Theoretical Derivation of $1/f$ Noise in Quantum Chaos

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It was recently conjectured that $1/f$ noise is a fundamental characteristic of spectral fluctuations in chaotic quantum systems. This conjecture is based on the power spectrum behavior of the excitation energy fluctuations, which is different for chaotic and integrable systems. Using random matrix theory, we derive theoretical expressions that explain without free parameters the universal behavior of the excitation energy fluctuations power spectrum. The theory gives excellent agreement with numerical calculations and reproduces to a good approximation the $1/f$ ($1/f^2$) power law characteristic of chaotic (integrable) systems. Moreover, the theoretical results are valid for semiclassical systems as well.

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Quantum chaos has not a precise definition as yet. It is often defined as the study of the quantum properties of systems with a chaotic classical analog. This point of view has made it possible to establish a clear relationship between the energy level fluctuation properties of a quantum system and the large time scale behavior of its classical analog [1]. The pioneering work of Berry and Tabor showed that the spectral fluctuations of a quantum system whose classical analog is fully integrable are well described by Poisson statistics [2]. On the other hand, Bohigas *et al.* conjectured that the fluctuation properties of quantum systems which in the classical limit are fully chaotic coincide with those of random matrix theory (RMT) [3]. Recently, a different approach to quantum chaos has been proposed [4]. Considering the sequence of energy levels as a discrete time series, it was conjectured that chaotic quantum systems are characterized by $1/f$ noise, whereas integrable ones exhibit $1/f^2$ noise. This conjecture is supported by numerical calculations which involve atomic nuclei and the paradigmatic classical random matrix ensembles (RME) [4]. Although $1/f$ noise is a ubiquitous feature of many complex systems, its origin is still an unsolved problem. However, the origin of the $1/f$ noise in the spectral fluctuations of chaotic quantum systems might be easier to understand because of the mathematical tractability of RMT.

In this Letter we present a theoretical derivation of $1/f$ and $1/f^2$ noises in chaotic and regular systems, respectively. We use RMT to derive these results, but they are also valid for semiclassical systems. We present the main steps of the derivation and compare the theoretical results with numerical calculations for RME, an atomic nucleus, and a quantum billiard, finding excellent agreement.

For any quantum system, the accumulated level density $N(E)$ can be separated into a smooth part $\overline{N}(E)$ and a fluctuating part $\tilde{N}(E)$. If we remove the main trend defined by the former, we can compare the statistical properties of different systems or different parts of the same spectrum. This can be done by means of a transformation, called unfolding, which consists in mapping the energy levels E_i onto new dimensionless levels $\epsilon_i = \overline{N}(E_i)$.

The analogy between the energy spectrum and a discrete time series is established in terms of the δ_q statistic, defined as

$$\delta_q = \sum_{i=1}^{q} (s_i - \langle s \rangle) = \epsilon_{q+1} - \epsilon_1 - q, \qquad (1)$$

where ϵ_i is the ith unfolded level and $s_i = \epsilon_{i+1} - \epsilon_i$. Note that δ_q represents the deviation of the excitation energy of the $(q+1)$th unfolded level from its mean value. Moreover, it is closely related to the level density fluctuations. Indeed, we can write

$$\delta_q = -\tilde{N}(E_{q+1}), \qquad (2)$$

if we appropriately shift the ground state energy; thus, it represents the accumulated level density fluctuations at $E = E_{q+1}$. Its power spectrum, defined as the square modulus of the Fourier transform, shows neat power laws $P_k^\delta \propto 1/k^\alpha$ both for fully chaotic and integrable systems, but level correlations change the exponent from $\alpha = 2$ for uncorrelated spectra to $\alpha = 1$ for chaotic quantum systems [4].

Notation.—We consider an interval of length $L \gg 1$ containing $N \simeq L$ unfolded energy levels. The fluctuating parts of the level and accumulated level densities are denoted $\tilde{\rho}(\epsilon)$ and $\tilde{n}(\epsilon)$, respectively. In addition to the δ_q statistic, we also introduce another discrete function $\tilde{n}_q = \tilde{n}(q)$, obtained by sampling the continuous function for integer values of the energy. The Fourier transforms and power spectra of these functions are defined in the usual way [5]. The differences between continuous and discrete Fourier transforms play a relevant role in the following. Our notation is summarized in Table I.

