Energetic Stability Criterion for a Nonlinear Spinorial Model

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The time evolution of expanded and contracted solitary waves of the Dirac field with scalar self-interaction is exhibited. It is shown that the positivity of the second variation of the energy functional is not a necessary condition for the stability of these waves as has been recently suggested.

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In the last few years, interest in particlelike solutions of Poincaré-invariant nonlinear equations has increased considerably.[1] One natural condition to require is the dynamical stability of these solutions. From a physical point of view, their possible relation with elementary objects in particle physics suggests itself. However, the fact that the only well-known stable solutions correspond to boson models is perhaps the reason for the limited interest in these models within this physical context. In fact, no stable elementary bosons other than the photon exist in nature. Interest in stable spinor solutions would seem in this sense more justified.

Recently, several authors[2] have studied the stability of spinor solitary waves against charge-preserving perturbations and have concluded that these waves are unstable. These studies are based on the assumption that if the spinor solitary waves are not a minimum of the energy functional for a fixed value of the charge, then these waves are unstable. We do not know of any rigorous mathematical proof of this statement for the spinorial case.[3] The assumption arises from a naive generalization to field theory of a well-known theorem of classical mechanics. However, there is a fundamental difference from familar classical mechanics of discrete systems: The spinor Lagrangian is not quadratic in the velocities.[4]

In this Letter we shall show that computer experiments are in contradiction with the mentioned studies for a typical spinorial model in (1+1)- and (1+3)-dimensional cases. The evolution of a Gaussian distribution to solitary waves was shown to take place in the (1+3)-dimensional case by use of explicit Lax-Wendroff simulation methods,[5] which allowed, however, in practice very limited reliable evolution times. Implicit difference schemes not only confirm those results but make it possible to follow accurately for long times the consequences of charge-preserving perturbations.[6] The field equation of the model is

$$i\gamma^{\mu}\partial_{\mu}\psi - m\psi + 2\lambda(\bar{\psi}\psi)\psi = 0\,, \qquad (1)$$

where λ is a positive real arbitrary parameter. Hereafter, we work in dimensionless units or, equivalently, we take $m=1$ and $\lambda=\frac{1}{2}$. The calculation accuracy was monitored against the conservation of charge and energy. The relative error of these quantities was less than 0.003%, so that the total charge and energy was conserved up to four decimals.

(a) (1+1)-dimensional case.—Our notation will be $g^{\mu\mu}=(1,-1)$, $\gamma^0=\sigma_3$, and $\gamma^1=i\sigma_2$. This model can be considered the massive one-component Gross-Neveu model.[7] The solitary waves of Eq. (1) have the form

$$\psi_s(x,t) = \varphi_s(x,\Lambda)e^{-i\Lambda t} \equiv \begin{pmatrix} A(x) \\ iB(x) \end{pmatrix} e^{-i\Lambda t}\,,$$

where $A(x)$ and $B(x)$ are known analytic functions.[8] The interaction dynamics of these waves was analyzed by Alvarez and Carreras.[9]

Let us consider the following Λ'-parameter charge-preserving family:

$$\phi(x,\Lambda,\Lambda') = \varphi_s(x,\Lambda')[Q_s(\Lambda)/Q_s(\Lambda')]^{1/2}\,, \qquad (2)$$

where $Q_s(\Lambda)$ stands for the solitary-wave charge at frequency Λ. It is known[2] that, for $\Lambda < 2^{-1/2}$, the functional energy has a maximum at $\Lambda'=\Lambda$. In our numerical study no manifestation of instability has been found. As an example, we have followed the temporal evolution of the initial data (2) with $\Lambda=0.5$ and $|\Lambda'-\Lambda| \leq 1$; the final states are solitary waves. From Fig. 1, where we plot $Q' = \int_0^{25} dx\,\rho_Q(x,t)$ and $E' = \int_0^{25} dx\,\rho_E(x,t)$

