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Documento de Trabajo

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No. 9411

Octubre 1994

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A RETRIAL SYSTEM WITH CONSTANT ATTEMPTS

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ABSTRACT

In this article we study a retrial queueing system in which customers in orbit join a queue with a FIFO discipline. We use the process $(M(t), N(t))$ where $M(t)$ represents the number of arrivals during a elapsed service time until t while $N(t)$ stands for the random number of customers in orbit at time t . We obtain the probability in the steady state in the $M/M/1$ case. The joint generating function in the $M/G/1$ case is also studied.

RESUMEN

En este artículo estudiamos un sistema de colas con reintentos en el cual los clientes en órbita se unen a una cola con disciplina FIFO. Usamos el proceso $(M(t), N(t))$ donde $M(t)$ representa el número de llegadas durante el tiempo de servicio hasta el instante t y $N(t)$ representa el número de individuos en órbita en el instante t . Obtendremos la probabilidad conjunta en el caso $M/M/1$ en estado estacionario. La función generatriz conjunta en el caso $M/G/1$ es también estudiada.

INTRODUCTION

Our retrial queueing system is characterized by the following feature: if the server is free at the instant of an arrival, the customer starts to be served immediately and leaves the system after his service completion (these customers are identified as primary customers or primary calls). If the channel is busy when a customer arrives he does not leave the system but joins a source of unsatisfied customers which we call the orbit. When a customer arrives into the orbit he must join a queue with a FIFO discipline, that is, after a random length of time only the first customer in orbit can repeat his demand.

We assume that primary customers arrive according to a Poisson process at the rate λ and the first customer in orbit reapplies for service according to a Poisson distribution with parameter μ . The service time distribution function, $B(x)$, is assumed to be the same for both primary and repeated calls. The flow of primary calls, the intervals between repetitions and the service time are taken to be mutually independent.

A review of the large literature on general retrial queues can be found in Falin [2] and Yang and Templeton [6]. Farahmand [4] studied this sort of systems by using the classical stochastic process $(C(t), N(t))$ where $C(t)=1$ or 0 according to whether the channel is busy or free at time t and $N(t)$ represents the number of customers in orbit at time t . Due to the structure of this process when a retrial arrives and finds the channel busy it goes back to the orbit and the state of the system does not change. In this article we will use a new model that generalize the classical model which store information about the useless attempts. This model was introduced by Falin, Rodrigo and Vázquez in [3]. In section-1 we study this model in the $M/M/1$ case by solving some difference finite equations in two variables. In section-2 we obtain a formula for the joint generating function in steady-state

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Nº C: X-53-230659-6

by using the theory of semiregenerative processes.

1. THE M/M/1 CASE

Let $B(x) = 1 - e^{-\nu x}$. We consider the process $(M(t), N(t))$ where $M(t)$ represents the number of primary and repeated calls during the elapsed service time until t . That is, if $M(t) = 0$ then the channel is free at time t . If $M(t) = m$, $m \geq 1$, then the channel is busy at time t and exactly m customers have arrived during the elapsed service time (including the customer being served). Our goal in this section is to obtain an explicit formula for the probability $P\{M_t = m, N_t = n\}$ in steady state.

Let

$$\hat{P}_{mn} = P\{C_t = m, N_t = n\}, \quad m = 0, 1, n \in \mathbb{N} \cup \{0\}$$

$$P_{mn} = P\{M_t = m, N_t = n\}, \quad m, n \in \mathbb{N} \cup \{0\}$$

be the probabilities in steady state. It is easy to see that

$$\hat{P}_{0n} = P\{C_t = 0, N_t = n\} = P_{0n} = P\{M_t = 0, N_t = n\}$$

$$\hat{P}_{1n} = P\{C_t = 1, N_t = n\} = \sum_{m=1}^{\infty} P_{mn} = P\{M_t \geq 1, N_t = n\}$$

