$ in $C([-1+\varepsilon,1];L^1(\Omega))$, we find that $\{U_n(-1+\varepsilon,\cdot)\}$ is a Cauchy sequence in $L^1(\Omega;\rho)$.

Then, by applying an estimate similar to (14) but for the parabolic problem (see estimate (1.26) of [60] easily adapted to the case in which L is given by (3.6)) we get

$$\begin{aligned}
&\|U_n - U_m\|_{L^1((-1+\varepsilon,1)\times\Omega)} + \lambda \|B_n - B_m\|_{L^1((-1+\varepsilon,1);L^1(\Omega;\rho))} \\
&\quad \le C\Big(\|F_n - F_m\|_{L^1((-1+\varepsilon,1);L^1(\Omega;\rho))} + \|H_n - H_m\|_{L^1((-1+\varepsilon,1);L^1(\partial\Omega))} \\
&\qquad + \|U_n(-1+\varepsilon) - U_m(-1+\varepsilon)\|_{L^1(\Omega;\rho)}\Big)
\end{aligned}$$

which proves that $\{\rho B_n\}$ is a Cauchy sequence in $L^1((-1+\varepsilon,1);L^1(\Omega))$ and so $\rho B_n \to \rho B_\infty$ strongly in $L^1((-1+\varepsilon,1);L^1(\Omega))$. Then, since β is maximal monotone we conclude (see [13]) that $B_\infty(s,x) \in \beta(U_\infty(s,x))$ for a.e. $(s,x) \in (-1+\varepsilon,1)\times\Omega$.

It only remains to be proved that $U_\infty(s,x) = u_\infty(x)$ with u_∞ the (unique) very weak solution of (3.7) with data f_∞, h_∞. Since

$$ess \sup_{t_n+s\in(0,+\infty)} \|u(t_n+s,.)\|_{L^1(\Omega:d)} \le C,$$

then there exists a (stationary) Radon measure $\mu_\infty \in M(\Omega;\rho)$ such that $u(t_n+s,.) \rightharpoonup \mu_\infty$ weakly in $M(\Omega;\rho)$. Moreover $u(t_n+s,.) \to U_\infty(s,x)$ (strongly) in $C([-1+\varepsilon,1];L^1(\Omega;\rho)) \subset C([-1+\varepsilon,1];M(\Omega;\rho))$. Then, by the uniqueness of the limit, we deduce that $U_\infty(s,.) = \mu_\infty(\cdot)$ for any $s \in [-1+\varepsilon,1]$, so that the singular part of the measure $\mu_\infty(\cdot)$ vanishes (i.e., $\mu_\infty \in L^1(\Omega;\rho)$). Let us denote now $u_\infty(x) \equiv \mu_\infty(x)$. Then $U_\infty(s,x) = u_\infty(x)$ for any $s \in [-1+\varepsilon,1]$. By the same reasons (thanks to the assumption on β) we get that $B_\infty(s,x) = b_\infty(x)$ for a.e. $s \in [-1+\varepsilon,1]$ for some $b_\infty \in L^1(\Omega;\rho)$ such that $b_\infty(x) \in \beta(u_\infty(x))$ for a.e. $x \in \Omega$. Finally, we take as test function $\varphi(s,x) = \psi(s)\zeta(x)$ with $\zeta \in W^{2,\infty}(\Omega)\cap W_0^{1,\infty}(\Omega)$ and $\psi \in C^1([-1,1])$ such that $\psi_{|[-1,-1+\varepsilon]} = \psi(1) = 0$ and such that $\int_{-1+\varepsilon}^{+1}\psi(s)ds = 1$ in the definition of very weak solution of the parabolic problem. Obviously such special $\varphi(s,x)$ is a correct test function since $\varphi \in W^{1,\infty}(-1,1;L^\infty(\Omega))\cap L^\infty(-1,1;W^{2,\infty}(\Omega)\cap W_0^{1,\infty}(\Omega))$ and $\varphi(1,.) = 0$. Then, from the definition of very weak solution, we get that

$$
\begin{aligned}
&- \int_{-1+\varepsilon}^{1} \int_\Omega U_n(s,x)\psi'(s)\zeta(x)\,\mathrm{d}x\mathrm{d}s + \int_{-1+\varepsilon}^{1} \int_\Omega \psi(s)U_n(s,x)L^*\zeta(x)\,\mathrm{d}x\mathrm{d}s \\
&\qquad + \lambda \int_{-1+\varepsilon}^{1} \int_\Omega B_n(s,x)\psi(s)\zeta(x)\,\mathrm{d}x\mathrm{d}s \\
&= \int_\Omega U_n(-1,x)\psi(-1)\zeta(x)\,\mathrm{d}x + \int_{-1+\varepsilon}^{1} \int_\Omega F_n(s,x)\psi(s)\zeta(x)\,\mathrm{d}x\mathrm{d}s \\
&\qquad - \int_{-1+\varepsilon}^{1} \int_{\partial\Omega} \psi(s)H_n(s,\sigma)\frac{\partial\zeta(\sigma)}{\partial n}\,\mathrm{d}\sigma\mathrm{d}s.
\end{aligned}
$$