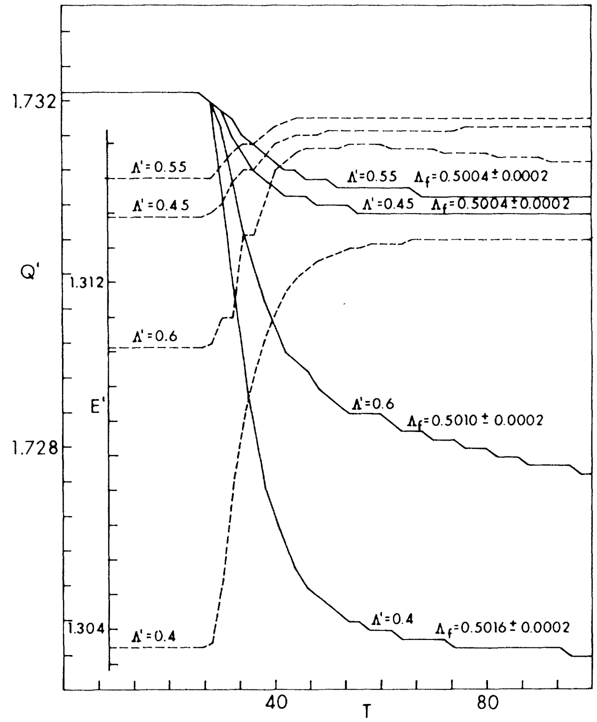

FIG. 1. Charge and energy of the central zone ($0 \leqslant x \leqslant 25$) for the state formed from the initial condition (2) with $\Lambda = 0.5$ and different values of Λ'. Likewise, the frequencies of the final solitary waves are shown.

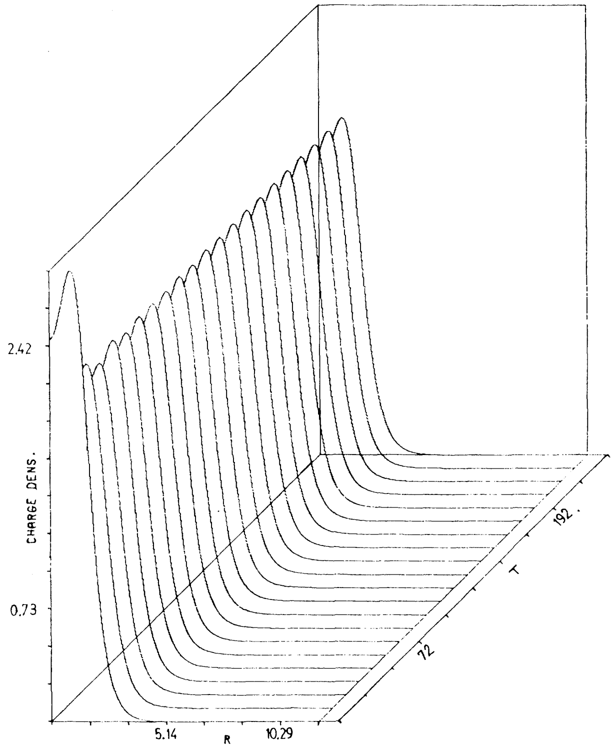

FIG. 2. Solitary-wave formation from a perturbated one with $\Lambda = 0.700$ and $\alpha = 1.1$.

(ρ_Q and ρ_E stand for the charge and energy densities), and the numerical fact $E(t=0)=E'(t=0)$ and $Q(t=0)=Q'(t=0)$, we deduce that the final solitary wave is reached via negative-energy-density emission to infinity.

(b) (1+3)-dimensional case.—We will adopt the Bjorken-Drell realization for the γ matrices and the metric tensor $g^{\mu\mu} = (1,-1,-1,-1)$. Solutions of Eq. (1) with minimum angular momentum can be found having the form

$$\psi_s(\vec{r},t) = \begin{pmatrix} G(r)\begin{pmatrix}1\\0\end{pmatrix} \\ iF(r)\begin{pmatrix}\cos\theta\\ \sin\theta\, e^{i\varphi}\end{pmatrix}\end{pmatrix} e^{-i\Lambda t}$$

with the radial functions F and G satisfying

$$\begin{aligned} &dG/dr + (1+\Lambda)F - (G^2 - F^2)F = 0\,, \\ &dF/dr + 2F/r + (1-\Lambda)G - (G^2-F^2)G = 0\,, \end{aligned} \qquad (3)$$

where $0 < \Lambda \leqslant 1$. Localized regular solutions of the system (3) with finite charge and energy have been found numerically by Finkelstein, Fronsdal, and Kaus.[10] We will analyze the dynamical stability of the nodeless solutions of Eqs. (3). There is a family of these solitary waves which depends continuously on the frequency Λ. The charge and energy spectra of this family present a minimum at $\Lambda = 0.936$.[11] García and Rañada[12] dealt with the static properties of the solitary waves of Eq. (1) coupled with other fields.