Spectral and ensemble averages will be denoted by $\langle \cdot \rangle$ or $\langle \cdot \rangle_\beta$ to distinguish ensemble averages performed in

TABLE I. Summary of the functions, Fourier transforms, and power spectra used in this Letter.

Domain	$\mathbb{R}$	$\mathbb{Z}$	
Function	$\tilde{n}(\epsilon)$	$\tilde{n}_q$	δ_q
Fourier transform	$\hat{n}(\tau)$	$\hat{n}_k$	$\hat{\delta}_k$
Power spectrum	$P^n(\tau)$	P^n_k	P^δ_k

different RME. Here, β stands for the repulsion parameter characterizing the ensemble. In this work we consider two RME: the Gaussian orthogonal ensemble (GOE) and the Gaussian unitary ensemble (GUE) [6]. We have $\beta = 1$ for GOE and $\beta = 2$ for GUE.

Spectral fluctuations.—The main object of this Letter is to obtain explicit expressions of the average value of P^δ_k for chaotic and integrable systems. Except for integrable systems, one of the main features of quantum spectra is that successive level spacings are not independent, but correlated quantities. This property makes it exceedingly difficult to work directly with the discrete δ_q sequence. To circumvent this difficulty we profit from the relationship (2). The statistical properties of $\tilde{n}(\epsilon)$ are usually measured in terms of the spectral form factor, defined as

$$K(\tau) = \left\langle \left| \int d\epsilon \tilde{\rho}(\epsilon) e^{-i2\pi\epsilon\tau} \right|^2 \right\rangle, \tag{3}$$

that is, as the power spectrum of the fluctuating part of the energy level density. Instead of $K(\tau)$, we can use the power spectrum $P^n(\tau)$ of $\tilde{n}(\epsilon)$ to analyze spectral fluctuations. Using the so-called differentiation theorem [5], it can be shown that for very large L values and $\tau, \tau' \neq 0$ both quantities are closely related by the equations [7]

$$\frac{\langle |\hat{n}(\tau)|^2 \rangle}{L} = \langle P^n(\tau) \rangle = \frac{K(\tau)}{4\pi^2\tau^2}, \tag{4a}$$

$$\frac{\langle \hat{n}^*(\tau)\hat{n}(\tau') \rangle}{L} = 0, \qquad \tau \neq \tau', \tag{4b}$$

valid for systems where $\langle \tilde{\rho}(\epsilon)\tilde{\rho}(\epsilon + \eta) \rangle$ goes to zero faster than $1/\eta$ as $\eta \to \infty$

The next step is to relate the spectral fluctuations of $\tilde{n}_q$ to those of the continuous function $\tilde{n}(\epsilon)$, as given by their power spectra. With the usual definitions for the continuous and discrete Fourier transforms we have [5]

$$\hat{n}_k = \frac{1}{\sqrt{N}} \sum_{q=-\infty}^{\infty} \hat{n}\left(\frac{k}{N} + q\right), \qquad k = 1, 2, \ldots, N-1, \tag{5}$$

and therefore

$$P^n_k = \sum_{p,q=-\infty}^{\infty} \frac{\hat{n}^*\left(\frac{k}{N} + p\right)\hat{n}\left(\frac{k}{N} + q\right)}{N}, \quad k = 1, 2, \ldots, N-1. \tag{6}$$

Considering separately the cases $p = q$ and $p \neq q$, using (4b) and the fact that $P^n(\tau)$ is symmetric in τ, one can easily deduce the average value of P^n_k for $N \gg 1$,

$$\langle P^n_k \rangle = \left\langle P^n\left(\frac{k}{N}\right)\right\rangle + \sum_{q=1}^{\infty} \left\langle P^n\left(\frac{k}{N} + q\right)\right\rangle + \sum_{q=1}^{\infty} \left\langle P^n\left(q - \frac{k}{N}\right)\right\rangle, \quad k = 1, 2, \ldots, N-1. \tag{7}$$