Passing to the limit, as $n \to +\infty$, and using that $\psi(-1+\varepsilon) = 0$ we arrive to

$$
\begin{aligned}
&- \left(\int_{-1+\varepsilon}^{1} \psi'(s)\,\mathrm{d}s\right)\left(\int_\Omega u_\infty(x)\zeta(x)\,\mathrm{d}x\right) + \int_{-1+\varepsilon}^{1} \psi(s)\,\mathrm{d}s \int_\Omega u_\infty(x)L^*\zeta(x)\,\mathrm{d}x \\
&\qquad + \lambda \int_{-1+\varepsilon}^{1} \int_\Omega B_\infty(s,x)\psi(s)\zeta(x)\,\mathrm{d}x \\
&= \int_{-1+\varepsilon}^{1} \psi(s)\,\mathrm{d}s \int_\Omega f_\infty(x)\zeta(x)\,\mathrm{d}x - \int_{-1+\varepsilon}^{1} \psi(s)\,\mathrm{d}s \int_{\partial\Omega} h_\infty(\sigma)\frac{\partial\zeta(\sigma)}{\partial n}\,\mathrm{d}\sigma.
\end{aligned}
$$

But

$$
\int_{-1+\varepsilon}^{1} \psi'(s)\mathrm{d}s = 0,
$$

and since $\int_{-1+\varepsilon}^{+1} \psi(s)\mathrm{d}s = 1$ we get that

$$
\begin{aligned}
&\int_\Omega u_\infty(x)L^*\zeta(x)\mathrm{d}x + \lambda \int_\Omega b_\infty(x)\zeta(x)\,\mathrm{d}x\mathrm{d}s \\
&\qquad = \int_\Omega f_\infty(x)\psi(s)\zeta(x)\,\mathrm{d}x - \int_{\partial\Omega} h_\infty(\sigma)\frac{\partial\zeta(\sigma)}{\partial n}\,\mathrm{d}\sigma
\end{aligned}
$$

which shows that u_∞ coincides with the (unique) very weak solution of the stationary problem.

□

Remark 3.2.1. Notice that the boundedness of the trayectories assumption is considerably weaker than the usual for weak solutions (see, e.g, [41]) which is of the type $u \in L^\infty(0,+\infty; H^1(\Omega))$. Notice also that this condition is necessary once we assume that conclusion of Theorem 3.0.15 holds.

A sufficient condition leading to the boundedness of the trajectories (assumption (3.20)) can be obtained by the method of super and subsolutions as in Proposition 3 of [41].

Proposition 3.2.2. *Assume that the stationary problem* (3.7) *admits a bounded weak solution* u_∞. *Let* f, f_∞ *and* h, h_∞ *satisfy* (3.18) *and* (3.19) *respectively. Suppose the existence of* $f_u, f_d \in L^1((0,T)\times\Omega)$ *for any* $T>0$ *and* $h_u, h_d \in L^1((0,T)\times\partial\Omega)$ *with* f_u, h_u (f_d, h_d) *non increasing (decreasing) in* t *such that*

$$-\overline{f}(x) \le f_d(t,x) \le f(t,x) \le f_u(t,x) \le \overline{f}(x),$$
$$-\overline{h}(x) \le h_d(t,x) \le h(t,x) \le h_u(t,x) \le \overline{h}(x),$$

for $0 \le \bar f = \operatorname{div}\mathbf{c}$ *with* $\mathbf{c}\in L^p(\Omega)^N$, $0\le \bar h \in L^1(\partial\Omega)$ *and*

$$\begin{aligned}\lim_{t\to\infty} f_u(t,x) &= \lim_{t\to\infty} f_d(t,x) = f_\infty(x) \text{ in } L^1(\Omega;\rho),\\ \lim_{t\to\infty} h_u(t,x) &= \lim_{t\to\infty} h_d(t,x) = h_\infty(x) \text{ in } L^1(\partial\Omega).\end{aligned} \tag{3.35}$$

Let u, u_u, u_d *be bounded weak solutions of* (3.9) *associated to the data* (f,h,u_0), $(f_u,h_u,\bar u_0)$ *and* $(f_d,h_d,\underline{u}_0)$ *with* $\bar u_0, \underline{u}_0$ *solutions of* (3.7) *with data* $\bar f, \bar h$ *and* $-\bar f, -\bar h$ *respectively. If* $u_u, u_d \in L^\infty((t_0,\infty);L^1(\Omega))$ *for some* $t_0>0$ *then the conclusion of Theorem 3.0.15 holds.*

3.3 On the half plane problems

3.3.1 Proof of Theorem 3.0.8.

Before starting with proving existence and uniqueness we shall show some results concerning the boundedness of the support of solutions. For this purpose we assume that

$$\beta(u) = u^q, \quad \lambda = 1, \quad \sup h = 1. \tag{3.36}$$

This hypothesis on λ and h is not a restriction as, taking it off, all calculations can be performed in the same way. Moreover the proof of Theorems 3.0.8 and 3.0.10 for the multivalued case (which formally corresponds to make $q=0$ in all the above expressions) follows, word by word, the same proof of the case in which $q\in(0,1)$ and replacing the identity symbol $=$ by the one of containing $\ni$. The details about local super and subsolutions can be seen also in the book [39] (Theorem 2.16 Chapter 2).

