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A SOLUTION METHOD FOR A CLASS OF *LEARNING BY DOING* MODELS WITH MULTIPLICATIVE UNCERTAINTY

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ABSTRACT

We present a solution method to find the closed form optimal solution for a class of *learning by doing* models when multiplicative uncertainty is introduced in the cost reduction function, which is assumed to be piecewise linear. Previous literature does not study the case with uncertainty in this function. We consider a monopolist, facing a linear demand function. The optimal policy for the resulting problem is piecewise linear. Furthermore, the optimal output increases with unit cost for certain values of the latter. Numerical examples are provided.

RESUMEN

Se presenta un método que permite encontrar la solución óptima en bucle cerrado para una familia de modelos *learning by doing*, cuando se introduce incertidumbre multiplicativa en la función de reducción de costes, que se supone lineal a trozos. En la literatura previa no se estudia el caso de incertidumbre en esta función. Se considera un monopolista, con una función de demanda lineal. La política óptima para el problema resultante es lineal a trozos. Además, el output óptimo crece con el coste unitario para ciertos valores del mismo. Se proporcionan ejemplos numéricos.

JEL CLASSIFICATION: C61, D20.
KEYWORDS: Mathematical Programming, Learning by Doing, Uncertainty.

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1 Introduction

The firms of some industries have output cost reduction over time due to learning. Through the repetition of their activity, that is, increasing the accumulated output, the firms improve their ability and so they reduce their production cost. This is known as *learning by doing*, and it has been observed in industries at their early stage.

In general, all of the papers on *learning by doing* can be grouped into three categories:

i) empirical evidence: these works look for empirical evidence on *learning by doing* using data for a specific industry. The first one dates back to Wright (1936), who observed that the time required for the construction of an aircraft decreased with the number of aircrafts already produced. Some more recent works are: Joskow (1979) and Lester and McCabe (1993) for electric power in nuclear plants, Dick (1991) for microconductors in Japan, Lieberman (1984) for the chemical industry in U.S.A., Lieberman (1987) for the pharmaceutical industry, Jarmin (1994) for the rayon industry, Argote et al. (1990) for the shipping industry.

ii) operations research: these works are summarized in the Encyclopedia of Operations Research and Management Science (Gass and Harris (1996)), in four topics: learning curves (Loerch), learning (Buck), cost analysis (Balut and Gullegde) and cost effectiveness analysis (Womer).

iii) economic theory: economic implications of *learning by doing* are studied. The first of these works is by Arrow (1962). Within this framework, the papers which are closely related to the present paper are: Fudenberg and Tirole (1983), Stokey (1986) and Dasgupta and Stiglitz (1988). These works ask the question: What is the optimal behaviour of firms when there is *learning by doing*? The answer depends on the industrial structure which is assumed, so different structures are considered: monopoly, social planner, oligopoly, and competitive equilibrium. In general, they find general properties of the optimal policy of the firms for each case, under assumptions for the demand function and cost reduction process. All of these works deal with deterministic models. Whenever functional forms for demand and cost reduction are not specified, only general results are available. On the other hand, in order to obtain greater insight in some aspects, specifying functional forms becomes necessary, and so Fudenberg and Tirole (1993) and Dasgupta and Stiglitz (1988) specify a linear demand and a piecewise linear cost reduction function, in a two period model. In such a model, *learning by doing* takes place in the first period, whereas the second period is interpreted as the mature stage of the industry.

In the present paper, we extend the deterministic optimization problem of a monopolist with *learning by doing* in a two period model, with linear demand function and piecewise linear cost reduction function (Dasgupta and Stiglitz (1988)), to a stochastic case where there is multiplicative uncertainty in the cost reduction process. We find the analytical solution to this problem, that is, the optimal output rule in every period as a function of the parameters. Two features of this analytical optimal policy are: i) the optimal output rule is piecewise linear and continuous; ii) the optimal output in the first period is an increasing function of the unit cost, for certain values of the latter. From the analytical solution, we can give an explanation for this second point.

