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SENSITIVITY ANALYSIS IN A CLASS OF DYNAMIC OPTIMIZATION MODELS[1]

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ABSTRACT

A general model of dynamic optimization, deterministic, in discrete time, and with infinite time horizon is considered. We suppose that there are parameters in the formulation of the model. Conditions for stability of the optimal solution are studied. Local analysis of steady state comparative statics and comparative dynamics are presented. In addition we apply these results to a quadratic case and to an economic example: a one sector growth model.

RESUMEN

Se considera un modelo general de optimización dinámica, determinístico, formulado en tiempo discreto y con horizonte temporal infinito. Se supone que existen diferentes parámetros en la formulación del modelo. Se estudian condiciones de estabilidad para la solución óptima. Se presentan análisis locales de estática y dinámica comparativa. Se aplican los resultados al caso cuadrático y a un ejemplo económico: un modelo de crecimiento unisectorial.

Key words
Dynamic optimization, sensitivity analysis, economic growth.

AMS classification
90C31, 90A16

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1. INTRODUCTION

In Stockey and Lucas (1989), recursive methods and applications in economic analysis are drawn together and presented in a systematic way, by the first time. As pointed out in that book, applications of these methods appear in almost every substantive area of economics: the theory of investment, the theory of the consumer, search theory, public finance, growth theory and so on. In economic analysis we find many optimization problems in which we have parameters, that is, values which are *a priori* known, but which can change in any moment (for instance prices, wages, interest rates, discount factors.). Quite often we are interested not only in obtaining the optimal path, but also studying how the optimal path changes when one of these parameters changes.

In this paper we deal with a broad family of dynamic optimization problems in discrete time, with infinite horizon. We consider the general deterministic model studied in Stokey and Lucas, for the unidimensional case, but introducing a vector of parameters in both the function and the correspondence of the problem, as in Santos (1992a, 1992b) or Araujo y Sheinkman (1979). It is presented a systematic approach to studying the qualitative properties of the family of models. This is achieved by studying, in order: (i) the local stability of the steady state, (ii) the steady state comparative statics, (iii) the local comparative dynamic properties of the model. The local stability analysis provides qualitative information which is necessary for the steady state comparative static analysis. In turn, this information is necessary for conducting the local comparative dynamic analysis.

This systematic approach is the one used by Caputo (1987, 1989) in continuous time models. This methodology is concentrated on obtaining all possible refutable implications of steady state comparative statics and comparative dynamics for any kind of parameter in the model. We obtain a set of theorems and corollaries which follow a similar sequence to those of Caputo. This battery of results is ready to be directly applied in concrete economic models. This methodology can only be applied to infinite-horizon problems that are autonomous in present-value or current-value terms. For other cases, Caputo (1990a, 1990b, 1990c, 1992) has generalised the primal-dual methodology developed by Silberberg for dynamic problems in continuous time.

Section 2 presents a formal description of the model. In section 3 we state necessary and sufficient optimality conditions, using the variational approach for discrete time. As pointed out by Stockey and Lucas, to apply the stability theory to the problem of characterising solutions to dynamic programs, the strategy that does work is to use a linear approximation to the Euler equation. In section 4 we study the local stability of the steady state. Also it is stated a theorem that assures that, under certain conditions, there exists a unique optimal solution to the problem, that, moreover, converges to the steady state. Following this there are sections on the steady state comparative static and local comparative dynamic properties of the model. In section 7 we study, particularly, the quadratic case, that in our opinion has special interest, because all the results are global. In section 8 a typical example is presented: a one sector optimal growth model. Some important developments of this model can be found in Becker (1985), Boldrin and Montrucchio (1986) and Amir et Al. (1991).

2. THE MODEL AND ASSUMPTIONS

The problem under consideration is to find a sequence $\{x_{t+1}\}_{t=0}^{\infty}$ that solves:

$$\max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}, \alpha)$$

$$\text{subject to: } x_{t+1} \in \Gamma(x_t, \alpha), \quad t = 0,1,2,\text{-} \qquad \text{(P)}$$
$$x_0 \in X, \text{ given}$$

where x_t is the state variable at time t.

Let: $X \subset \Re$ be the set of possible values for the state variable x,

$A \subset \Re^m$ be the set of possible values for the vector of parameters,

Γ: $X \times A \rightarrow X$ be the correspondence describing the feasibility constraints: that is, for each $(x, \alpha) \in XxA$, $\Gamma(x,\alpha)$ is the set of feasible values for the state variable next period if the current state is x, and the vector of parameters is a,

G be the graph of G: $G = \{(x, y, \alpha) \in X \times X \times A,\ y \in \Gamma(x, \alpha)\}$,

F: $G \rightarrow \Re$ be the one-period return function,

$\beta \in B = (0,1) \subset \Re_+$ be the discount factor.

Call any sequence $\{x_{t+1}\}_{t=0}^{\infty}$ in X a plan. Given $x_0 \in X$, and $\alpha \in A$, let

$$\Pi(x_0, \alpha) = \left\{\{x_t\}_{t=0}^{\infty}:\ x_{t+1} \in \Gamma(x_t, \alpha),\ t = 0,1,2,\ldots\right\}$$

be the set of plans that are feasible from x_0, given the set of parameters $\alpha \in A$.

In addition, we suppose that the following hypothesis always hold:

(H.1) $Int[\Gamma(x,\alpha)]$ is non-empty, for all $(x,\alpha) \in X \times A$.

(H.2) For all $x_0 \in X, \alpha \in A$, and $\{x_t\}_{t=0}^{\infty} \in \Pi(x_0, \alpha)$, $\sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}, \alpha)$ is finite.

Clearly, hypothesis (H.1) implies that $\Gamma(x,\alpha)$ is non-empty. We suppose that $Int[\Gamma(x,\alpha)]$ is non-empty because we want to characterise interior optimal solutions of the stated problem. Hypothesis (H.2) is satisfied if the function F is bounded (both hypotheses appear, for example, in Santos (1993)).

3. OPTIMALITY CONDITIONS

In this section we characterise interior optimal solutions for problem (P). It is proved that Euler condition is necessary, and that Euler and transversality conditions are sufficient, according to Stockey and Lucas (1989), with some small changes. In order to apply the variational approach for problem (P), let's add the following assumption:

(A.1) F continuos and continuously differentiable in Int(G) with respect to (x,y), for each $\alpha \in A$.