In this first part of this section we study some comparison functions which are essential, first of all, to give sense to the formulation of problem (3.15), especially to its boundary conditions, fundamental in the definition of "limit very weak solution". Secondly, they are important also in the study of the behaviour of solutions near the origin.

Assuming (3.36), we repeat the same procedure of [48], looking for local supersolutions, which are solutions of

$$\begin{cases} -Lu + u^q = 0 & \text{in } B_R(x_0), \\ u = 1 & \text{on } \partial B_R(x_0), \end{cases} \tag{3.37}$$

where $B_R(x_0)$ is the ball with radius R and centered in x_0. The problem is that, differently from [48], we do not know exact radial solution for (3.37). So we introduce a family of radial supersolution for (3.37).

Assume that $y(x) = \eta(|x - x_0|)$ is a radially symmetric function defined in $B_R(x_0)$. Then, if we denote with $r = |x - x_0|$, we have

$$Ly(x) = \eta'' \sum_{ij} a_{ij} \frac{x_i x_j}{r^2} + \frac{\eta'}{r} \left(\sum_i a_{ii} - \sum_{ij} a_{ij} \frac{x_i x_j}{r^2} \right). \tag{3.38}$$

Considering that

$$\sum_i a_{ii} - \nu \leq \sum_i a_{ii} - \sum_{ij} a_{ij} \frac{x_i x_j}{r^2} \leq \sum_i a_{ii} - \mu, \tag{3.39}$$

we can define the quantity

$$B_A = \sup_{x \in \Omega} \left(\sum_i a_{ii} - \sum_{ij} a_{ij} \frac{x_i x_j}{r^2} \right).$$

In particular, if we assume $\eta', \eta'' \geq 0$

$$-Ly \geq -\nu \eta'' - \frac{B_A}{r} \eta'. \tag{3.40}$$

We introduce the operator

$$L_\nu(\eta) = \nu \eta'' + \frac{B_A}{r} \eta' \tag{3.41}$$

which operates on functions of a real scalar variable and we study the properties of the solutions to the problem

$$\begin{cases} -L_\nu \eta + \eta^q = 0 & r \in (0, R), \\ \eta(0) = 0, \ \eta(R) = 1 \end{cases} \tag{3.42}$$

whenever $R \in \mathbb{R}^+$. We set the constants

$$C_0 = \left(\frac{(1-q)^2}{2\nu(q+1) + 2B_A(1-q)} \right)^{1/(1-q)}, \qquad R_0 = \frac{1}{C_0^{(1-q)/2}} \tag{3.43}$$

and introduce the function

$$\eta_0(r) = C_0 r^{\frac{2}{1-q}}, \quad r \in [0, R_0].$$

It is a direct computation to see that η_0 is the solution of (3.42). For $R > R_0$ we do not know the analytic form of the solution η_R of (3.42) but we know that the function

$$\overline{\eta}_R(r) = \begin{cases} 0 & r \in [0, R - R_0], \\ \eta_0(r - (R - R_0)) & r \in [R - R_0, R] \end{cases} \tag{3.44}$$

is a supersolution. The next lemma gives the proof of this fact.

Lemma 3.3.1. *The function $\overline{\eta}_R$, for $R > R_0$, defined by* (3.44) *is a supersolution of* (3.42).

Proof. For $r \in (0, R - R_0)$, $-L_\nu \overline{\eta}_R + \overline{\eta}_R^q = 0$. For $r \in (R - R_0, R)$, calling $s = r - (R - R_0)$,

$$\begin{aligned} -L_\nu \overline{\eta}_R + \overline{\eta}_R^q &= -\nu \eta_0''(s) - \frac{B_A}{s + R - R_0} \eta_0'(s) + \eta_0^q(s) \\ &\geq -\nu \eta_0''(s) - \frac{B_A}{s} \eta_0'(s) + \eta_0^q(s) \\ &= -L_\nu \eta_0 + \eta_0^q = 0, \end{aligned}$$

where the inequality is due to the fact that $\eta_0' \geq 0$ and $R - R_0 \geq 0$. This inequality combined with the values of $\overline{\eta}_R$ at the boundary, i.e., $\overline{\eta}_R(0) = 0$ and $\overline{\eta}_R(R) = 1$ makes of $\overline{\eta}_R$ a spersolution for problem (3.42). □

The following lemma is the conclusion of this line of reasoning.