In section 2 we state the problem. In section 3 we present the solution method. In section 4 we show that the optimal policy is a continuous function in every period and we discuss some of its properties. In section 5 we present numerical examples. Finally we present conclusions in section 6.

2 The model

We consider a monopolist, risk neutral, facing no entry, with *learning by doing* in two periods, labelled as 0 and 1. The monopolist chooses output in every period, denoted by $q(0)$ and $q(1)$, so as to maximize the discounted expected profit flow. The discount parameter is λ. The demand function is constant over time and linear, concretely, the inverse demand function is:

$$p(t) = a - bq(t) \quad \text{for } t = 0,1 \tag{1}$$

where $p(t)$ is the price in period t.

In period 0, $q(0)$ is produced at a unit cost $c(0)$, which is given. There are no fixed costs. That output is sold at a price given by **(1)** for $t=0$, and then a realization of a random variable, $\beta(0)$, takes place, such that the unit cost in period 1, $c(1)$, is given by: $c(1)=\max\{\tau,c(0)-\beta(0)q(0)\}$. After $c(1)$ is generated by this equation, period 1 starts. In this period the monopolist observes $c(1)$ and then chooses $q(1)$, which is sold at a price given by **(1)** for $t=1$, and the problem finishes. The *learning by doing* effect is contained in: $c(1)=\max\{\tau,c(0)-\beta(0)q(0)\}$. It is a natural extension to a stochastic case of the one proposed by Dasgupta and Stiglitz (1988), where $\beta(0)$ is a fixed constant. We assume that $\beta(0)$ has a discrete probability distribution with a finite number of possible realizations.

So, the problem is:

Problem (P):

$$\underset{q(0),q(1)}{\text{MAX}} \quad E\{\sum_{t=0}^{1} \lambda^t (a-bq(t)-c(t))q(t) \;/\; c(0)\} \tag{2}$$

subject to:

$$c(1) = \max\{\tau, c(0)-\beta(0)q(0)\} \tag{3}$$

$$c(0) \text{ is given} \tag{4}$$

$$q(t) \geq 0 \; ; \; t = 0,1 \tag{5}$$

■

The probability distribution of $\beta(0)$ is known. The possible realizations for this variable are $\beta_1,\ldots,\beta_n$, such that:

$$0 < \beta_1 < \ldots < \beta_n \tag{6}$$

and $\text{Prob}(\beta(0)=\beta_i)=p_i$, with $p_i>0$, for $i=1,\ldots,n$; and $\sum_{i=1}^{n} p_i = 1$.

Other assumptions are:

$$c(0) > \tau \tag{7}$$

$$a > c(0) \tag{8}$$

$$\tau > 0 \; ; \; b > 0 \; ; \; \lambda \in [0,1] \tag{9}$$

The assumption **(6)** ensures that the cost cannot increase, and also, along with **(7)**, it ensures that the cost can decrease with positive output. The assumption **(8)** is usual in the literature on *learning by doing*, and it ensures an optimal positive output in every period. Finally, in **(9)** we assume that: the minimum attainable cost is positive ($\tau>0$), the demand function is decreasing ($b>0$), and the discount parameter lies between 0 and 1.

For the **problem (P)**, defined by **(2)** to **(5)**, assuming also **(6)** to **(9)** our target is to find the analytical solution, as a function of the parameters: λ, a, b, τ, $c(0)$, $\beta_1,\ldots,\beta_n$, $p_1,\ldots,p_n$. The problem stated is a generalization to a stochastic problem from a deterministic one previously studied (Alvarez and Cerdá (1996)). Now it is assumed that the ability to learn, measured by $\beta(0)$, is random. In Appendix I, **figure 1** represents the order in which the state variable, $c(t)$, the control variable, $q(t)$, and the random variable, $\beta(0)$, appear.

3 Solution for problem (P)

The **problem (P)** is a two period stochastic dynamic optimization problem, in discrete time, with complete observation. We solve the problem by using Dynamic Programming, that is, by finding the analytical solution to the Bellman's equation.