Proposition 1 (Necessity of Euler condition). In problem (P), for a fixed $\alpha \in A$, with assumption (A.1) holding, if there exists an interior optimal solution $\{x_t^*\}_{t=0}^{\infty}$ (that is, an optimal solution such that $x_{t+1}^* \in Int\left[\Gamma(x_t^*,\alpha)\right]$, $\forall t = 0,1,2\ldots$) then, it must verify the Euler condition:

$$0 = F_y(x_t, x_{t+1}, \alpha) + \beta\ F_x(x_{t+1}, x_{t+2}, \alpha), \text{ for } t = 0,1,2,\text{-} \qquad [1]$$

where F_x is the partial derivative of F with respect to its first argument, and F_y is the partial derivative of F with respect to its second argument.

To assure that Euler and transversality conditions are sufficient for the maximization of (P), let's add the following assumptions:

(A.2) $X \subset \Re^+ \cup \{0\}$.

(A.3) For each y,α, $F(.,y,\alpha)$ is strictly increasing in its first argument.

(A.4) For each $\alpha \in A$, F is concave in x,y, that is,

$$F(\theta(x,y,\alpha)+(1-\theta)(x',y',\alpha)) \geq \theta F(x,y,\alpha)+(1-\theta)F(x',y',\alpha)$$

$$\forall x \in X,\ y \in \Gamma(x,\alpha),\ x' \in X,\ y' \in \Gamma(x',\alpha)$$

In addition, the inequality is strict if $x \neq x'$.

The transversality condition is: $$\lim_{t\to\infty} \beta^t\ F_x(x_t, x_{t+1}, \alpha).x_t = 0 \qquad [2]$$

Proposition 2 (Sufficiency of Euler and transversality conditions). Let's consider problem (P), for a fixed $\alpha \in A$, with assumption (A.1) to (A.4) holding and assuming that there exists an interior solution $\{x_t^*\}_{t=0}^{\infty}$ (that is, a solution such that $x_{t+1}^* \in Int\left[\Gamma(x_t^*,\alpha)\right]$, $\forall t = 0,1,2\ldots$).

If $\{x_t^*\}_{t=0}^{\infty}$ verifies [1] and [2], then it is the unique global optimal solution of (P).

4.- STEADY STATE AND STABILITY

For problem (P), with given $\alpha \in A$, it has been proved that if it exists an interior optimal solution $\{x_t^*\}_{t=0}^{\infty}$, it has to verify the Euler equation [1], a second order difference equation. We define the steady state x^* of problem (P) as the steady state of this difference equation that is defined by:

$$0 = F_y(x^*,x^*,\alpha) + \beta\ F_x(x^*,x^*,\alpha) \qquad [3]$$

obtained from [1], with $x_t^*=x_{t+1}^*=x_{t+2}^*=x^*$.

Let us assume now that:

(A.5) For each $\alpha \in A$, $\beta \in (0,1)$, there exists a unique $x^* \in X$, in an open interval with centre at x^*, such that $0 = F_y(x^*,x^*,\alpha) + \beta F_x(x^*,x^*,\alpha)$.

In order to make the problem analytically tractable, let's define the following function:

$$\phi(x_t, x_{t+1}, x_{t+2}) = F_y(x_t, x_{t+1}, \alpha) + \beta F_x(x_{t+1}, x_{t+2}, \alpha) \qquad [4]$$

and let's impose an additional assumption on (P):

(A.6) $F \in C^{(2)}$, in Int(G).

Now, we can expand the right hand side of [4] about the steady state using a Taylor polynomial of ϕ and retaining only linear terms:

$$F_{yx}^*(x_t - x^*) + [F_{yy}^* + \beta F_{xx}^*](x_{t+1} - x^*) + \beta F_{xy}^*(x_{t+2} - x^*) = 0 \qquad [5.a]$$

Since $F^*_{yx}=F^*_{xy}$, and combining terms, we obtain:

$$\beta F_{xy}^* x_{t+2} + [F_{yy}^* + \beta F_{xx}^*] x_{t+1} + F_{xy}^* x_t = [F_{yy}^* + \beta F_{xx}^* + F_{xy}^*(1+\beta)]\ x^* \qquad [5.b]$$

where $F_{xx}^* = F_{xx}(x^*,x^*,\alpha)$, $F_{yy}^* = F_{yy}(x^*,x^*,\alpha)$, $F_{xy}^* = F_{xy}(x^*,x^*,\alpha)$.

To check the stability of the steady state, we examine the characteristic equation of [5.b] to determine its roots. The associated homogeneous equation is:

$$\beta F_{xy}^* x_{t+2} + [F_{yy}^* + \beta F_{xx}^*] x_{t+1} + F_{xy}^* x_t = 0 \qquad [6]$$

A) Suppose that $F^*_{xy}=0$. From assumptions (A.4) and (A.5), $F^*_{yy}<0$, $F^*_{xx}<0$ so [6] reduces to:

$$[F_{yy}^* + \beta F_{xx}^*]\ x_{t+1} = 0 \quad \Rightarrow \quad x_{t+1} = 0, \ \forall t = 0,1,2,\ldots$$

The solution to the homogeneous equation is $x_0=x_0$ given, $x_1=x_2=\ldots=x_t=\ldots=0$. The complete solution is $x_0=x_0$ given, $x_1=x_2=\ldots=x_t=\ldots=x^*$.

B) Now suppose that $F_{xy}^* \neq 0$. Then [6] can be written:

$$x_{t+2} + \frac{[F_{yy}^* + \beta F_{xx}^*]}{\beta F_{xy}^*} x_{t+1} + \frac{1}{\beta} x_t = 0 \qquad [7]$$

For this case, we obtain the following results:

Proposition 3: Let's consider problem (P), with assumptions (A.2) to (A.6) holding. Let's assume that $F_{xy}^* \neq 0$. Then:

(1) It is not possible that the steady state be asymptotically stable.

(2) If the roots of the characteristic polynomial are complex, then the steady state exhibits instability.

(3) If the roots of the characteristic polynomial are real and equal, then the steady state exhibits instability.