Lemma 3.3.2. *Suppose that u_R is solution of* (3.37)*. Then the function $y_R(x) = \overline{\eta}_R(|x - x_0|)$ satisfies $y_R \geq u_R$ in $\overline{B}_R(x_0)$.*

Proof. The proof uses the comparison principle. Just notice, recalling (3.40) and Lemma 3.3.1, that

$$-L y_R + y_R^q \geq -L_\nu \overline{\eta}_R + \overline{\eta}_R^q \geq 0 = -L u_R + u_R^q.$$

Checking the boundary conditions we obtain the statement. □

Using the results on local supersolution just given, we can find a function $\rho : (R_0, \infty) \to \mathbb{R}^+$ such that if $R > R_0$ and if η_R is the solution of (3.42) in $(0, R)$ then

$$\eta_R = 0 \text{ in } [0, \rho(R)] \quad \text{and} \quad \eta_R > 0 \text{ in } (\rho(R), R].$$

We define the function

$$d(R) = R - \rho(R), \quad R > R_0, \tag{3.45}$$

and the following properties hold:

Lemma 3.3.3. *It holds*

$$(i)\ R_0 < R_1 < R_2 \Rightarrow d(R_1) \geq d(R_2);$$
$$(ii)\ \lim_{R\to\infty} d(R) = c_{\nu q}.$$

Proof. Point (i) is exactly the same as in [48]. The second statement follows the same line too but we sketch it to show the details. We already know that on $[0, \rho(R)]$ the solution is zero since $\overline{\eta}_R$ is zero. We focus on the problem

$$\begin{cases} -L_\nu \eta + \eta^q = 0 \quad r \in (\rho(R), R), \\ \eta(\rho(R)) = 0, \quad \eta(R) = 1. \end{cases}$$

We set $w(r) = \eta(R - r)$ and we transform the problem into

$$\begin{cases} -\nu w'' + \dfrac{B_s}{R - r} w' + w^q = 0, \\ w \geq 0, \\ w(0) = 1, \quad w(d(R_0)) = 0. \end{cases} \tag{3.46}$$

We note that the second extreme of the domain in (3.46) should be $d(R) = R - \rho(R)$, which, according to point (i), is smaller than $d(R_0)$. This does not affect the result since $w = 0$ in $(d(R), d(R_0))$. We multiply (3.46) by w' and integrate over $(r, d(R_0))$

$$\begin{aligned} 0 &= -\frac{\nu}{2} \int_r^{d(R_0)} \frac{d}{ds}(w')^2 \, \mathrm{d}s + \int_r^{d(R_0)} \frac{B_s}{R - r}(w')^2 \, \mathrm{d}s + \frac{1}{q+1} \int_r^{d(R_0)} \frac{d}{ds} w^{q+1} \, \mathrm{d}s \\ &\geq \frac{\nu}{2}(w')^2(r) - \frac{1}{q+1} w^{q+1}(r). \end{aligned}$$

The inequality in the second line is due to the non negativity of the second term of the right-hand side in the first line. Since $0 \leq w \leq 1$ we conclude that

$$|w'(r)| \leq \left(\frac{2}{\nu(q+1)} \right)^{1/2}, \quad 0 \leq r \leq d(R_0).$$

We have found that the family of solutions $\{w(r,R) : R > R_0\}$ is equicontinuous in $[0, d(R_0)]$ and from compactness argument we can extract a subsequence $\{w(\cdot, R_n)\}$ with $R_n \to \infty$ and a function $\bar{w} \in C([0,\infty))$ such that $w(\cdot, R_n) \to \bar{w}$ uniformly on compact sets. Actually what really matters is the convergence on $[0, d(R_0)]$ as $w(r,R) = 0$ for $r > d(R_0)$ and $R > R_0$. In the limit, the equation for $\bar{w}$ becomes

$$\begin{cases} -\nu w'' + w^q = 0, \\ w \geq 0, \\ w(0) = 1, \quad w(\infty) = 0. \end{cases} \tag{3.47}$$

whose unique solution is

$$z(t) = \left[\left(1 - \frac{t}{c_{\nu q}}\right)_+\right]^{2/(1-q)}, \tag{3.48}$$

with

$$c_{\nu q} = \frac{(2\nu(1+q))^{1/2}}{1-q}. \tag{3.49}$$

The convergences of $w(\cdot, R)$ to the solution of (3.47) implies that $d(R) \to c_{\nu q}$ as $R \to \infty$. □

Lemma 3.3.4. *Let u_R be the solution of* (3.37). *Then $u_R(x) \leq y_R(x) = \eta_R(|x - x_0|)$.*

Remark 3.3.5. If in (3.37) we set the value at the boundary to be ε instead of one, all results just shown change just in the value of the constants. To be more specific, the constant C_0 appearing in (3.43) remains the same while R_0 should be changed in $R_\varepsilon = \varepsilon^{(1-q)/2} C_0$ and $c_{\nu q}$ in $c_{\nu q \varepsilon} = \varepsilon^{(1-q)/2} c_{\nu q}$.