The key of the solution method is to identify the role of τ as a binding constraint. It may occur that, under the optimal solution, we have $\text{Prob}(c(1)=\tau)=0$. This is so when, in terms of the other parameters of the problem, the difference between $c(0)$ and τ is large enough. In this case τ does not represent any constraint. If the difference between $c(0)$ and τ is slightly smaller, then there is a positive probability for $c(1)=\tau$, under the optimal solution. Likewise, we may consider all of the possible cases, even that in which $\text{Prob}(c(1)=\tau)=1$. The solution method that we present identifies which case actually occurs (what is the probability for $c(1)=\tau$ under the optimal solution) given the parameters, and it solves in every case the corresponding problem.

Now we present the solution method. To begin with, we give some definitions referring to **problem (P)**.

Definition 1. Let $\pi(0)=\{q(0),q(1)\}$ be. We say that $\pi(0)$ is a feasible policy if $q(i)\geq 0$ for $i=0,1$. We denote by $S(0)$ the set of all feasible policies.

Definition 2. Let $\pi(0)\in S(0)$ and $c(0)$ be given, we define:

$$V(c(0),\pi(0),0) = E\left\{ \sum_{t=0}^{1} \lambda^t(a-bq(t)-c(t))q(t) \;/\; c(0) \right\}$$

where $c(1)$ is given by **(3)**.

The previous definitions characterize the set of all possible decisions that the monopolist can take in the two periods, $S(0)$, and, for one of them, $\pi(0)$, the expected profit, $V(c(0),\pi(0),0)$. Now

we characterize the optimal decisions.

Definition 3. Let $\pi^*(0)=\{q^*(0),q^*(1)\}\in S(0)$ be. We say that $\pi^*(0)$ is an optimal policy if: $V(c(0),\pi(0),0)\leq V(c(0),\pi^*(0),0)$ for every $\pi(0)\in S(0)$.

Definition 4. We define the value function as:

$$V^*(c(1),1)=(a-bq^*(1)-c(1))q^*(1) \quad \text{(for the second period)}$$
$$V^*(c(0),0)=V(c(0),\pi^*(0),0) \quad \text{(for the first period)}$$

The function we define now summarizes all the relevant information, in terms of **problem (P)**, of the probability distribution of $\beta(0)$.

Definition 5. For $k=0,1,2$ and $j=0,\ldots,n$, we define:

$$h_k(j) = \begin{cases} \sum_{i=1}^{j} \beta_i^k p_i & \text{if } j \in \{1,\ldots,n\} \\ 0 & \text{if } j = 0 \end{cases}$$

Note that, for any given probability distribution for $\beta(0)$, the latter function is well defined and its values can be easily calculated.

Now we present the results which allow us to find the optimal policy for **problem (P)**. We show that, given the interval of possible values for $c(0)$ (which is given by **(7)** and **(8)**), this optimal policy for period 0, that is $q^*(0)$, is a piecewise linear function of $c(0)$.

The next theorem shows when $\text{Prob}(c(1)=\tau)=0$ under $\pi^*(0)$, and it also obtains the optimal policy in this case. As we have mentioned, this occurs when the difference between $c(0)$ and τ is *large enough*. So the theorem defines firstly what *large enough* is. Next notation is used later in the theorem.

$$K(1) = \frac{1}{4b} \;;\quad \phi(1) = \frac{1}{2b} \tag{10}$$

$$\phi_{n+1}(0) = \frac{1+2\lambda K(1)h_1(n)}{2b-2\lambda K(1)h_2(n)} \tag{11}$$

$$K_{n+1}(0) = \lambda K(1)+\frac{1}{2}[1+2\lambda K(1)h_1(n)]\phi_{n+1}(0) \quad \textbf{(12)}$$