The generating function $P_0(z)$ of \hat{P}_{0n} in the M/G/1 case is given by [4]

$$P_0(z) = \sum_{n=0}^{\infty} \hat{P}_{0n} z^n = \frac{(1-\rho)\mu(\beta(\lambda-\lambda z) - z)}{(\lambda z + \mu)\beta(\lambda-\lambda z) - (\lambda + \mu)z}$$

where $\beta(s)$ is the Laplace transform for the service time distribution, $\rho = \lambda(\lambda + \mu)\beta_1\mu^{-1}$ and β_1 is the first moment of the service time. From this result and with a little algebra we obtain in the M/M/1 case the following expression for the probability \hat{P}_{0n} :

$$\hat{P}_{0n} = \lambda(\lambda + \mu)^{-1}(1-\rho)\rho^n, \quad n > 0, \quad \hat{P}_{00} = (1-\rho) \quad (1)$$

We now derive the joint probability $P_{mn}, m > 0$, in the steady-state.

Theorem 1

If the M/M/1 retrial queue is in steady-state the joint probability $P_{mn} = P\{M_t = m, N_t = n\}$, $m > 0$, is given by

$$\begin{aligned} P_{10} &= \frac{\lambda}{\nu}(1-\rho) \\ P_{mn} &= \frac{\lambda}{(\lambda + \mu)} \left(\frac{\lambda}{\nu}\right)^m (1-\rho)\rho^{n-m+1}, \quad m \leq n, \quad m > 0 \\ P_{mn} &= \frac{\lambda}{\nu}(1-\rho) \binom{m-2}{n-1} \left(\frac{\lambda}{\lambda + \mu + \nu}\right)^n \left(\frac{\mu}{\lambda + \mu + \nu}\right)^{m-n-1} + \\ &+ \frac{\lambda}{(\lambda + \mu)} \frac{\lambda}{\nu} (1-\rho) \rho^n \left(\frac{\mu}{\lambda + \mu + \nu}\right)^{m-1} \sum_{s=0}^{n-1} \binom{m-1}{s} \left(\frac{\nu}{\lambda + \mu}\right)^s, \quad m \geq n+1, \quad n > 0 \end{aligned}$$

Proof.

The process $(M(t), N(t))$ is Markovian. We can see that P_{mn} verifies the Kolmogorov equations

$$(I) \quad (\lambda + \mu)P_{0n} = \nu \sum_{m=1}^{\infty} P_{mn}$$

$$(II) \quad (\lambda + \mu + \nu)P_{1n} = \lambda P_{0n} + \mu P_{0n+1}, \quad (\lambda + \nu)P_{10} = \lambda P_{00} + \mu P_{01}$$

$$(III) \quad (\lambda + \mu + \nu)P_{mn} = \lambda P_{m-1, n-1} + \mu P_{m-1, n}, \quad m \geq 2, \quad n > 0.$$

Since $P_{0n} = \hat{P}_{0n}$, P_{0n} is given by (1). Solving equation (II) we get

$$P_{1n} = \frac{\lambda}{(\lambda + \mu)} \frac{\lambda}{\nu} (1-\rho)\rho^n \quad (2)$$

Equation (III) can be rewritten as

$$P_{mn} = \frac{\lambda}{(\lambda + \mu + \nu)} P_{m-1n-1} + \frac{\mu}{(\lambda + \mu + \nu)} P_{m-1n} \quad (3)$$

and a specific solution for P_{mn} can be obtained since the initial conditions are given by

$$\begin{aligned} P(1, n) &= P_{1n} \\ P(m, 0) &= 0, \quad m \geq 2 \end{aligned}$$

(Note that when more than one call has been made during a elapsed service time, there must be at least one customer in orbit).