We remind that the Heaviside function H_v is given by

$$H_v(x_1) = \begin{cases} 0 & x_1 \in (-\infty, 0), \\ 1 & x_1 \in (0, \infty). \end{cases} \tag{3.50}$$

Lemma 3.3.6. *Let u be a solution of* (3.7) *in the setting of H_{hp} with h the Heaviside function. Then*

$$\mathcal{S}(u) \subset \{(x,y) \in \Omega : x \geq 0, y < c_{\nu q}\} \cup \{(x,y) \in \Omega : x \leq 0, r < c_{\nu q}\}.$$

Proof. The proof is the same as the one proposed in [48] and uses the technique of local super-solution. We start by giving a bound in the x_2 direction. For $R > R_0$ we consider the function y_R as in Lemma 3.3.4 and we set

$$\bar{u}(x_1, x_2; \xi) = y_R\,(x_1 - \xi, x_2 - R)$$

defined in $B_R(\xi, R)$. Since $\bar{u} = 1$ in $\partial B_R(\xi, R)$, by the comparison principle we obtain that $\bar{u} \geq u$ in $B_R(\xi, R)$. As we have chosen $\xi \in \mathbb{R}$ arbitrarily we deduce that

$$u(x_1, x_2) = 0 \quad \text{for all } x_1 \in \mathbb{R},\ d(R) \leq x_2 \leq 2R - d(R).$$

Letting $R \to \infty$ and thanks to Lemma 3.3.3 we have that

$$u(x_1, x_2) = 0 \quad \text{for all } x_1 \in \mathbb{R},\ x_2 \geq c_{\nu q}.$$

The boundedness in for $x_1 < 0$ works similarly. Again we set

$$\bar{u}(x_1, x_2; \theta) = y_R\,(x_1 - R\cos\theta, x_2 - R\sin\theta)\,, \quad \text{for } \pi/2 \leq \theta \leq \pi$$

this time defined on $\Sigma_\theta = B_R(R\cos\theta, R\sin\theta) \cap \Omega$. The boundary of Σ_θ consists of the part of $\partial B_R(R\cos\theta, R\sin\theta)$ which is in Ω and where $\bar{u} = 1$ and $u \leq 1$ and a part of $\partial\Omega$ with $x_1 < 0$ where $\bar{u} \geq 0$ and $u = 0$. Once again, because we can move $\theta \in [\pi/2, \pi]$, by the comparison principle we obtain that

$$u(x_1, x_2) = 0 \quad \text{in } \{(x_1, x_2) : x_1 \leq 0,\ d(R) < r < 2R - d(R)\},$$

with $r = (x_1^2 + x_2^2)^{1/2}$. Letting $R \to \infty$

$$u(x_1, x_2) = 0 \quad \text{in } \{(x_1, x_2) : x_1 \leq 0,\ r \geq c_{\nu q})\}.$$

□

Remark 3.3.7. As in Remark 3.3.5, if in Lemma 3.3.6 we substitute H_v with εH_v, the result is the same but with $c_{\nu q}$ replaced by $c_{\nu q \varepsilon}$.

Proof of Proposition 3.0.8. We start by reminding that Ω_n is any sufficiently smooth regularization of the rectangular

$$R_n = \{(x_1, x_2) : |x_1| < n, 0 < x_2 < \bar{x}\}.$$

We set $\bar{x} > c_{\nu q}$ where $c_{\nu q}$ is given by (3.49). We call u_n the unique solution of (3.15), which we know exists due to Proposition 3.0.6. From Lemma 3.3.6, $u_n = 0$ for $x_1 \leq -c_{\nu q}$, so we extend them to Ω_∞ by setting $u_n(x_1, x_2) = 0$ for $x_1 \leq n$, $0 \leq x_2 \leq \bar{x}$ and $u_n(x_1, x_2) = z(x_2)$ for $x_1 \geq n$, $0 \leq x_2 \leq \bar{x}$. Applying the comparison principle we see that

$$u_n \leq z \quad \text{and} \quad 0 \leq u_{n+1} \leq u_n \quad \text{for all } n \in \mathbb{N}.$$

Hence we can define punctually

$$u(x_1, x_2) = \lim_{n\to\infty} u_n(x_1, x_2). \tag{3.51}$$

To show that u is a local weak solution, we select a general test function whose support will be contained in $\bar{\Omega}_n$ for all n sufficiently big (since all functions are zero for $x_2 > c_{\nu q}$ this is not a loss of generality). Passing to the limit in (3.11) we obtain that u satisfies (3.11) for all $\varphi \in C_0^2(\bar{\Omega})$.

Uniqueness follows from the uniqueness of the limit in (3.51) and the comparison principle from the same property of the truncated solutions. □

3.3.2 Proof of Theorem 3.0.10

The proof of Theorem 3.0.10 start by showing (ii) for $h = \varepsilon H_v$. We will use this result later to prove (ii) for a general boundary data inside the assumption of (ii). In a second moment we will show (i) and give a numerical representation of the behaviour of $\overline{C}(q)$ and $\underline{C}(q)$ when $\varepsilon = 1$.

3.3.3 Heaviside function

Problem (3.7) under assumption H_{hp} and with $h = \varepsilon H_v$ is

$$\begin{cases} -Lu + u^q = 0 & \text{in } \Omega, \\ u(x_1, 0) = \varepsilon H_v(x_1) & x_1 \in \mathbb{R}. \end{cases} \tag{3.52}$$

We remind that the symmetric matrix A

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

satisfies

$$\mu|\xi|^2 \le \xi^T A\xi \le \nu|\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^2, \tag{3.53}$$

for some $\mu, \nu > 0$.