Theorem 1 If:

$$c(0) > \frac{\tau+\beta_n\phi_{n+1}(0)a}{1+\beta_n\phi_{n+1}(0)} \quad \textbf{(13)}$$

$$b > \lambda K(1)h_2(n) \quad \textbf{(14)}$$

then, for **problem (P)**:

i) $q^*(0)=\phi_{n+1}(0)(a\text{-}c(0))$; $V^*(c(0),0)=K_{n+1}(0)(a\text{-}c(0))^2$, and $q^*(1)=\phi(1)(a\text{-}c(1))$; $V^*(c(1),1)=K(1)(a\text{-}c(1))^2$; where $\phi_{n+1}(0)$, $K(1)$, $K_{n+1}(0)$, $\phi(1)$ are defined in **(10)** to **(12)**.

ii) $\text{Prob}(c(1)=\tau)=0$, under $\pi^*(0)$.

■

The proof of all theorems and propositions is left to Appendix II.

Theorem 1, under **(14)**, gives the optimal policy when $c(0)$ is greater than a given value (condition **(13)**), and it ensures that in this case, the unit cost does not reach τ under $\pi^*(0)$. The condition in **(14)** ensures the concavity of the objective function in the Bellman equation.

Now the question is: What does occur when c(0) is smaller than the quantity on the right hand side of **(13)**? The next proposition gives the answer.

Proposition 1. If **(14)** holds and **(13)** does not hold, then, under $\pi^*(0)$, $\text{Prob}(c(1)=\tau)\geq p_n$.

■

So, if **(13)** and **(14)** hold, **problem (P)** is solved by **theorem 1**. If **(13)** does not hold, but **(14)** still holds, the problem is not solved yet, but we already know that, under $\pi^*(0)$, $\text{Prob}(c(1)=\tau)\geq p_n$ holds. If this is the case, then we go to the next theorem (taking $j=n$). Next notation is used later in the theorem.

Let $j=2,\dots,n$, be; and also:

$$\phi_j(0) = \frac{1+2\lambda K(1)h_1(j-1)}{2b-2\lambda K(1)h_2(j-1)} \quad \textbf{(15)}$$

$$K_j(0) = \lambda K(1)h_0(j-1)+\frac{1}{2}[1+2\lambda K(1)h_1(j-1)]\phi_j(0) \quad \textbf{(16)}$$

$$K_{j,0}(0) = \lambda K(1)[1-h_0(j-1)](a-\tau)^2 \quad \textbf{(17)}$$

Theorem 2. Let $j=2,\dots,n$, be; if $\text{Prob}(c(1)=\tau)\geq 1\text{-}h_0(j\text{-}1)$ and also:

$$\frac{\tau+\beta_{j-1}\phi_j(0)a}{1+\beta_{j-1}\phi_j(0)} < c(0) \leq \frac{\tau+\beta_j\phi_j(0)a}{1+\beta_j\phi_j(0)} \quad \textbf{(18)}$$

$$b > \lambda K(1)h_2(j-1) \quad \textbf{(19)}$$

then, for **problem (P)**:

i) $q^*(0)=\phi_j(0)(a\text{-}c(0))$; $V^*(c(0),0)=K_j(0)(a\text{-}c(0))^2+K_{j,0}(0)$, and $q^*(1)=\phi(1)(a\text{-}c(1))$; $V^*(c(1),1)=K(1)(a\text{-}c(1))^2$; where $\phi_j(0)$, $K_j(0)$ and $K_{j,0}(0)$ are defined in **(15)** to **(17)**, and $\phi(1)$ and $K(1)$ are defined in **(10)**.

ii) $\text{Prob}(c(1)=\tau)=1\text{-}h_0(j\text{-}1)$, under $\pi^*(0)$.