Let

$$P_{mn} = \left(\frac{\lambda}{\lambda + \mu + \nu} \right)^n g_{mn}$$

Then, equation (3) becomes in

$$g(m, n) = g(m-1, n-1) + \frac{\mu}{\lambda + \mu + \nu} g(m-1, n) \quad (4)$$

$$g(1, n) = \left(\frac{\lambda + \mu + \nu}{\lambda} \right)^n P_{1n}$$

$$g(m, 0) = 0, \quad m \geq 1$$

Morán [5] shows that the general equation

$$g(m, n) = g(m-1, n-1) + h(m, n)g(m-1, n) \quad (IV)$$

is a linear manifolds whose basis verifies the following lemma:

Lemma 1

The set of solutions α_k, γ_k for (IV) given initial conditions $\alpha_k(k, 0) = 1, \alpha_k(j, 0) = 0 \quad j \neq k, \alpha_k(1, j) = 0 \quad \forall j$ and $\gamma_k(j, 0) = 0 \quad \forall j, \gamma_k(1, j) = 0 \quad j \neq k, \gamma_k(1, k) = 1$ constitute a basis for (IV).

Hence the solution of our equation given the initial conditions g_{1n}, g_{m0} is given by

$$g(m, n) = \sum_{k=1}^{\infty} g(1, k) \gamma_k(m, n) + \sum_{k=1}^{\infty} g(k, 0) \alpha_k(m, n)$$

We now derive an explicit formula for the basis of our linear manifold. By using the induction method in (4) we obtain for $\gamma_k(m, n)$

$$\gamma_k(m, k+p) = \binom{m-1}{p} \left(\frac{\mu}{\lambda + \mu + \nu} \right)^{m-1-p}, \quad m-1 \geq p$$

and then

$$\gamma_k(m, n) = \binom{m-1}{n-k} \left(\frac{\mu}{\lambda + \mu + \nu} \right)^{m-1-n+k}, \quad m-1 \geq n-k$$

To obtain an expression for $\alpha_k(m, n) \quad \forall k > 0$, observe that we only need $\alpha_1(m, n)$ since $g(m, 0) = 0 \quad \forall m > 1$. By induction in (4)

$$\alpha_1(m, n) = \binom{m-2}{n-1} \left(\frac{\mu}{\lambda + \mu + \nu} \right)^{m-n-1}, \quad m-n-1 \geq 0$$

Therefore

$$g(m, n) = \sum_{k=\max(1, 1+n-m)}^n g(1, k) \gamma_k(m, n) + g(1, 0) \alpha_1(m, n) I_{m \geq n+1}$$

Since $P_{mn} = \left(\frac{\lambda + \mu + \nu}{\lambda} \right)^n g_{mn}$ we get

$$P_{m,n} = \sum_{k=1+n-m}^n P_{1k} \binom{m-1}{n-k} \left(\frac{\lambda}{\lambda + \mu + \nu} \right)^{n-k} \left(\frac{\mu}{\lambda + \mu + \nu} \right)^{m-1-n+k} \quad m \leq n, \quad m \geq 2$$

$$P_{m,n} = \sum_{k=1}^n P_{1k} \binom{m-1}{n-k} \left(\frac{\lambda}{\lambda + \mu + \nu} \right)^{n-k} \left(\frac{\mu}{\lambda + \mu + \nu} \right)^{m-1-n+k} +$$

$$+P_{10} \binom{m-2}{n-1} \left(\frac{\lambda}{\lambda+\mu+\nu} \right)^n \left(\frac{\mu}{\lambda+\mu+\nu} \right)^{m-n-1}, m=n+1, n>0$$

Substituting P_{1k} for his expression in (2) and after some algebra the theorem follows. ■

The probability for M_t alone is easily calculated from the above theorem. We obtain

$$P(M_t=0) = \left(1 - \frac{\lambda}{\nu}\right), \quad P(M_t=1) = \left(1 - \frac{\lambda}{\nu}\right) \frac{\lambda}{\nu}$$

$$P(M_t=m) = \frac{\lambda}{\nu} \frac{\lambda}{\lambda+\mu+\nu} \left(\frac{\lambda+\mu}{\lambda+\mu+\nu} \right)^{m-2}, \quad m>1$$