Let us consider the charge-preserving solitary-wave deformation $\psi(\vec{r},0) = \alpha^{3/2}\psi_s(\alpha\vec{r},0)$. It is known[2] that the functional energy has a maximum at $\alpha = 1$. However, as we will see below, this is not a condition for instability.

The computer results can be summarized in the following three assertions: (i) The solitary waves with $\Lambda < 0.936$ are stable against the above deformation; (ii) in this stability range, the perturbed solitary waves with less frequency tend to the final state faster than the ones with $\Lambda \leqslant 0.936$; (iii) the solitary waves with $\Lambda > 0.936$ are unstable against the mentioned deformations.

The radial charge density ρ_Q is plotted versus time and space coordinates (T an R in the figures) for some relevant cases. Define

$$\Lambda_1(t = 0.5\int_0^{25} dr\, r\rho_Q(r,t)\,[\,\int_0^{25} dr\, r^2 |g(r,t)f(r,t)|\,]^{-1}$$

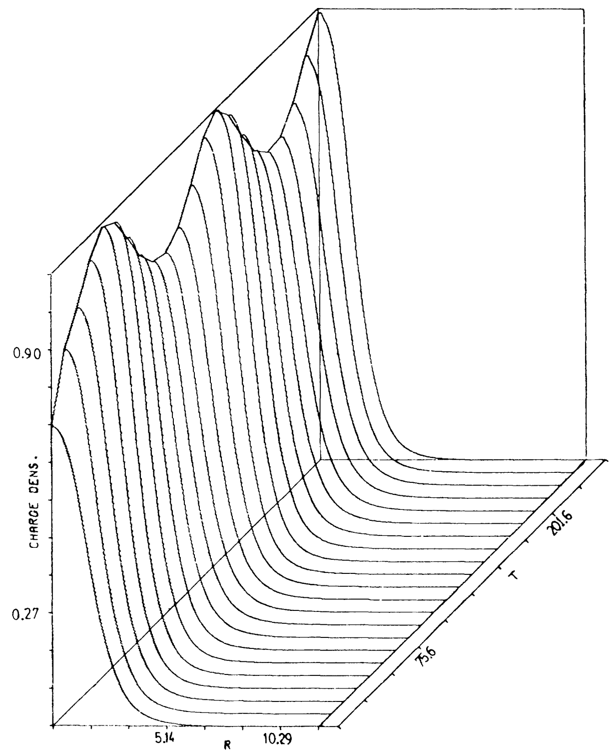

FIG. 3. The initial parameters are $\Lambda = 0.920$ and $\alpha = 0.9$. The final state tends slowly to a solitary wave.

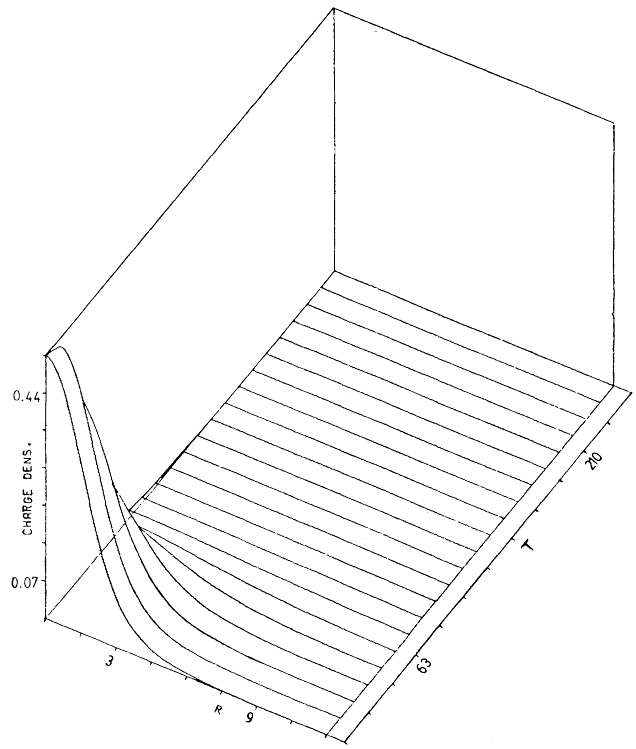

FIG. 4. Time evolution of the expanded ($\alpha = 0.9$) soliatry wave $\Lambda = 0.950$.