Proof:

(1) Let's consider equation [7]. Let $\gamma = \dfrac{F_{yy}^* + \beta F_{xx}^*}{\beta F_{xy}^*}$. It is $\mathrm{sign}[\gamma] = -\mathrm{sign}[F_{xy}^*]$.

The characteristic equation is: $\lambda^2 + \gamma \lambda + \dfrac{1}{\beta} = 0$

This can be factored in the following way:

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1 \lambda_2 = 0$$

For these expressions to be identical, it must be verified that

$$\lambda_1 + \lambda_2 = -\gamma$$

$$\lambda_1 \lambda_2 = \frac{1}{\beta} > 1 \Rightarrow \text{sign}[\lambda_1] = \text{sign}[\lambda_2]$$

Since 0<β<1, the product of the roots must be positive and bigger than the unity, which rules out the possibility that $|\lambda_1| < 1$, $|\lambda_2| < 1$, and hence the asymptotical stability.

(2) Let's assume that λ_1 and λ_2 are complex. Suppose that:

$$\lambda_1 = a + bi$$
$$\lambda_2 = a - bi$$

Then:

$$\lambda_1 + \lambda_2 = 2a$$

$$\lambda_1 \lambda_2 = a^2 + b^2 = \frac{1}{\beta} > 1$$

$$\|\lambda_1\| = \|\lambda_2\| = \sqrt{a^2 + b^2} = \frac{1}{\sqrt{\beta}} > 1$$

and then the steady state is unstable.

(3) If $\lambda_1 = \lambda_2$ real, then

$$\gamma^2 - \frac{4}{\beta} = 0 \Leftrightarrow \gamma^2 = \frac{4}{\beta} \Rightarrow |\gamma| = \frac{2}{\sqrt{\beta}}$$

$$\lambda_1 = \lambda_2 = -\frac{\gamma}{2}$$

$$|\lambda_1| = |\lambda_2| = \frac{|\gamma|}{2} = \frac{1}{\sqrt{\beta}} > 1$$

and then the steady state is instable. ■

The main result about stability of the steady state is the following:

Theorem 1: Let's consider problem (P), with assumptions (A.2) to (A.6) holding. Let's assume that $F^*_{xy} \neq 0$. Then, the necessary and sufficient condition for the steady state to exhibit saddlepoint stability is that:

$$F^*_{yy} + \beta F^*_{xx} + |F^*_{xy}| (1 + \beta) < 0 \qquad [8]$$

Proof: In proposition 3 we have seen that, in order to obtain saddlepoint stability, it is necessary that the roots of the characteristic polynomial are real and distinct. The roots of the characteristic polynomial are:

$$\lambda_1 = \frac{-\gamma}{2} + \frac{1}{2}\sqrt{\gamma^2 - \frac{4}{\beta}}; \quad \lambda_2 = \frac{-\gamma}{2} - \frac{1}{2}\sqrt{\gamma^2 - \frac{4}{\beta}}$$

It must be verified that:

$$\gamma^2 - \frac{4}{\beta} > 0 \Leftrightarrow \gamma^2 > \frac{4}{\beta} \Rightarrow |\gamma| > \frac{2}{\sqrt{\beta}}$$

Suppose that this condition is holding. If, at the same time,

a) $\gamma > 0$, then is obvious that $\lambda_2 < \lambda_1 < 0$. For saddlepoint stability to exist it is required that:

$$\lambda_2 < -1 < \lambda_1 < 0 \Leftrightarrow$$

$$\Leftrightarrow -\frac{\gamma}{2} - \frac{1}{2}\sqrt{\gamma^2 - \frac{4}{\beta}} < -1 < -\frac{\gamma}{2} + \frac{1}{2}\sqrt{\gamma^2 - \frac{4}{\beta}} < 0 \Leftrightarrow$$

$$\Leftrightarrow |\gamma - 2| < \sqrt{\gamma^2 - \frac{4}{\beta}} \Leftrightarrow$$

$$\Leftrightarrow |\gamma| = \gamma > 1 + \frac{1}{\beta}$$

b) $\gamma < 0$, then $0 < \lambda_2 < \lambda_1$. Saddlepoint stability requires that:

$$0 < \lambda_2 < 1 < \lambda_1 \Leftrightarrow$$

$$\Leftrightarrow -\frac{\gamma}{2} - \frac{1}{2}\sqrt{\gamma^2 - \frac{4}{\beta}} < 1 < -\frac{\gamma}{2} + \frac{1}{2}\sqrt{\gamma^2 - \frac{4}{\beta}} < 0 \Leftrightarrow$$

$$\Leftrightarrow |\gamma + 2| < \sqrt{\gamma^2 - \frac{4}{\beta}} \Leftrightarrow$$

$$\Leftrightarrow |\gamma| = -\gamma > 1 + \frac{1}{\beta}$$

Thus, in order to exhibit saddlepoint stability, the steady state must satisfy:

$$\text{(i)} \ |\gamma| > \frac{2}{\sqrt{\beta}}$$

$$\text{(ii)} \ |\gamma| > 1 + \frac{1}{\beta}$$

Let us see now that if (ii) is verified, then (i) is verified:

$$|\gamma| > 1 + \frac{1}{\beta} \Rightarrow |\gamma| > \frac{2}{\sqrt{\beta}} \text{ because } 1 + \frac{1}{\beta} > \frac{2}{\sqrt{\beta}} \Leftrightarrow$$

$$\Leftrightarrow 1 + \frac{1}{\beta} - \frac{2}{\sqrt{\beta}} > 0 \Leftrightarrow \left(1 - \frac{1}{\sqrt{\beta}}\right)^2 > 0$$

which is evident.

Therefore, assuming that $F^*_{xy} \neq 0$, the necessary and sufficient condition for the steady state to have saddlepoint stability is that:

$$|\gamma| > 1 + \frac{1}{\beta} \Leftrightarrow \frac{|F^*_{yy} + \beta \ F^*_{xx}|}{\beta \ |F^*_{xy}|} > 1 + \frac{1}{\beta} \Leftrightarrow$$

$$\Leftrightarrow F^*_{yy} + \beta \ F^*_{xx} + |F^*_{xy}| \ (1 + \beta) < 0$$

and the theorem is proved.■

Note: In the formulation of the model, in section 2, it is assumed that 0<β<1. Let us see what happens if β=1, that is, if there is no discount. If β=1, according to Levhari-Liviatan (1972) there is with absolute certainty saddlepoint stability. It is also proved immediately that the necessary and sufficient condition of saddlepoint stability of theorem 1 holds, since the hessian matrix is negative defined for each $\alpha \in A$.