2 THE M/G/1 CASE

Let N_1 be the number of customers in orbit at the instant of the i th departure η_1 . $\{N_1\}$ is a markov chain whose steady state distribution $\pi_n = \lim_{t \rightarrow \infty} P(N_1=n)$ verifies the kolmogorov equations

$$\pi_n = \pi_0 k_n + \lambda(\lambda+\mu)^{-1} \sum_{m=1}^n \pi_n k_{n-m} + \mu(\lambda+\mu)^{-1} \sum_{m=1}^{n+1} \pi_n k_{n-m} \quad (5)$$

where

$$k_m = \int_0^{\infty} e^{-\lambda x} \frac{(\lambda x)^m}{m!} dB(x), \quad m \in \mathbb{N} \cup \{0\},$$

represents the probability of m arrivals during a service time.

Let

$$\Pi(z) = \sum_{n=0}^{\infty} \pi_n z^n$$

be the generating function of the steady state distribution $\{\pi_n\}$. Using expression (5) we obtain that $\Pi(z)$ is given by

$$\Pi(z) = \sum_{n=0}^{\infty} \pi_n z^n = \pi_0 \frac{\mu(1-z)\beta(\lambda-\lambda z)}{(\lambda z + \mu)\beta(\lambda-\lambda z) - (\lambda+\mu)z} \quad (6)$$

Moreover taking limits in (6) when z tends to 1 we obtain for π_0

$$\pi_0 = 1 - \lambda(\lambda+\mu)\beta_1 \mu^{-1} = 1 - \rho \quad (7)$$

and

$$\pi_1 = \lim_{z \rightarrow 0} \frac{\pi(z) - \pi_0}{z} = (\lambda+\mu)\mu^{-1} \left(\frac{1-\beta(\lambda)}{\beta(\lambda)} \right) \pi_0 \quad (8)$$

One can see that $\{\pi_n\}_{n \in \mathbb{N}}$ is a probabilistic distribution if and only if $\rho < 1$.

Consider the process $Z(t) = (M(t), N(t))$ where $M(t)$ is defined in the same way as section-1. $Z(t)$ is a semiregenerative process with respect to the markovian renewal process (N_1, η_1) . Then [1]

$$\lim_{t \rightarrow \infty} p(Z_t \in A) = (\pi m)^{-1} \int_0^{\infty} \int_0^{\infty} p(Z_t \in A, \eta_1 > t / N_0 = j) dt \quad (9)$$

where

$$\pi m = \sum_{n=0}^{\infty} \pi_n m_n, \quad \text{and } m_n = E[\eta_1 / N_0 = n].$$

Let

$$P_0(z) = \sum_{n=0}^{\infty} p(M_t=0, N_t=n) z^n$$

$$P(y, z) = \sum_{l=1}^{\infty} y^l \sum_{n=0}^{\infty} p(M_t=l, N_t=n) z^n$$

be the joint generating functions of the process (M_t, N_t) in steady state. By formula (9) we get

$$P(y, z) = (\pi m)^{-1} \sum_l y^l \sum_n z^n \int_0^{\infty} \pi_j P(M_t=l, N_t=n, \eta_1 > t / N_0=j) dt \quad (10)$$

We can now established our main Theorem:

Theorem 2

If our M/G/1 retrial queue is in steady state the generating

functions $P_0(z)$ and $P(y, z)$ are given by

$$P_0(z) = \frac{\mu}{\lambda + \mu} \pi_0 + \frac{\lambda}{\lambda + \mu} \Pi(z)$$

$$P(y, z) = \left[\frac{-\lambda^2(1-\rho)y^2z}{\mu(1-y)-\lambda yz} - \frac{\lambda\mu^2y(1-y)}{(\mu(1-y)-\lambda yz)(\lambda+\mu)} \pi_1 + \frac{(\lambda^2z+\lambda\mu)y}{(\lambda+\mu)z} (\Pi(z)-\pi_0) \right] \times \left[\frac{1-\beta(\lambda+\mu-\lambda yz-\mu y)}{(\lambda+\mu-\lambda yz-\mu y)} + \frac{(\mu^2\pi_1 + \mu(\lambda+\mu)\pi_0)(1-y)y}{(\mu(1-y)-\lambda yz)(\lambda+\mu)} \right] (1-\beta(\lambda))$$

where $\Pi(z)$, π_0 and π_1 are given by (6), (7) and (8) respectively.