and

$$\Lambda_2(t) = \int_0^{25} dr\, r^2 \{\rho_E(r,t) - \tfrac{1}{2}[\,|g(r,t)|^2 - |f(r,t)|^2]^2\}[\int_0^{25} dr\, r^2 \rho_Q(r,t)]^{-1},$$

where ρ_E is the radial energy density. Here $g(r,t)$ and $f(r,t)$ are the time evolution of the initial functions $\alpha^{3/2}G(\alpha r) \equiv g(r,0)$ and $\alpha^{3/2}F(\alpha r) \equiv f(r,0)$. If $\psi(\vec{r},t)$ tends to a solitary wave, and both $\Lambda_{1,2}(t)$ tend to a constant Λ, this will be taken as the solitary wave frequency.

The final states of contracted ($\alpha = 1.1$) or expanded ($\alpha = 0.9$) solitary waves with $\Lambda = 0.700$ are solitary waves with frequencies 0.7015 ± 0.0005 and 0.7025 ± 0.0003, respectively. The former case is presented in Fig. 2. As for the (1+1)-dimensional case, the field can radiate away negative energy if necessary to adjust itself to the final state. An example of the evolution of an expanded ($\alpha = 0.9$) solitary wave with $\Lambda \lesssim 0.936$ is shown in Fig. 3. The initial data evolve slowly to a solitary wave with $\Lambda = 0.92 \pm 0.01$. A similar result is obtained for a contracted solitary wave.

The inevitable numerical errors at $t = 0$ are sufficient to demonstrate the instability of the solitary wave with $\Lambda = 0.950$. The final state oscillates around a solitary wave with $\Lambda = 0.92 \pm 0.02$; this state is reached via emission of radiation to infinity (0.05% of charge and energy was radiated at $t = 324$) and compression of the initial state. This last fact suggests, and the experiments confirm (Figs. 4 and 5), that the solitary waves with $\Lambda > 0.936$ are more unstable against expansions than against contractions.

The different behavior of the solitary waves with frequencies greater than or less than 0.936 comes from the fact that for a fixed solitary-wave charge there are two solitary-wave energies. The solitary wave with $\Lambda > 0.936$ has greater energy than the one with $\Lambda < 0.936$ and therefore the latter is stable whereas the former evolves to a solitary wave with frequency less than 0.936. Similar phenomena occur for the bell solitary waves of the Klein-Gordon equation with a cubic nonlinearity.[1]

Coupling of the field (1) with the electromagnetic field has also been carried out. The resulting model presents similar numerical features to those reported here. These results will be published elsewhere.

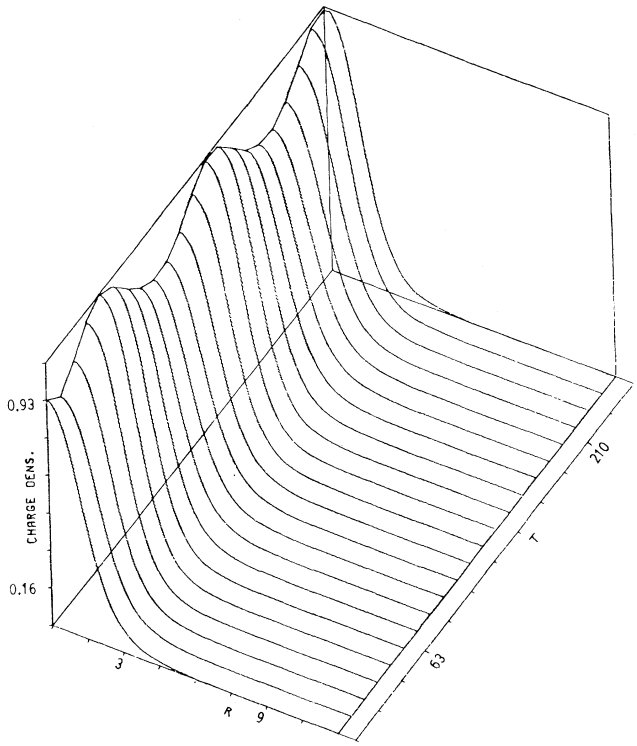

FIG. 5. The contracted ($\alpha = 1.1$) solitary wave $\Lambda = 0.950$ tends to a solitary wave with frequency $\Lambda = 0.92 \pm 0.01$.

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