Remark 3.3.8. The existence of $\mu > 0$ such that condition (3.53) holds is equivalent to the fact that

$$a_{12}^2 < a_{22}a_{11}. \tag{3.54}$$

Indeed, if we set $\xi = (\xi_1 \xi_2)$, we have

$$\begin{aligned}\xi^T A\xi &= a_{11}\xi_1^2 + 2a_{12}\xi_1\xi_2 + a_{22}\xi_2^2 \\ &= \mu(\xi_1^2 + \xi_2^2) + \left((a_{11}-\mu)\xi_1^2 + 2a_{12}\xi_1\xi_2 + (a_{22}-\mu)\xi_2^2\right).\end{aligned}$$

In order for the second term to be greater or equal to zero for all $\xi_1, \xi_2 \in \mathbb{R}$, it must hold

$$a_{12}^2 < (a_{11}-\mu)(a_{22}-\mu) < a_{11}a_{22}.$$

This fact is used later on to obtain a subsolution.

In order to study the positivity set for the solution of (3.52) we look for a proper subsolution. What we are really interested in is the behavior of the solution in a neighborhood of the origin, i.e. the point $(0,0)$. In fact, we will show that, although the boundary datum εH_v is zero for $x_1 < 0$, the solution is positive for $x_1 > -\zeta_{\mu\varepsilon}$ and x_2 sufficiently small, for some $\zeta_{\mu\varepsilon} > 0$.

The procedure to obtain a proper lower bound $\underline{u}$ for u is the same as the one in [48]. We look for a $\underline{u}$ solution of

$$\begin{cases} -L\underline{u} = -\varepsilon^q & \text{in } \mathbb{R}\times(0, c_{\nu q}), \\ \underline{u}(x_1,0) = \varepsilon H(x_1) & x_1 \in \mathbb{R}, \\ \underline{u}(x_1, c_{\nu q\varepsilon}) \le 0 & x_1 \in \mathbb{R}. \end{cases} \tag{3.55}$$

If such $\underline{u}$ exists, and remembering the bounds $0 \le u \le \varepsilon$ where u is solution of (3.52), we get

$$-L(\underline{u} - u) = -\varepsilon^q + u^q \le 0.$$

This property, by the comparison principle, assures that $u \ge \underline{u}$.

To find out an explicit formula for $\underline{u}$, we split $\underline{u} = u_1 + u_2$ with

$$\begin{cases} -Lu_1 = -\varepsilon^q & x_1 \in \mathbb{R}, 0 < x_2 < c_{\nu q\varepsilon}, \\ u_1(x_1,0) = 0, \quad u_1(x_1, c_{\nu q\varepsilon}) = -\varepsilon & x_1 \in \mathbb{R}, \end{cases} \tag{3.56}$$

and

$$\begin{cases} -Lu_2 = 0 & x_1 \in \mathbb{R}, 0 < x_2 < c_{\nu q\varepsilon}, \\ u_2(x_1,0) = \varepsilon H(x_1), \quad u_2(x_1, c_{\nu q\varepsilon}) \le \varepsilon & x_1 \in \mathbb{R}. \end{cases} \tag{3.57}$$

We try, as in [48], to find a u_2 which depends only on the ratio $m = x_1/x_2$. For such a function f we have

$$\sum_{i,j=1}^{2} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} f(x_1/x_2) = \frac{1}{x_2^2} \left((a_{22}m^2 - 2a_{12}m + a_{11})f'' + 2(a_{22}m - a_{12})f' \right). \tag{3.58}$$

We look for solution of the differential equation

$$(a_{22}m^2 - 2a_{12}m + a_{11})f'' + 2(a_{22}m - a_{12})f' = 0 \tag{3.59}$$

we see that it is equivalent to $gf'' + g'f' = (gf')' = 0$ where $g(m) = \bar{a}_{22}m^2 - 2\bar{a}_{12}m + \bar{a}_{11}$. We deduce that f'^{-1} where C is a constant. Hence f is the indefinite integral of it,

$$f = C \int \frac{1}{a_{22}m^2 - 2a_{12}m + a_{11}} \tag{3.60}$$

From (3.54) we know that $a_{12}^2 < a_{11}a_{22}$, which implies that

$$f(m) = \bar{C} + C \frac{1}{(a_{11}a_{12} - a_{12}^2)^{1/2}} \arctan \left(\frac{a_{22}m - a_{12}}{a_{11}a_{12} - a_{12}^2} \right).$$

We set

$$u_2(x_1, x_2) = f_\varepsilon(x_1/x_2),$$

where f_ε is f with the constant $\bar{C} = \bar{C}_\varepsilon$ and $C = C_\varepsilon$ chosen for the specific problem. Let us check the boundary conditions: we fix $x_1 \neq 0$ and send $x_2 \to 0$. If $x_1 < 0$, then $x_1/x_2 \to -\infty$ and $f_\varepsilon(x_1/x_2) \to C_1 - C\frac{\pi}{2}(a_{11}a_{12} - a_{12}^2)^{-1/2}$. Setting $C = C_\varepsilon = \varepsilon\pi^{-1}(a_{11}a_{12} - a_{12}^2)^{1/2}$ and $\bar{C} = \bar{C}_\varepsilon = \varepsilon/2$ we have that $f_\varepsilon(x_1/x_2) \to 0$. For $x_1 > 0$, we see that $f_\varepsilon(x_1/x_2) \to \varepsilon$. Since, with these values for the constants, $0 \leq f_\varepsilon \leq \varepsilon$, also the other boundary condition is satisfied.