Proof

$P_0(z)$ is known [4]. Then by (6) the result is easily derived. Moreover $P_0(1)=1-\lambda\beta_1$, so by using formula (9) we obtain

$$\pi m = \lambda^{-1} \tag{11}$$

To calculate the conditional probabilities given in formula (10) for $m>0$ we distinguish the following cases:

b) $M_t = 1$

$$1. \int_0^\infty P(M_t=1, N_t=0, \eta_1 > t / N_0=0) dt = \int_0^\infty e^{-\lambda y} (1-B(y)) dy.$$

$$2. \int_0^\infty P(M_t=1, N_t=0, \eta_1 > t / N_0=1) dt = \frac{\mu}{\lambda+\mu} \int_0^\infty e^{-\lambda y} (1-B(y)) dy.$$

$$3. \int_0^\infty P(M_t=1, N_t=n, \eta_1 > t / N_0=n) dt = \frac{\lambda}{\lambda+\mu} \int_0^\infty e^{-(\lambda+\mu)y} (1-B(y)) dy, \quad n \geq 1.$$

$$4. \int_0^\infty P(M_t=1, N_t=n, \eta_1 > t / N_0=n+1) dt = \frac{\mu}{\lambda+\mu} \int_0^\infty e^{-(\lambda+\mu)y} (1-B(y)) dy, \quad n > 0.$$

c) $M_t \geq 1$.

$$1. \int_0^\infty P(M_t=m, N_t=n, \eta_1 > t / N_0=0) dt = \frac{\lambda^n \mu^{m-n-1}}{(n-1)!(m-n-1)!} \int_0^\infty e^{-\lambda y} \left[\frac{(m-2)!}{\mu^{m-1}} - e^{-\mu y} \sum_{j=0}^{m-2} \frac{1}{j!} \frac{(m-1-i)y^{m-j-2}}{\mu^{j+1}} \right] (1-B(y)) dy, \quad n > 0, m > n.$$

$$2. \int_0^\infty P(M_t=m, N_t=n, \eta_1 > t / N_0=j) dt = \frac{\lambda}{(\lambda+\mu)} \int_0^\infty e^{-(\lambda+\mu)y} \frac{(\lambda y)^{n-j}}{(n-j)!} \frac{(\mu y)^{m-1-n+j}}{(m-1-n+j)!} (1-B(y)) dy + \frac{\mu}{(\lambda+\mu)} \int_0^\infty e^{-(\lambda+\mu)y} \frac{(\lambda y)^{n-j+1}}{(n-j+1)!} \frac{(\mu y)^{m-2-n+j}}{(m-2-n+j)!} (1-B(y)) dy, \quad j > 1.$$

$$3. \int_0^\infty P(M_t=m, N_t=n, \eta_1 > t / N_0=1) dt = \frac{\lambda}{(\lambda+\mu)} \int_0^\infty e^{-(\lambda+\mu)y} \frac{(\lambda y)^{n-1}}{(n-1)!} \frac{(\mu y)^{m-n}}{(m-n)!} (1-B(y)) dy + \frac{\mu \lambda^n \mu^{m-n-1}}{(n-1)!(m-n-1)!(\lambda+\mu)} \times \int_0^\infty e^{-\lambda y} \left[\frac{(m-2)!}{\mu^{m-1}} - e^{-\mu y} \sum_{j=0}^{m-2} \frac{1}{j!} \frac{(m-1-i)y^{m-j-2}}{\mu^{j+1}} \right] (1-B(y)) dy$$

Substituting this conditional probabilities in (10) an after some algebra, we are done. ■

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