Stability of the minimum solitary wave of a nonlinear spinorial model

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It is shown that the minimum solitary wave of a nonlinear spinorial model is unstable under charge-preserving deformations via a radiative decay. We analyze the cause of this decay.

In a recent work[1] the stability under dilations of the spinorial waves of a nonlinear Dirac equation in 1+3 dimensions was analyzed in the framework of the Shatah-Strauss formalism.[2] The field equation of the analyzed model is, in dimensionless units,

$$i\gamma^\mu \partial_\mu \psi - \psi + (\bar{\psi}\psi)\psi = 0 \tag{1}$$

with

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix},$$

where σ^k are the Pauli matrices. The solitary waves of Eq. (1) are

$$\psi_s(\mathbf{r},t) = \begin{pmatrix} G(r)\begin{pmatrix}1\\0\end{pmatrix} \\ iF(r)\begin{pmatrix}\cos\theta \\ \sin\theta e^{i\phi}\end{pmatrix} \end{pmatrix} e^{-i\omega t}$$

with $F(r)$ and $G(r)$ satisfying

$$\begin{aligned} &dG/dr + (1+\omega)F - (G^2-F^2)F = 0 , \\ &dF/dr + 2F/r + (1-\omega)G - (G^2-F^2)G = 0 , \end{aligned} \tag{2}$$

where $0<\omega<1$. The conclusion of the above study is that the solitary waves of the model are dilation stable for $0<\omega<0.936$ and dilation unstable for $0.936<\omega<1$. This result agrees with previous numerical results[3] and corroborates that the energetic stability criterion is not necessarily correct for spinorial models as some authors claim.[4]

In the present work we study the stability under dilations of the solitary wave with $\omega=\omega_c=0.936$. The solitary wave with this frequency has the minimum energy and charge, and these values are, respectively (excepting a constant factor arising from the integration of angular variables), $E_s(\omega_c)=3.7561$ and $Q_s(\omega_c)=3.6598$. From a physical point of view, the analysis of this minimum solitary wave against perturbations is more interesting than for the other values of the frequency, since this localized wave has been proposed as a model of extended fermions.[5]

To study the charge-preserving dilations of the minimum solitary wave, a code was constructed in spherical geometry. It consists of two parts: the initial and the evolution modules. The former uses a sixth-order eight-stage Runge-Kutta method for calculating the solitary wave $\psi_s(\mathbf{r},0)$ with $\omega=\omega_c=0.936$. Varying the value $G(0)$ [$F(0)$ is always zero] we obtain the solution of Eq. (2). In order to save on computational cost, the search was terminated whenever we obtained a $G(r)$ and an $F(r)$ without nodes and with exponential decay in the interval $r\in(0,25]$. For $r\in(25,\infty)$ we put $G(r)=F(r)=0$. To check the accuracy of this numerical method, we calculate

$$\omega_1 = \int_0^{25} dr\, r\rho_Q(r) \left[2\int_0^{25} dr\, r^2 GF \right]^{-1}$$

and

$$\omega_2 = \int_0^{25} dr\, r^2[\rho_E(r) - (G^2-F^2)^2/2] \left[\int_0^{25} dr\, r^2 \rho_Q(r) \right]^{-1},$$

where $\rho_E(r)$ and $\rho_Q(r)$ are the energy and charge densities. If $G(r)$ and $F(r)$ are the solution of Eq. (2), then $\omega_1=\omega_2=\omega$. In our calculation we obtain $\omega_1=\omega_2=0.936 \pm 0.0002$. The output of the initial module is the charge-preserving deformation $\psi(\mathbf{r},0)=\alpha^{3/2}\psi_s(\alpha r,0)$ of the previous constructed solitary wave ψ_s, where α stands for a perturbative parameter. The evolution module starts from the output of the initial one, and it uses a second-order accurate implicit scheme for time stepping. Its exactness was checked against charge and energy conservation. The relative error of these quantities was less than 0.003%.

The code result implies that the condition $\delta^2E>0$, with the fixed charge as a constraint, is relevant for stability of the minimum solitary wave of Eq. (1). However, the reason has nothing to do with the energetic stability criterion. When this wave is subject to the above charge-preserving deformation, the initial state has less energy than the minimum solitary wave and though a negative energy density is emitted (Fig. 1), the top of the lump, after a time $\sim$80, goes to zero with increasing time (Fig. 2).