For u_1 we choose

$$u_1(x_1, x_2) = -\left(\frac{\varepsilon}{c_{\nu q \varepsilon}} + \frac{\varepsilon^q c_{\nu q \varepsilon}}{2} \right) x_2 + \frac{\varepsilon^q}{2} x_2^2. \tag{3.61}$$

It is immediate to verify that u_1 given by (3.61) is solution of (3.56).

Now we check that $\underline{u}$ is display the property of positivity in a neighbourhood of the origin. We know that for $x_2 = 0$ and $x_1 < 0$, $\underline{u}$ is zero. We want to understand if it is positive for some $x_2 > 0$. We compute

$$\frac{\partial \underline{u}}{\partial x_2} = -\left(\frac{\varepsilon}{c_{\nu q\varepsilon}} + \frac{c_{\nu q\varepsilon}}{2}\right) + x_2 - \frac{C_\varepsilon x_1}{a_{22}x_1^2 - a_{12}x_1x_2 + a_{11}x_2^2}, \tag{3.62}$$

and evaluate it in $x_2 = 0$

$$\frac{\partial \underline{u}}{\partial x_2}(x_1, 0) = -\left(\frac{\varepsilon}{c_{\nu q\varepsilon}} + \frac{c_{\nu q\varepsilon}}{2}\right) - \frac{C_\varepsilon}{a_{22}x_1}. \tag{3.63}$$

One can see that it is strictly positive for $x_1 < 0$ sufficiently close to zero, precisely in a neighbourhood of $(-\zeta_{\nu\varepsilon}, 0) \times \{0\}$, where

$$\zeta_{\nu\varepsilon} = \frac{2C_\varepsilon c_{\nu q\varepsilon}}{a_{22}(2\varepsilon + c_{\nu q\varepsilon}^2)}$$

3.3.4 General data

In this part we show that the behaviour displayed by the solution of (3.52) can be found also in solution of (3.1) where h is continuous, depending on the decay rate of h near zero. If h is sufficiently big when $x_1 \sim 0$, then the free boundary is not connected with $\mathcal{S}(h)$.

The function $h_\varepsilon \in C(\mathbb{R})$ has the form

$$h_\varepsilon(x_1) = \begin{cases} 0 & x_1 \leq 0, \\ Cx^{\frac{2}{1-q}} & 0 < x_1 \leq x_\varepsilon, \\ 1 & x > x_\varepsilon, \end{cases} \tag{3.64}$$

This boundary data differ from H_v just in a right neighbourhood of the origin. Indeed for $x_1 \leq 0$ and for $x_1 \geq x_\varepsilon$ they coincide. Hence the only relevant discrepancy in the behaviour of the solution may be registered close to the origin.

Remark 3.3.9. We use the family of functions h_ε as representative of the whole family of boundary data described in hypothesis H_{hp} since what really matters is the behaviour of the data in a right neighbourhood of zero.

proof of Theorem 3.0.10. For the proof of (i) it is enough to consider the special case of $h_\varepsilon(x_1) = \overline{C}x_1^{\frac{2}{1-q}}$ for a.e. $x_1 \in (0, \overline{x}_{1,\varepsilon})$ and $h_\varepsilon(x_1) = \overline{\varepsilon}$ for a.e. $x_1 \in (\overline{x}_{1,\varepsilon}, +\infty)$. Indeed, if $h \geq h_\varepsilon$ then we know that the correspondent limit very weak solutions u and u_ε satisfy that $u \geq u_\varepsilon \geq 0$. So, if the *expansion on the boundary of the support property* holds for u_ε with more reason it also holds for u.

Consider problem the solution $\underline{u}_\varepsilon$ of (3.52) with $\varepsilon = 1$ and boundary data given by $H(x_1 - x_\varepsilon)$. It is immediate to check that $H(x_1 - x_\varepsilon) \le h_\varepsilon(x_1)$ for all $x_1 \in \mathbb{R}$. This implies that $\underline{u}_\varepsilon \le u_\varepsilon$ where u_ε is the solution of (3.52) with boundary data h_ε. So if

$$x_\varepsilon = \left(\frac{1}{C}\right)^{\frac{1-q}{2}}$$

is such that $x_\varepsilon < \zeta_\nu$ then there exists V, neighbourhood of $(0,0)$ in $\mathbb{R}^2$, such that $\underline{u}_\varepsilon > 0$ in $V \cap \{x_1 < 0, x_2 > 0\}$. But this is true whenever

$$C > \left(\frac{1}{\zeta_\nu}\right)^{\frac{2}{1-q}}.$$

It is enough to set $\overline{C} = \zeta_\nu^{-2/(1-q)}$ and $\overline{x}_{1,\varepsilon} = (1/\overline{C})^{(1-q)/2}$. The statement follows since $\underline{u}_\varepsilon \le u_\varepsilon$.