■

Let us suppose that, from **proposition 1**, we know that $\text{Prob}(c(1)=\tau)\geq p_n$ under $\pi^*(0)$. Then we go to **theorem 2** and check if **(18)** and **(19)** hold for $j=n$ (the former is a condition which identifies values for $c(0)$ such that $\text{Prob}(c(1)=\tau)=1\text{-}h_0(j\text{-}1)$ under $\pi^*(0)$, the latter is a concavity condition similar to that in **(14)**). If so, **problem (P)** is already solved: the analytical solution is given by the theorem and we also know from the theorem that $\text{Prob}(c(1)=\tau)=p_n$, under $\pi^*(0)$. On the other hand, if **(18)** does not hold but **(19)** does hold, then we go to **proposition 2**, which is stated now, and we check if **(20)** holds for $j=n$. If this latter case occurs, **proposition 2** says that in fact, under $\pi^*(0)$, not only does $\text{Prob}(c(1)=\tau)\geq p_n$ but also $\text{Prob}(c(1)=\tau)\geq p_n+p_{n-1}$, and so we go back to **theorem 2** to check if the hypothesis of the theorem holds for $j=n\text{-}1$.

Proposition 2. Let $j=2,\dots,n$, be. If $\text{Prob}(c(1)=\tau)\geq 1\text{-}h_0(j\text{-}1)$, **(19)** holds and furthermore:

$$c(0) \leq \frac{\tau+\beta_{j-1}\phi_j(0)a}{1+\beta_{j-1}\phi_j(0)} \tag{20}$$

then, under $\pi^*(0)$, $\text{Prob}(c(1)=\tau) \geq 1-h_0(j\text{-}2)$.

■

If we go this way from $j=n$ to 2 and the hypothesis of **proposition 2** holds for $j=2$, then, under $\pi^*(0)$, $\text{Prob}(c(1)=\tau)=1$. In this case **theorem 2** is no longer useful, and we go to **theorem 3**, which is presented now. Previously, we introduce some notation.

$$\phi_1(0) = \frac{1}{2b} \tag{21}$$

$$K_1(0) = \frac{1}{4b} \; ; \quad K_{1,0}(0) = \lambda K(1)(a-\tau)^2 \tag{22}$$

Theorem 3. If, under $\pi^*(0)$, $\text{Prob}(c(1)=\tau)=1$, and also:

$$c(0) \leq \frac{\tau+\beta_1\phi_1(0)a}{1+\beta_1\phi_1(0)} \tag{23}$$

then, for **problem (P)**:

i) $q^*(0)=\phi_1(0)(a-c(0))$; $V^*(c(0),0)=K_1(0)(a-c(0))^2+K_{1,0}(0)$, and $q^*(1)=\phi(1)(a-c(1))$; $V^*(c(1),1)=K(1)(a-c(1))^2$; where $\phi_1(0)$, $K_1(0)$ and $K_{1,0}(0)$ are defined in **(21)** and **(22)** and $\phi(1)$ and $K(1)$ are defined in **(10)**.

■

Theorems 1, 2 and **3** establish a partition for the interval of possible values for $c(0)$, that is (τ,a), such that, if $c(0)$ belongs to some of the intervals of the partition, we can apply some of the previous theorems, obtaining the optimal policy for **problem (P)**. The **figure 2**, in Appendix I, shows this partition, with the theorem that must be applied in each case. Now we show that the ordering of the points which define the partition in (τ,a) is the one which is drawn in that figure.

Theorem 4. Let $b>\lambda K(1)h_2(j\text{-}1)$ for $j=1,\dots,n+1$, be, then:

i) $\tau < \dfrac{\tau+\beta_1\phi_1(0)a}{1+\beta_1\phi_1(0)}$

ii) $\dfrac{\tau+\beta_{j-1}\phi_j(0)a}{1+\beta_{j-1}\phi_j(0)} < \dfrac{\tau+\beta_j\phi_j(0)a}{1+\beta_j\phi_j(0)}$; for $j = 2,\dots,n$

iii) $\dfrac{\tau+\beta_j\phi_j(0)a}{1+\beta_j\phi_j(0)} < \dfrac{\tau+\beta_j\phi_{j+1}(0)a}{1+\beta_j\phi_{j+1}(0)}$; for $j = 1,\dots,n$

iv) $\dfrac{\tau+\beta_n\phi_{n+1}(0)a}{1+\beta_n\phi_{n+1}(0)} < a$

where $\phi_{n+1}(0)$ is defined in **(11)**, $\phi_j(0)$ is defined in **(15)**, and $\phi_1(0)$ is defined in **(21)**.