The cause of this decay is that, the charge density being positive definite, no other state is available after the negative-energy radiation is produced because the solitary wave with $\omega=\omega_c$ also minimizes the charge. Since the origin of the spinorial decay of the minimum solitary wave is the positive-definite character of the charge density, it seems reasonable to conjecture that the coupling of the spinor field with its charge-conjugate one would prevent

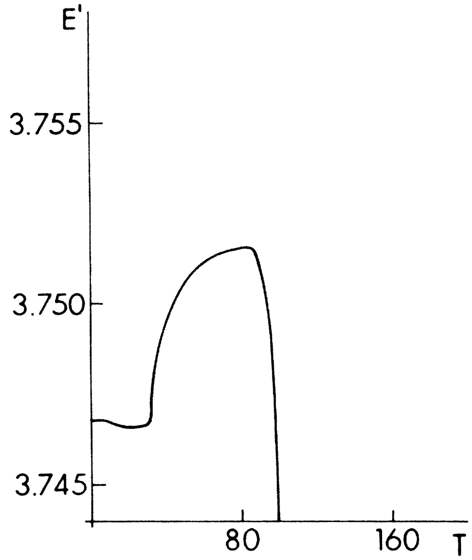

FIG. 1. Energy of the central zone ($0 \leq r \leq 25$ in dimensionless units) for a perturbed minimum soliton. The initial conditions are $\psi(\mathbf{r},0) = \alpha^{3/2}\psi_s(\alpha\mathbf{r},0)$ with $\alpha = 0.9$.

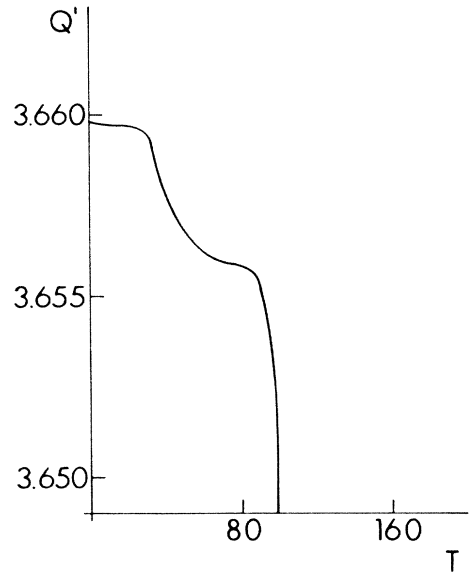

FIG. 2. Time evolution of the central charge for the initial conditions of Fig. 1.

the minimum solitary wave instability. It is very important to eliminate this decay because it has been recently shown[6] that this dissipative channel is switched on in the head-on collision of the minimum solitary waves. To finish, it is interesting to note two differences with the instability of some scalar solitary waves:[7] First, the spinorial instability is always radiative, that is, there is no singular mode decay; and second, the long time of the spinorial decay (2 for the scalar case versus 80 for the spinorial one).

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[1]W. Strauss and L. Vázquez, this issue, Phys. Rev. D **34**, 641 (1986).

[2]J. Shatah, Commun. Math. Phys. **91**, 313 (1983); Trans. Amer. Math. Soc. **290**, 701 (1985); J. Shatah and W. Strauss, Commun. Math. Phys. **100**, 173 (1985).

[3]A. Alvarez and M. Soler, Phys. Rev. Lett. **50**, 1230 (1983).

[4]I. L. Bogolubsky, Phys. Lett. **73A**, 87 (1979); J. Werle, Acta Phys. Pol. B **12**, 601 (1981); P. Mathieu and T. F. Morris, Phys. Lett. **126B**, 74 (1983); **155B**, 156 (1985).

[5]M. Soler, Phys. Rev. D **1**, 2766 (1970); see, for a review, A. F. Rañada, in *Quantum Theory, Groups, Fields and Particles*, edited by A. O. Barut (Mathematical Physics Studies 4) (Reidel, Dordrecht, 1983).

[6]A. Alvarez, Phys. Rev. D **31**, 2701 (1985).

[7]D. L. T. Anderson and G. H. Derrick, J. Math. Phys. **11**, 1336 (1970).