One can also try to repeat the same proof with $\varepsilon H(x_1 - x_\varepsilon)$ and compare the results to find the best lower bound for which the *expansion on the boundary of the support property* holds.

For the proof of (*ii*) it is enough to consider the special case of $h_\varepsilon(x_1) = \underline{C} x_1^{\frac{2}{1-q}}$ for a.e. $x_1 \in (0, \underline{x}_{1,\varepsilon})$ and $h_\varepsilon(x_1) = \underline{\varepsilon}$ for a.e. $x_1 \in (\underline{x}_{1,\varepsilon}, +\infty)$. Indeed, if $h \le h_\varepsilon$ then we know that the correspondent limit very weak solutions u and u_ε satisfy that $0 \le u \le u_\varepsilon$. So, if the *nondiffusion of the boundary of the support property* holds for u_ε then it also holds for u.

Consider the function $\overline{u} = w(x_1 + vx_2)$ where

$$w(s) = C s_+^{\frac{2}{1-q}},$$

with $C > 0$. We compute

$$\begin{aligned} -L\overline{u} + \overline{u}^q &= -(a_{11} + 2a_{12}v + a_{22}v^2)w'' + w^q \\ &= x_+^{2/(1-q)} \left(-(a_{11} + 2a_{12}v + a_{22}v^2) \frac{2(1+q)}{(1-q)^2} C + C^q \right). \end{aligned} \tag{3.65}$$

If take

$$C \le \frac{(1-q)^2}{2(1+q)(a_{11} + 2a_{12}v + a_{22}v^2)},$$

we have that $-L\overline{u} + \overline{u}^q \ge 0$, hence $\overline{u}$ is a supersolution. We notice that

$$\overline{u}(\theta, r) = 0 \quad \text{for } \theta \in \Theta = (-\pi/2, \arctan(-v)),$$

where $\theta = \arctan(x_1/x_2)$. Again, since $u \le \overline{u}$ we obtain that u is zero in the sector Θ. □

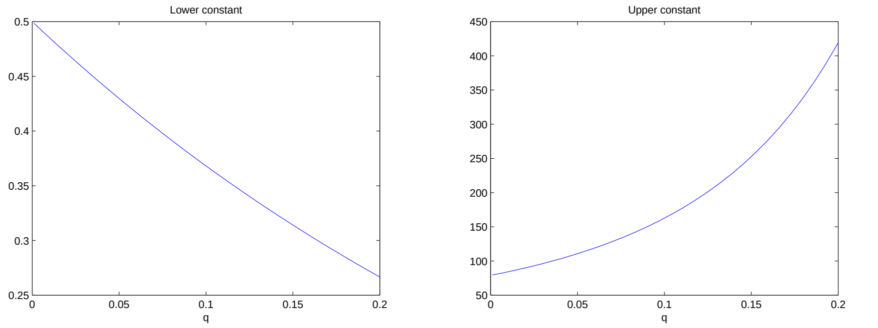

Figure 3.5: Graphs of $\underline{C}(q)$ and $\overline{C}(q)$

Of course for consistency we expect $\underline{C} \leq \overline{C}$. But can we affirm that equality holds? The answer in general is no. Let us consider for example the case of $L = \Delta$, which means $A = Id$. We have

$$\overline{C} = \left(\frac{\pi(q^2 - q + 2)}{(1-q)\sqrt{2(q+1)}} \right)^{\frac{2}{1-q}} \quad \text{and} \quad \underline{C} = \frac{(1-q)^2}{2(1+q)}.$$

We see from the graphs in Figure 3.5 that $\underline{C} < \overline{C}$ for $q \in (0, 0.2)$, and the difference is quite big. For $q \in (0.2, 1)$ the difference becomes even bigger as $\underline{C}$ is decreasing while $\overline{C}$ is increasing.

The question of what happen when $\underline{C} < C < \overline{C}$ is still open and a different approach or a finer analysis is needed.

Proof of Corollary 3.0.11. The additional properties are a by product of the proof of Theorem 3.0.10. Indeed, for (1), since the supersolution w already satisfies this property, so does the solution because it is non negative. Point (2) is true because we have shown that $u \geq u_1 + u_2$ with u_1 and u_2 solutions of the problems (3.56) and (3.57) respectively and $\underline{u} = u_1 + u_2$ satisfies the required properties. □

3.3.5 Proof of Corollary 3.0.16

In the case of (i) we can use the same supersolution (which denote now by $\overline{u}(x)$) than for the stationary case. Since as our initial condition is $u_0 = 0$ then, applying the comparison result, we get that $0 \leq u(t, x) \leq \overline{u}(x)$ for any $t > 0$ and a.e. $(x_1, x_2) \in \mathbb{R} \times [0, \infty)$.
The proof of part $ii)$ comes from the fact that $u_1(x_1, x_2) + u_2(x_1, x_2)$ is a subsolution for the parabolic problem for $t \in (0, T]$.

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