We wonder now under which conditions we can assure that the equation of the steady state [3] defines implicitly x^* as a continuously differentiable function of α and β. We have the answer in the following proposition:

Proposition 4: Let's consider problem (P) for $\alpha \in Int(A)$, with assumptions (A.2) to (A.6) holding. Let's assume that $F^*_{xy} \neq 0$. Then:

If the steady state exhibits saddlepoint stability, there exists a unique continuously differentiable function $x^*=x^*(\alpha,\beta)$, defined in an appropriate neighbourhood in AxB, such that:

$$F_y(x^*(\alpha,\beta),x^*(\alpha,\beta),\alpha) + \beta \ F_x(x^*(\alpha,\beta),x^*(\alpha,\beta),\alpha) = 0$$

Proof: Let's consider equation [3] of the steady state. Let's see that we can apply the implicit function theorem. It is evident that conditions 1 and 2 of the implicit function theorem hold. The third condition requires that

$$F^*_{yy} + \beta \ F^*_{xx} + F^*_{xy} \ (1 + \beta) \neq 0$$

(Notice similarity with [8]).

1) If $|F^*_{xy}| = F^*_{xy}$, that is, if $F^*_{xy} > 0$, then from [8] we have:

$$F^*_{yy} + \beta\, F^*_{xx} + F^*_{xy}\,(1+\beta) < 0, \text{ so } \neq 0.$$

2) If $|F^*_{xy}| = -F^*_{xy}$, that is, if $F^*_{xy} < 0$, then, as also $F^*_{yy} < 0$, $F^*_{xx} < 0$ we have:

$$F^*_{yy} + \beta\, F^*_{xx} + F^*_{xy}\,(1+\beta) < 0, \text{ so } \neq 0.$$

By direct application of the implicit function theorem, the result is obtained .■

Now it is stated and proved theorem 2 that shows that, if there is saddlepoint stability and if x_0 belongs to some interval with centre at x^*, then the unique path that converges to the steady state from x_0 is the optimal solution of problem (P), assuming that the optimal solution of the problem is an interior solution. It is assumed, therefore, that:

(A7) If $\{x^*_t\}_{t=0}^{\infty}$ is a solution that verifies the Euler condition for problem (P), it verifies that $x_{t+1} \in Int[\Gamma(x_t, \alpha)]$.

This assumption is standard, since it is found, for example, in the studies of Boldrin and Montrucchio (1989), Santos (1991, 1992, 1993), Araujo (1991) or Caputo (1987).

Theorem 2: Let's consider problem (P), with assumptions (A.2) to (A.7) holding. Let's assume that $F^*_{xy} \neq 0$, and $F^*_{yy} + \beta F^*_{xx} + |F^*_{xy}|(1+\beta) < 0$.

Then: for a fixed $\alpha \in A$, there exists U, open interval with centre at x^*, such that if $x_0 \in U$, there exists $\{x^*_t\}_{t=0}^{\infty}$ with $x_0^* = x_0$, being $\{x^*_t\}_{t=0}^{\infty}$ the unique optimal solution of problem (P). Moreover, $\lim_{t\to\infty} x^*_t = x^*$.

Proof: Consider the Euler equation associated to problem (P), for a fixed $\alpha \in A$. It can be written as:

$$0 = F_y(x,y,\alpha) + \beta\, F_x(y,z,\alpha)$$

By direct application of the implicit function theorem at point $x=x^*$, $y=x^*$, $z=x^*$, we obtain that there exists a continuously differentiable function g(y,x), defined in a neighbourhood of (x^*,x^*), verifying $g(x^*,x^*)=x^*$, with

$$0 = F_y(x,y,\alpha) + \beta\, F_x(y,g(y,x),\alpha)$$

We have that:

$$g_x(x^*,x^*) = -\frac{1}{\beta}; \quad g_y(x^*,x^*) = -\frac{F^*_{yy} + \beta F^*_{xx}}{\beta F^*_{xy}}$$

Define the first order dynamic system:

$$Z_{t+1} = (x_{t+2}, x_{t+1}) = (g(x_{t+1}, x_t), x_{t+1}) = h(Z_t), \quad \text{for } t = 0,1,2,\ldots \qquad [9]$$

where $Z_t=(x_{t+1},x_t)$, for which $Z^*=(x^*,x^*)$ is a steady state.

The Jacobian matrix of h at (x^*,x^*) is:

$$A = \begin{pmatrix} -\dfrac{F^*_{yy} + \beta F^*_{xx}}{\beta F^*_{xy}} & -\dfrac{1}{\beta} \\ 1 & 0 \end{pmatrix}$$

that verifies $|I - A| = [F^*_{yy} + \beta F^*_{xx} + (1+\beta)F^*_{xy}] \,/\, (\beta F^*_{xy}) \neq 0$, being one of the eigenvalues of A less than 1 in absolute value, an the other greater than 1 in absolute value.

Applying to dynamic system [9] the results of theorem 6.6 in Stockey and Lucas (1989), it can be assured that there is a neighbourhood V of $Z^*=(x^*,x^*)$, and a continuously differentiable function $\phi: V \to \Re$ for which the matrix $(\phi_y(Z^*), \phi_x(Z^*))$ has rank 1, such that if $\{Z_t\}_{t=0}^{\infty}$ is a solution of [9] with $Z_0 \in V$ and $\phi(Z_0)=0$, then $\lim_{t\to\infty} Z_t = Z_t^*$.

Therefore there exists U, open interval with centre at x^*, such that $\forall x_0 \in U$ there is a unique x_1^*, such that $\phi(x_0,x_1^*)=0$, and a unique $\{x_t^*\}$, with $x_{t+2}^*=g(x_{t+1}^*,x_t^*)$, $x_0^*=x_0$ and $x_1^*=x_1^*(x_0)$, such that $\{x_t^*\}_{t=0}^{\infty}$ verifies the Euler equation and $\lim_{t\to\infty} x_t^* = x^*$.