Using Eq. (4b) and taking into account that $K(\tau) \simeq 1$ for $\tau \geq 1$, which is exactly equal for systems with GUE and Poisson statistics, and a good approximation for systems with GOE-like spectral fluctuations, (7) simplifies to

$$\langle P^n_k \rangle = \frac{N^2}{4\pi^2} \left\{ \frac{K(\frac{k}{N}) - 1}{k^2} + \frac{K(1 - \frac{k}{N}) - 1}{(N-k)^2} \right\} + \frac{1}{4\sin^2(\frac{\pi k}{N})}, \quad k = 1, 2, \ldots, N-1. \tag{8}$$

Finally, in order to obtain $\langle P^\delta_k \rangle$ from Eq. (8) we use the relationship between the variances of δ_q and $\tilde{n}(\epsilon)$, which depends strongly on the system fluctuations. For fully chaotic systems ($\beta > 0$) we have $\langle \delta_q^2 \rangle - \langle \tilde{n}(q)^2 \rangle = -1/6$, $q > 0$, an expression which is essentially valid for the three classical RME and their interpolations [8]. On the other hand, for systems with Poisson statistics ($\beta = 0$), $\langle \delta_q^2 \rangle = \langle \tilde{n}(q)^2 \rangle$. From these simple expressions we obtain a new relationship between the covariance matrices of δ_q and $\tilde{n}(q)$,

$$\langle \delta_p \delta_q \rangle - \langle \tilde{n}(p)\tilde{n}(q) \rangle = \begin{cases} -\frac{1+\delta^c_{p,q}}{12}, & p, q > 0, \ \beta > 0, \\ 0, & \beta = 0, \end{cases} \tag{9}$$

where δ^c is the Kronecker delta. Inserting this equation into the definitions of $\langle P^\delta_k \rangle$ and $\langle P^n_k \rangle$, we get

$$\langle P^\delta_k \rangle = \begin{cases} \langle P^n_k \rangle - \frac{1}{12} & \text{for chaotic systems}, \\ \langle P^n_k \rangle & \text{for integrable systems}, \end{cases} \tag{10}$$

where $k = 1, 2, \ldots, N-1$ and $N \gg 1$.

Collecting (10) and (8) one obtains an analytical expression for $\langle P^\delta_k \rangle$ in terms of $K(k/N)$, valid for fully chaotic and integrable systems.

Quantum Chaos.—One of the most important features of fully chaotic systems is the universal behavior of their spectral fluctuations that are well described by RMT for large energy windows. Exact analytical expressions are known [6] for $K(\tau)$ in all RME,

$$K(\tau)_{\beta=2} = \begin{cases} \tau, & \tau \leq 1, \\ 1, & \tau \geq 1, \end{cases} \tag{11}$$

$$K(\tau)_{\beta=1} = \begin{cases} 2\tau - \tau\log(1 + 2\tau), & \tau \leq 1, \\ 2 - \tau\log(\frac{2\tau+1}{2\tau-1}), & \tau \geq 1. \end{cases} \tag{12}$$

Equation (10), together with (8) and (12) or (11), gives explicit expressions of $\langle P^\delta_k \rangle$ for GOE and GUE, which we do not show for the sake of space. For generic chaotic systems we can apply GOE or GUE results depending on their symmetries.

Integrable systems.—In the case of integrable systems we have $K(\tau) = 1$ [1], instead of (11) or (12). Using again (10) and (8), we see that $\langle P_k^\delta \rangle$ reduces to the last term of Eq. (8).