■

Given the inequality **iii)** in this theorem we have that, such as is shown in **figure 2** in Appendix I, there are subintervals in (τ,a) for which **problem (P)** is not solved by the previous theorems. The next theorem gives the optimal policy for this case. First we introduce some notation.

Let $j=1,\dots,n$, and also:

$$K_{Fj,2}(0) = -\frac{1}{\beta_j}-\frac{b}{\beta_j^2}+\lambda K(1)\left[h_0(j-1)-\frac{2h_1(j-1)}{\beta_j}+\frac{h_2(j-1)}{\beta_j^2}\right] \tag{24}$$

$$K_{Fj,1}(0) = \frac{a+\tau}{\beta_j}+\frac{2b\tau}{\beta_j^2}-2\lambda aK(1)\left[h_0(j-1)-\frac{h_1(j-1)}{\beta_j}\right]+\frac{2\lambda\tau K(1)}{\beta_j}\left[h_1(j-1)-\frac{h_2(j-1)}{\beta_j}\right] \tag{25}$$

$$K_{Fj,0}(0) = -\frac{a\tau}{\beta_j}-\frac{b\tau^2}{\beta_j^2}-2\lambda aK(1)\left[\tau(1-h_0(j-1))-\frac{\tau h_1(j-1)}{\beta_j}\right]+\lambda K(1)\left[\tau^2[1-h_0(j-1)]+\frac{\tau^2h_2(j-1)}{\beta_j^2}+a^2\right] \tag{26}$$

Theorem 5. Let $j=1,\dots,n$, be. If **(14)** and **(19)** hold, and also:

$$\frac{\tau+\beta_j\phi_j(0)a}{1+\beta_j\phi_j(0)} < c(0) \leq \frac{\tau+\beta_j\phi_{j+1}(0)a}{1+\beta_j\phi_{j+1}(0)} \tag{27}$$

then:

i) $q^*(0)=\beta_j^{-1}(c(0)-\tau)$; $V^*(c(0),0)=K_{Fj,0}(0)+K_{Fj,1}(0)c(0)+K_{Fj,2}(0)c(0)^2$, and $q^*(1)=\phi(1)(a\text{-}c(1))$; $V^*(c(1),1)=K(1)[a\text{-}c(1)]^2$; where $\phi_j(0)$ is defined in **(15)**, $K_{Fj,0}(0)$, $K_{Fj,1}(0)$ and $K_{Fj,2}(0)$ are defined in **(24)** to **(26)**, and $\phi(1)$ and $K(1)$ are defined in **(10)**.

ii) $\text{Prob}(c(1)=\tau)=1-h_0(j-1)$, under $\pi^*(0)$.

∎

This completes the solution method for **problem (P)**.

4 Continuity of the optimal policy and an explanation for its increasing pieces

From the theorems in the last section it is obvious that the optimal output in period 1 is a linear and decreasing function of the cost for that period. However, the optimal output in the first period is piecewise linear and, as we move from the lowest to the highest possible value of $c(0)$, it increases or decreases with $c(0)$ (we illustrate this with numerical examples in the next section). This is due to the maximum function in equation **(3)**. In the next result we show that the optimal output in the first period is a continuous function of $c(0)$.

Theorem 6. If **(14)** and **(19)** hold, then $q^*(0)$ is a continuous function of $c(0)$.