Let's see that the solution obtained from the Euler equation verifies transversality condition [2]:

$\lim_{t\to\infty} \beta^t \, F_x(x_t^*, x_{t+1}^*, \alpha) \, x_t^* = 0$ since,

$0<\beta<1$ implies that $\beta^t \to 0$ as $t \to \infty$

$\{x_t^*\}_{t=0}^{\infty} \to x^*$, so $\{x_t^*\}_{t=0}^{\infty}$ is bounded,

F_x is a continuous function. $x_t^* \in \{x_t^*\}_{t=0}^{\infty} \cup \{x^*\}$ and $x_{t+1}^* \in \{x_t^*\}_{t=0}^{\infty} \cup \{x^*\}$, being $\{x_t^*\}_{t=0}^{\infty} \cup \{x^*\}$ a compact set. Therefore $F_x(x_t^*,x_{t+1}^*,a)$ is bounded.

By assumption (A.7) the solution is interior and, by applying proposition 2, it follows that $\{x_t^*\}_{t=0}^{\infty}$ is the unique global optimal solution of problem (P). ■

5. THE STEADY STATE COMPARATIVE STATICS

In this section we try to answer the following question: How does steady state change when there is a small change in some parameters in problem (P)?. The steady state comparative statics are found by taking partial derivatives in [3] with respect to the parameters (α, β) using the chain rule. Results are given by the following theorem:

Theorem 3: In the problem (P) for $\alpha \in Int(A)$, with assumptions (A.2) to (A.7) holding and with $F^*_{xy} \neq 0$, the effects of variations in the parameters (α,β) on the optimal steady state choice function $x_t = x^*(\alpha,\beta)$ are found from:

$$\frac{\partial x^*}{\partial \alpha_i} = \frac{-F^*_{y\alpha_i} - \beta F^*_{x\alpha_i}}{F^*_{yy} + \beta F^*_{xx} + F^*_{xy}(1+\beta)}, \qquad i = 1,2\ldots m$$

$$\frac{\partial x^*}{\partial \beta} = \frac{-F^*_x}{F^*_{yy} + \beta F^*_{xx} + F^*_{xy}(1+\beta)}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots \alpha_m)$

Proof: The derivation of equation [3] with respect to α_i yields to:

$$F^*_{yx}\frac{\partial x^*}{\partial \alpha_i} + F^*_{yy}\frac{\partial x^*}{\partial \alpha_i} + \beta\left[F^*_{xx}\frac{\partial x^*}{\partial \alpha_i} + F^*_{xy}\frac{\partial x^*}{\partial \alpha_i}\right] + F^*_{y\alpha_i} + \beta F^*_{x\alpha_i} = 0$$

Solving this equation gives: $$\frac{\partial x^*}{\partial \alpha_i} = \frac{-F^*_{y\alpha_i} - \beta F^*_{x\alpha_i}}{F^*_{yy} + \beta F^*_{xx} + F^*_{xy}(1+\beta)}$$

Similarly, deriving [3] with respect to β we obtain:

$$F^*_{yx}\frac{\partial x^*}{\partial \beta} + F^*_{yy}\frac{\partial x^*}{\partial \beta} + \beta\left[F^*_{xx}\frac{\partial x^*}{\partial \beta} + F^*_{xy}\frac{\partial x^*}{\partial \beta}\right] + F^*_x = 0$$

Solving this equation gives: $$\frac{\partial x^*}{\partial \beta} = \frac{-F_x^*}{F_{yy}^* + \beta F_{xx}^* + F_{xy}^* (1+\beta)}$$ ■

At this level of generality, no signs are implied for this expressions. However, from the assertion that the steady state exhibits saddle-point stability, the denominator of both expressions is negative. Then, it can be proved the following corollary:

Corollary 3: Under the assumptions of Theorem 3, if the steady state exhibits saddle-point stability, then:

(1) $sign\left(\frac{\partial x^*}{\partial \beta}\right) = sign(F_x^*)$.

(2) $sign\left(\frac{\partial x^*}{\partial \alpha_i}\right) = sign(F_{y\alpha_i}^* + \beta \bullet F_{x\alpha_i}^*)$

(3) If any parameter α_i enters the function F such that it is attached to x_t only, when evaluated at the steady state, that is, $F_{y\alpha_i}^* = 0$, then $sign\left(\frac{\partial x^*}{\partial \alpha_i}\right) = sign(F_{x\alpha_i}^*)$.

(4) If any parameter α_i enters the function F such that it is attached to x_{t+1} only, when evaluated at the steady state, that is, $F_{x\alpha_i}^* = 0$, then $sign\left(\frac{\partial x^*}{\partial \alpha_i}\right) = sign(F_{y\alpha_i}^*)$.

6. LOCAL COMPARATIVE DYNAMICS

In the previous section we have seen how does steady state change when there is a small change in some parameters in problem (P). Steady state changes from x^1 to x^2 as a consequence of a small change in the value of some parameter. In this section we try to answer the following question: Which trajectory will be the optimal solution in moving from x^1 to x^2?.

Suppose that there exists saddle-point stability. In this case, the solutions of the characteristic equation of [5b] are real roots λ_1 and λ_2 (non-zero). Assume, for example, that $|\lambda_1| < 1$, $|\lambda_2| > 1$. The general solution to the linearized equation [5a] is therefore:

$$z(t;\alpha,\beta,x_0) = A_1 \cdot \lambda_1^t + A_2 \cdot \lambda_2^t + x^*(\alpha,\beta)$$

To find the specific path that maximises (P) near the steady state it must be taken into account the boundary condition, $x(0) = x_0$, and the result from Theorem 2:

$$\lim_{t\to\infty} x_t = x^*(\alpha,\beta)$$

In this case and near the steady state, the optimal solution can be approximated by the solution of the linearized equation [5a]. Since $|\lambda_2| > 1$, this limit result holds if and only if $A_2=0$.