Semiclassical systems.—On the other hand, using periodic orbit theory and semiclassical mechanics, it is possible to calculate $K(\tau)$ for semiclassical systems. For integrable systems the semiclassical theory predicts $K(\tau) = 1$, while for chaotic systems it gives $K(\tau) = 2\tau/\beta$, which coincides with the short time behavior of Eqs. (11) and (12). These expressions are valid when $\tau_{\min} \ll \tau \ll \tau_H$, where $\tau_{\min}$ is the period of the shortest periodic orbit, and τ_H is the Heisenberg time of the system, related to the time a wave packet takes to explore the complete phase space of the system [9,10]. Therefore, the expressions derived above are generic results applicable to chaotic and integrable quantum systems for τ between these two values. Nevertheless, in semiclassical systems (like quantum billiards) there can be some deviations at the lowest frequencies due to the behavior of the shortest periodic orbits. Generally speaking, we expect smaller values $\langle P_k^\delta \rangle$ for $k \leq N\tau_{\min}$ than those of Eq. (10).

$1/f$ and $1/f^2$ noises.— A Taylor expansion of Eq. (8) shows that its first term becomes dominant when $k \ll N$ and $N \gg 1$, so we can write [7]

$$\langle P_k^\delta \rangle_\beta = \begin{cases} \frac{N}{2\beta\pi^2 k}, & \text{for chaotic systems.} \\ \frac{N^2}{4\pi^2 k^2}, & \text{for integrable systems.} \end{cases} \quad (13)$$

This expression shows that for small frequencies, the excitation energy fluctuations exhibit $1/f$ noise in chaotic systems and $1/f^2$ noise in integrable systems. As we see below, these power laws are also approximately valid through almost the whole frequency domain, due to partial cancellation of higher order terms. Only near $k = N/2$, the so-called Nyquist frequency [5], the effect of these terms, becomes appreciable.

To test all these theoretical expressions we have compared their predictions to numerical results obtained for different ensembles and systems. Figure 1 displays the theoretical values of $\langle P_k^\delta \rangle$ for GUE and integrable systems, as given by the appropriate previous equations, together with the numerical average values for 500 GUE matrices and 500 Poisson level sequences. In order to enlarge the high frequency region, where the numerical results show small deviations from the $1/f^\alpha$ power law behavior, an upper right panel is added to the figure. The theoretical curve describes perfectly the power laws, characteristic of small and intermediate frequencies, as well as the deviations observed at the highest frequencies (note that there are no free parameters in the analytical result).

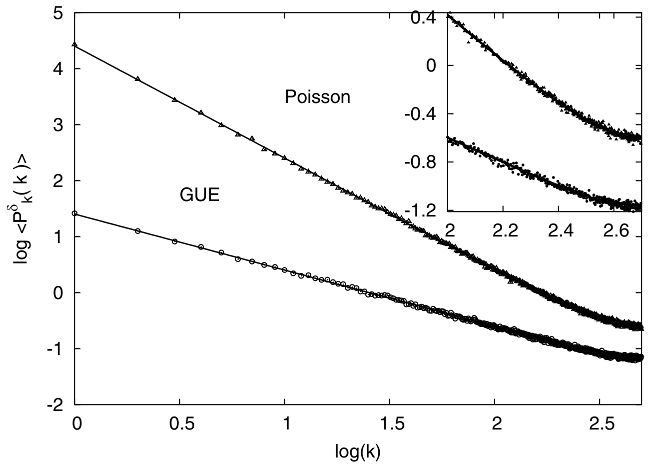

FIG. 1. Theoretical power spectrum of the δ_q function for GUE and integrable systems (solid lines), compared to numerical averages calculated using 500 GUE matrices of dimension $N = 1000$ (circles) and 500 Poisson level sequences of length $N = 1000$ (triangles).

We have also compared our predictions to the power spectra of δ_q for two physical systems: an atomic nucleus (chaotic) and a rectangular billiard (integrable).