∎

The explanation for the decreasing pieces of the optimal output in the first period is simple. Let us suppose that $c(0)$ is close to τ, such that the optimal policy is defined by **theorem 3**. Then, it is optimal to reach τ w.p.1 in period 1 and, furthermore, this can be done by producing the quantity which maximizes current profit in period 0 ($q^*(0)=\phi_1(0)(a-c(0))$). This quantity decreases when $c(0)$ increases, so $q^*(0)$ is a decreasing function of $c(0)$ when the latter is very close to τ. Now, if we take a value for $c(0)$ slightly higher such that optimal policy is defined by **theorem 5** (for $j=1$), then it is still optimal to reach τ w.p.1 in period 1, but now the quantity which maximizes current output in period 0 ($\phi_1(0)(a-c(0))$) is not enough for that, and so the monopolist will produce in period 0 a quantity higher than $\phi_1(0)(a-c(0))$. More specifically, he will produce as much as necessary to reach τ w.p.1 in period 1, that is $\beta_1^{-1}(c(0)-\tau)$. Note that this quantity *increases* when $c(0)$ increases, that is, the higher $c(0)$, the higher the necessary quantity to reach τ w.p.1 in period 1. So we have an interval of values for $c(0)$ where $q^*(0)$ is increasing in $c(0)$. If we now take a new value for $c(0)$ higher than the previous one and such that the optimal policy is now given by **theorem 2** (for $j=2$), then it is not longer optimal to reach τ in period 1 w.p.1, but with a lower probability, say $1-h_0(1)$. Now, in period 0 the monopolist maximizes the current and expected discounted profit, this leads to $q^*(0)=\phi_2(0)(a-c(0))$, and this quantity is decreasing in $c(0)$. Let us take $c(0)$ again slightly higher than before, such that the optimal policy is now given by **theorem 5** (for $j=2$). As in the previous case, now it is still optimal to reach τ in period 1 w.p. $1-h_0(1)$, but now $\phi_2(0)(a-c(0))$ is not enough for that. Then the monopolist will produce a higher quantity which ensures that τ is reached in period 1 w.p. $1-h_0(1)$, that is $\beta_2^{-1}(c(0)-\tau)$, so we have again that $q^*(0)$ increases with $c(0)$. If we continue through the whole interval of possible values for $c(0)$, we find that increasing and decreasing functions for $q^*(0)$ keep coming up alternately.

5 Examples with specific parameter values

In this section, we present the closed form optimal policy for some examples with specific parameter values. Since the solution method allows for any finite probability distribution for $\beta(0)$, we solve **problem (P)** for different probability distributions, and we fix all other parameters. We take $a=20$, $b=10$, $\lambda=0.9$ and $\tau=1$. More specifically, we concentrate on the changes which take place in the optimal policy when we consider different probability distributions where all of them have the same mean. This exercise has no obvious results *a priori* since the monopolist is assumed to be risk neutral. We take probability distributions for $\beta(0)$ which have $E(\beta(0))=2$ and have different values for variance and asymmetry. We consider seven probability distributions, which are specified in the **table 1** in Appendix III, and, for each, the graphs in that appendix show $q^*(0)$ as a function of $c(0)$.

The distributions in the table are explained now. All of the distributions have positive probability on five points. Furthermore, denoting by $\beta^i(0)$ the random variable defined by the distribution in example labelled as i, we have: i) $Var(\beta^3(0))>Var(\beta^1(0))>Var(\beta^2(0))$, though these distributions have the same extreme points and are symmetric; ii) $Var(\beta^4(0))>Var(\beta^1(0))>Var(\beta^5(0))$, and these distributions do not have the same extreme points though they are symmetric. If we look at the corresponding graphs in Appendix III we see that: a) the higher the variance is (regardless of whether we keep the extreme points fixed or not) the larger is the smallest subinterval in the horizontal axis (values of $c(0)$) containing all points where $q^*(0)$ is increasing in $c(0)$; b) any effect of the variance on the range of values that $q^*(0)$ takes it is not clear (i.e. considering distributions from 1 to 5, number 5 has the lowest variance and number 4 has the highest one, and there are not large differences in the values on the vertical axis for these distributions).

The distributions in examples 6 and 7 are not symmetric, though they have their mean equal to two (like all other distributions considered). If we look at the corresponding graphs, the asymmetry does not seem to play any role itself, as variance does, in determining the length of the smallest subinterval which contains all points of $c(0)$ where $q^*(0)$ is increasing in $c(0)$.