Therefore: $$z(t;\alpha,\beta,x_0) = A_1 \cdot \lambda_1^t + x^*(\alpha,\beta)$$

and $$A_1 = x_0 - x^*(\alpha,\beta)$$

The specific equilibrium path around the steady state takes the following form:

$$z(t;\alpha,\beta,x_0) = [x_0 - x^*(\alpha,\beta)] \cdot \lambda_1^t + x^*(\alpha,\beta) \qquad \textbf{[10]}$$

From this expression we can obtain the following local comparative dynamics result:

Theorem 4: Be the problem (P) for $\alpha \in Int(A)$, with assumptions (A.2) to (A.7); let us assume also that $F_{xy}^* \neq 0$ and $F_{yy}^* + \beta \bullet F_{xx}^* + |F_{xy}^*| \bullet (1+\beta) < 0$. Then, the effects of variations in the parameters (x_0, α, β) on the equilibrium path $z(t;\alpha,\beta,x^*(\alpha,\beta))$ around the steady state are found from:

$$\frac{\partial z_t}{\partial x_0} = \lambda_1^t \quad ; \quad \frac{\partial z_t}{\partial \alpha_i} = \frac{\partial x^*}{\partial \alpha_i}(1 - \lambda_1^t) \quad ; \quad \frac{\partial z_t}{\partial \beta} = \frac{\partial x^*}{\partial \beta}(1 - \lambda_1^t)$$

where λ_1 is the root of the characteristic equation of [5b], with $\lambda_1 < 1$.

Proof: These local comparative dynamics results follow from the derivation of equation [10] with respect to the parameters (x_0, α, β) and setting $x_0 = x^*(\alpha,\beta)$. Consider first parameter x_0:

$$\frac{\partial z_t}{\partial x_0} = \lambda_1^t \qquad [11]$$

Consider now parameter α_i:

$$\frac{\partial z_t}{\partial \alpha_i} = -\frac{\partial x^*}{\partial \alpha_i}\lambda_1^t + \left[x_0 - x^*(\alpha,\beta)\right] t\, \lambda_1^{t-1}\frac{d\lambda_1}{d\alpha_i} + \frac{\partial x^*}{\partial \alpha_i}$$

This derivative must be evaluated at $x_0 = x^*(\alpha,\beta)$, so:

$$\frac{\partial z_t}{\partial \alpha_i} = \frac{\partial x^*}{\partial \alpha_i}(1 - \lambda_1^t) \qquad [12]$$

Finally, consider the parameter β, evaluating the derivative at $x_0 = x^*(\alpha,\beta)$:

$$\frac{\partial z_t}{\partial \beta} = -\frac{\partial x^*}{\partial \beta}\lambda_1^t + \left[x_0 - x^*(\alpha,\beta)\right] t\, \lambda_1^{t-1}\frac{d\lambda_1}{d\beta} + \frac{\partial x^*}{\partial \beta}$$

$$\frac{\partial z_t}{\partial \beta} = \frac{\partial x^*}{\partial \beta}(1 - \lambda_1^t) \qquad [13]$$

■

We can evaluate these derivatives when $t \to \infty$, resulting in the following intuitive results:

$$\lim_{t\to\infty}\frac{\partial z}{\partial x_0}(t;\alpha,\beta,x^*(\alpha,\beta)) = 0$$

$$\lim_{t\to\infty}\frac{\partial z}{\partial \alpha_i}(t;\alpha,\beta,x^*(\alpha,\beta)) = \frac{\partial x^*}{\partial \alpha_i}$$

$$\lim_{t\to\infty}\frac{\partial z}{\partial \beta}(t;\alpha,\beta,x^*(\alpha,\beta)) = \frac{\partial x^*}{\partial \beta}$$

From the assertion that the steady state exhibits saddle-point stability, $|\lambda_1| < 1$. Then, its obvious the following corollary:

Corollary 4:

(1)

$$sign\left(\frac{\partial z_t}{\partial x_0}\right) = +, \quad if \quad \gamma < 0 \Leftrightarrow F_{xy}^* > 0$$

$$sign\left(\frac{\partial z_t}{\partial x_0}\right) = alternately\ positive\ and\ negative,\ if \quad \gamma > 0 \Leftrightarrow F_{xy}^* < 0$$

(2) $$sign\left(\frac{\partial z_t}{\partial \alpha_i}\right) = sign\left(\frac{\partial x^*}{\partial \alpha_i}\right).$$

(3) $$sign\left(\frac{\partial z_t}{\partial \beta}\right) = sign\left(\frac{\partial x^*}{\partial \beta}\right).$$

7. THE QUADRATIC CASE

In this section we study a particular case and apply the results obtained in previous sections. In this case function F in the objective is quadratic; this is a very interesting case since it does not require that the initial x_0 be around the steady state and we can be sure of the existence of an optimal solution: results have a global nature rather than a local one.

We consider the following problem:

$$\underset{\{x_{t+1}\}_{t=0}^{\infty}}{Max} \sum_{t=0}^{\infty} \beta^t F[x_t, x_{t+1}, a,b,c,d,e,f] \qquad \text{with } x(0) = x_0 \text{ given} \qquad \textbf{(QP)}$$

where $F[x_t, x_{t+1}, a,b,c,d,e,f] = \frac{1}{2}ax_t^2 + \frac{1}{2}bx_{t+1}^2 + cx_tx_{t+1} + dx_t + ex_{t+1} + f$

In this case: $X = \Re^+ \cup \{0\}$; $\qquad A = \{\alpha = (a,b,c,d,e,f)\} \subset \Re^6$

$\Gamma(x,\alpha) = \Re, \ \forall x \in X, \ \forall \alpha \in A$; $\qquad \beta \in (0,1)$

We assume that $a<0, b<0, ab-c^2>0$, so that F is strictly convex in x,y, for any $\alpha \in A$. In order to assure that assumption (A.3) hold, we assume that for any $y \in X, \alpha \in A$ is $F_x(x,y,\alpha)=ax+cy+d>0$. The Euler equation is in this case:

$$cx_t + bx_{t+1} + e + \beta(ax_{t+1} + cx_{t+2} + d) = 0, \quad \forall t = 0,1,2\ldots \qquad [14]$$

Therefore, the steady state is defined by:

$$x^* = -\frac{e+\beta d}{c(1+\beta)+b+\beta a} \qquad [15]$$

which exists and it is unique if and only if $c(1+\beta)+b+\beta a \neq 0$.