In the first case, we have performed a shell-model calculation for ^{34}Na using an adequate realistic interaction and the shell-model code NATHAN (see [11] and references therein). The Hamiltonian matrices for different angular momenta, parity, and isospin were fully diagonalized. Then, 25 sets of 256 consecutive high energy levels of the same quantum numbers $J^\pi T$ were selected, and the average power spectrum of the δ_q function was calculated numerically. Figure 2 shows the result of this calculation together with the theoretical values of Eq. (10). An excellent agreement between the

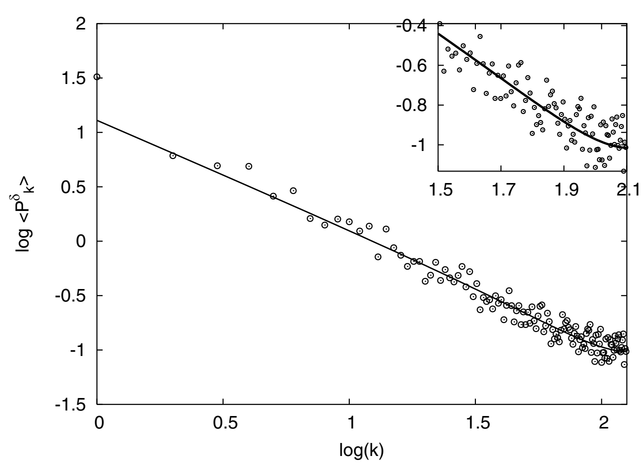

FIG. 2. Numerical average power spectrum of the δ_q function for ^{34}Na, calculated using 25 sets of 256 consecutive levels from the high level density region, compared to the parameter free theoretical values (solid line) for GOE.

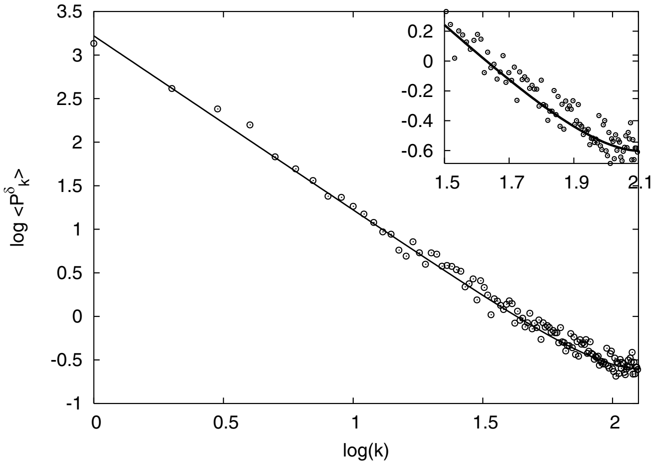

FIG. 3. Numerical average power spectrum of δ_q for a rectangular billiard, calculated using 25 sets of 256 consecutive levels, compared to the parameter free theoretical values (solid line) for integrable systems.

theoretical and numerical results is obtained through the whole frequency interval.

As an example of an integrable system we have chosen a rectangular billiard with sides of length $a = \sqrt{\lambda}$ and $b = 1/\sqrt{\lambda}$, with $\lambda = (\sqrt{5} + 1)/2$; this geometry gives rise to an irrational ratio $a/b = \lambda$, and thus there are no degeneracies in the spectrum. We have calculated the spectrum and selected 25 sets of 256 consecutive very high energy levels in order to avoid, as far as possible, the influence of short periodic orbits. Figure 3 shows the results for the average power spectrum of δ_q and our theoretical values. As in the previous cases, we can see that the agreement between the theoretical and numerical results is very good in the whole frequency domain.

In summary, we have derived random matrix theoretical expressions for the power spectrum of the δ_q function both for regular and chaotic quantum systems. These expressions are universal and do not contain any free parameter. We have compared our theoretical predictions with numerical results for RME, a rectangular billiard, and an atomic nucleus, obtaining excellent agreement for all these systems. The theory reproduces the power laws of type $1/f$ for chaotic systems, and $1/f^2$ for regular ones, observed in the power spectrum of the excitation energy fluctuations up to frequencies very close to the Nyquist limit. Although these results are derived in RMT, they are also valid for semiclassical systems except in the low frequency region.

The power spectrum P_k^δ of δ_q gives direct information on the spectral form factor $K(\tau)$, with the advantage that it can be applied to discrete level spectra. Perhaps the most important feature of P_k^δ is that it provides an intrinsic characterization of quantum chaos, without any reference to RMT, and all chaotic quantum systems exhibit $1/f$ noise, regardless of the system symmetries.

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