Euler equation (17) can also be written as:

$$\beta c x_{t+2} + (b+\beta a)x_{t+1} + cx_t = (\beta c + b + \beta a + c)x^* \qquad [16]$$

Assume $c \neq 0 (i.e. F^*_{xy} \neq 0)$. The necessary and sufficient condition for saddlepoint stability is:

$$b + \beta a + |c|(1+\beta) < 0$$

This condition guarantees that there is only one steady state x*. The roots of the characteristic equation associated with (16) are:

$$\lambda_1 = \frac{1}{2}\left(-\frac{b+\beta a}{\beta c} + \sqrt{\left(\frac{b+\beta a}{\beta c}\right)^2 - \frac{4}{\beta}}\right) \; ; \; \lambda_2 = \frac{1}{2}\left(-\frac{b+\beta a}{\beta c} - \sqrt{\left(\frac{b+\beta a}{\beta c}\right)^2 - \frac{4}{\beta}}\right) \qquad (17)$$

Assuming there is saddlepoint stability: $x_t^* = z(t;\alpha,\beta,x_0) = A_1\lambda_1^t + A_2\lambda_2^t + x^*(\alpha,\beta)$

- if c>0, then $0<\lambda_2<1<\lambda_1$. Making $A_1=0$, $x(0)=x_0=A_2+x^*$, we have

$$x_t^* = (x_0 - x^*)\lambda_2^t + x^*$$

- if c<0, then $\lambda_2 < -1 < \lambda < 0_1$. Making $A_2=0$, $x(0)=x_0=A_1+x^*$, we have

$$x_t^* = (x_0 - x^*)\lambda_1^t + x^*$$

Therefore,

$$\forall x_0 \in X, \; x_t^* = (x_0 - x^*)\lambda_i^t + x^*, \quad \text{with, } i=1 \text{ if } c<0, \text{ and } i=2 \text{ if } c>0 \qquad [18]$$

is the only optimal solution to (QP), which also verifies $\lim_{t\to\infty} x_t^* = x^*$.

In order to study the steady state comparative statics we could directly calculate the partial derivatives of x* with respect to parameters a,b,c,d,e,f,β from equation (15). We could also use Theorem 3. Corollary 3 gives us some useful results:

1. $sign\left(\frac{\partial x^*}{\partial \beta}\right) = sign(F_x^*) = sign((a+c)x^* + d)$
2. Since $F^*_{ya} = F^*_{yd} = F^*_{yf} = 0$, we obtain:

   $sign\left(\frac{\partial x^*}{\partial a}\right) = sign(F^*_{xa}) = sign(x^*)$

   $sign\left(\frac{\partial x^*}{\partial d}\right) = sign(F^*_{xd}) = sign(1)$, which is positive.

   $sign\left(\frac{\partial x^*}{\partial f}\right) = sign(F^*_{xf}) = 0$
3. Since $F^*_{xb} = F^*_{xe} = 0$, we obtain:

   $sign\left(\frac{\partial x^*}{\partial b}\right) = sign(F^*_{yb}) = sign(x^*)$

   $sign\left(\frac{\partial x^*}{\partial e}\right) = sign(F^*_{ye}) = sign(1)$, which is positive

In order to obtain comparative dynamics results we use equations (18) and (17). We apply Theorem 4 to obtain:

$$\frac{\partial x_t^*}{\partial x_0} = \lambda_i^t ;$$

$$\frac{\partial x_t^*}{\partial \alpha_i} = \frac{\partial x^*}{\partial \alpha_i}(1 - \lambda_i^t) \quad \text{for } \alpha_i = a,b,c,d,e,f ;$$

$$\frac{\partial x_t^*}{\partial \beta} = \frac{\partial x^*}{\partial \beta}(1 - \lambda_i^t)$$

with i=1 if c<0, and i=2 if c>0. We could also directly use Corollary 4.

8. AN ECONOMIC EXAMPLE: OPTIMAL GROWTH

One of the most well known economic applications of dynamic optimization in discrete time is the one sector model of optimal growth. We could write this family of models as the following maximization problem:

$$\underset{\{k_{t+1}\}_{t=0}^{\infty}}{Max} \sum_{t=0}^{\infty} \beta^t U[C_t]$$

$$s.t. \quad C_t \leq C[k_t, k_{t+1}, \delta]$$
$$C_t \geq 0 \quad k_t \geq 0$$

where:

- $k_t \geq 0$ is the capital per worker at period "t". k_0 is given.
- $U(C_t)$ is the utility function of the representative consumer (the standard assumptions are imposed: U'>0, U''<0).The inequality in the restriction becomes equality because the utility function is always increasing.
- $C_t(k_t, k_{t+1}, \delta) \geq 0$ is the consumption function for which we assume $C_x>0$, $C_y<0$. In order to simplify this example we will assume the following form for this consumption function: $C_t(k_t, k_{t+1}, \delta) = f(k_t) - \delta k_{t+1}$

- f is the production function and δ is the capital depreciation coefficient, (the standard assumptions, f'>0, f''<0, are imposed).

Therefore, the maximization problem becomes:

$$\underset{\{k_{t+1}\}_{t=0}^{\infty}}{Max} \sum_{t=0}^{\infty} \beta^t U[f(k_t) - \delta k_{t+1}]$$

$$s.t. \quad 0 \leq k_{t+1} \leq \frac{1}{\delta} \cdot f(k_t)$$

For convenience we write now the first and second derivatives of the utility function:

$$F(k_t, k_{t+1}, \delta) = U[f(k_t) - \delta k_{t+1}]$$

$$F_x = U'[f(k_t) - \delta k_{t+1}] f'(k_t) > 0$$

$$F_y = -\delta U'[f(k_t) - \delta k_{t+1}] < 0$$

$$F_{xx} = U''[.][f'(k_t)]^2 + U'[.]f''(k_t) < 0$$

$$F_{yy} = \delta^2 U''[.] < 0$$

$$F_{xy} = -\delta U''[.]f'(k_t) > 0$$

Steady State

Using the formulas above, we can write the Euler condition in the steady state as:

$$F_y^* + \beta F_x^* = 0 \Leftrightarrow -\delta U'^* + \beta f'^* U'^* = 0$$

By the assumptions of positive marginal utility and decreasing marginal productivity we obtain the only possible steady state amount of Capital from:

$$f'(k^*) = \frac{\delta}{\beta} \Rightarrow k^* = (f')^{-1}\left(\frac{\delta}{\beta}\right)$$

We can use Theorem 1 to test for saddle-point stability:

$$F^*_{yy} + \beta F^*_{xx} + (1+\beta)\left|F^*_{xy}\right| =$$

$$= \delta^2 U''^* + \beta\left[U''^*\left(f'^*\right)^2 + U'^* f''^*\right] - (1+\beta)\delta U''^* f'^* =$$

$$= \beta U'^* f''^* < 0$$

We conclude that there is saddle-point stability in this problem. In addition, from $F_{xy} > 0$ we know that the roots of the characteristic equation are real and positive and therefore, near the steady state, there is no oscillation of the state variable (capital) in the optimal path: if the initial amount of capital is higher (lower) than the steady state level, the optimal path will be monotonously decreasing (increasing).

Comparative Statics

Using the corresponding corollaries above, we can sign the partial derivatives of the steady state level of capital. Beginning with parameter β, from Corollary 3 (1):

$$sign\left(\frac{\partial k^*}{\partial \beta}\right) = sign(F^*_x) \Rightarrow \frac{\partial k^*}{\partial \beta} > 0$$

which means that an increase in the valuation of the utility obtained in the future will increase the optimal steady state level of capital. The optimal decision consists on consuming less in the first period, in order to be able to produce and consume more in the following periods.

With respect to the capital depreciation coefficient δ, we obtain from Corollary 3 (2):

$$sign\left(\frac{\partial k^*}{\partial \delta}\right) = sign(F^*_{y\delta} + \beta\, F^*_{x\delta})$$

Since:

$$F^*_{y\delta} + \beta\, F^*_{x\delta} = -U'^* + \delta U''^* k^* - \beta f'^* U''^* k^* = -U'^* < 0$$

we conclude:

$$\frac{\partial k^*}{\partial \delta} < 0$$

That is, if capital depreciation rate increases, then the steady state level of capital will be reduced. The optimal decision will be to consume more in the first period, reducing the level of production and consumption in the future.

Both steady state comparative statics results could also be obtained directly from the above expression for the steady state level of capital and using the assumptions of positive and decreasing productivity of capital (f'>0, f''<0).

Local Comparative Dynamics

From Corollary 4 we obtain the following local comparative dynamics results:

$$\frac{\partial k_t}{\partial k_0} = \lambda_1^t > 0, \text{ with } 0 < \lambda_1 < 1$$

$$sign\left(\frac{\partial k_t}{\partial \delta}\right) = sign\left(\frac{\partial k^*}{\partial \delta}\right) \Rightarrow \frac{\partial k_t}{\partial \delta} < 0$$

$$sign\left(\frac{\partial k_t}{\partial \beta}\right) = sign\left(\frac{\partial k^*}{\partial \beta}\right) \Rightarrow \frac{\partial k_t}{\partial \beta} > 0$$

The sign of the effects of changes in δ and β on the optimal level of capital in each period (k_t^* with t>0) are the same as those of the effects on the steady state level of capital (k^*). However, from Theorem 4 we know that the amount of these effects is smaller for k_t^* (with t finite) than for k^*. We can represent both steady state comparative statics results and local comparative dynamics results in the following figure:

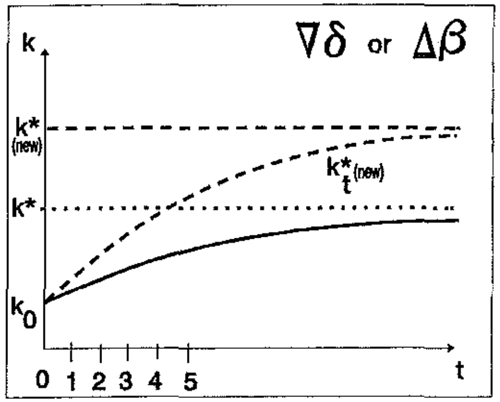

Figure 1

Interpretation of the Euler Condition

Euler condition can be written as:

$$\beta F^*_x(t+1) = - F^*_y(t)$$

This is a typical marginalist result. Capital in period (t+1) affects utility in both period (t) (in a negative way) and period (t+1) (in a positive way). Optimality in the path of the stock of capital requires that the positive marginal utility of capital (t+1) in period (t+1) (corrected by the discount factor β), should be equal to the negative marginal utility of capital (t+1) in period (t). If the former were larger than the later, it would be profitable to increase the amount of capital (t+1), and therefore, we would not be in an optimal path; in the optimal path, capital (t+1) would be increased until its total inter temporal marginal utility is zero.

Interpretation of the sign of F_{xy}

The concavity assumption about the general function F requires:

$$F^*_{xx} < 0,\ F^*_{yy} < 0,\ F^*_{xx}\ F^*_{yy} > (F^*_{xy})^2$$

These conditions hold in our example. In general the cross second derivative F^*_{xy} could have any sign, as far as its absolute value is not too high. However in our example, under the standard assumptions on the utility function (U''<0) and the production function (f'>0), we have.

$$F^*_{xy} = -\delta U''^* f'^* > 0$$

This sign can intuitively be explained be the following chain of reasoning:

$$\Delta k_{t+1} \rightarrow \nabla C_t \rightarrow \Delta U'(C_t) \rightarrow \Delta F_x$$
$$\textit{with } \Delta = \textit{increase},\ \nabla = \textit{decrease}$$

In this case we already know there is no oscillation, which can also be understood from the condition for an optimal path (Euler condition):

$$\phi(k_t,k_{t+1},k_{t+2}) = F_y(k_t,k_{t+1}) + \beta F_x(k_{t+1},k_{t+2}) = 0$$

The partial derivatives of ϕ have the following signs: $\phi_{k_t} > 0,\ \phi_{k_{t+1}} < 0,\ \phi_{k_{t+2}} > 0$. We also know that in the steady state Euler condition must hold:

$$\phi(k^*,k^*,k^*) = 0$$

Therefore, if $k_0 > k^*$, then $k^*_1 > k^*$ in order to maintain $\phi = 0$. And if $k^*_1 > k^*$, then $k^*_2 > k^*$. And so on. So, there is no oscillation.

In an optimal growth model like the one we have developed in this example, there is not much economic sense in either $F^*_{xy} = 0$ nor $F^*_{xy} < 0$. In the former, the link between the optimisation decisions in the different periods would be broken and, therefore, optimal decision would be the same in all periods. In the latter oscillations would rise in the optimal path. In both cases the standard assumptions on the utility and the production functions would not hold.

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