6 Conclusions

In this paper we find the analytical solution for a class of *learning by doing* models with multiplicative uncertainty, formulated in discrete time. We consider a monopolist, risk neutral, facing no entry, who operates in a market with linear demand. The monopolist has cost reduction over time as long as he accumulates experience (*learning by doing*). We take the cost reduction function from Dasgupta and Stiglitz (1988), where future unit cost is a piecewise linear function of the current unit cost and output, and we introduce multiplicative uncertainty in it. Previous literature does not consider uncertainty in the cost reduction function. Given the demand and the cost reduction functions, the monopolist chooses quantities so as to maximice the discounted expected profit thoroughout two periods. The first period is interpreted as the *enfant* phase (Stokey (1986)) whereas the second is the mature phase of the industry.

The analytical solution is found by solving Bellman's equation associated with the problem stated. Two remarkable features of the function which gives the optimal output of the first period for every possible value of the initial unit cost are: i) it is piecewise linear and continuous; ii) there are subintervals of possible values of the initial unit cost where *it* is increasing. An explanation for these properties is found on the basis of the solution method.

Some examples with specific parameter values are provided.

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Appendix I

figure 1:

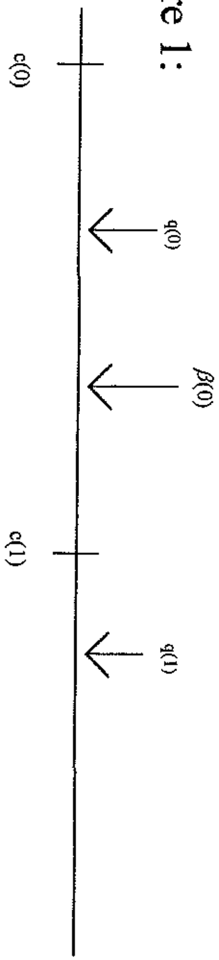
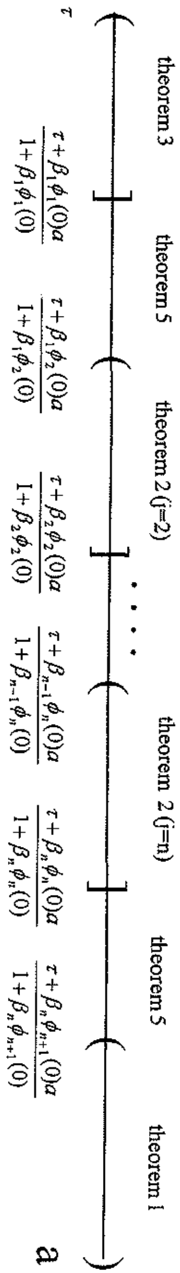


figure 2:



Appendix II

This Appendix contains the proof of all theorems and propositions in the paper. Before that, we present two lemmas which are used later in the proofs. The first one refers to the distribution of the random variable $c(1)$ conditional on $c(0)$ and $q(0)$. The second one will be proved to be the maximization problem which faces the monopolist in every period.

Lemma 1. Let the random variable $c_1=\max\{\tau, c-\beta q\}$ be, with $0<\tau<c$, $q\geq 0$, and β is a random variable which takes values in $\{\beta_1,\ldots,\beta_n\}$ with probabilities: $\text{Prob}(\beta=\beta_i)=p_i$, with $0<\beta_1<\ldots<\beta_n$. Let the function $h_k(j)$ be as in **definition 5**.

If $c-\beta_j q\leq\tau<c-\beta_{j-1}q$, for $j\in\{1,\ldots,n+1\}$, then:

$E(c_1/c,q)=\tau[1-h_0(j-1)]+ch_0(j-1)-qh_1(j-1)$;

$E(c_1^2/c,q)=\tau^2[1-h_0(j-1)]+c^2h_0(j-1)+q^2h_2(j-1)-2cqh_1(j-1)$;

where, for a given distribution for β, we take: $\beta_0=0$ and β_{n+1} any finite value satisfying $\beta_{n+1}>\beta_n$, with: $\text{Prob}(\beta=\beta_0)=p_0=\text{Prob}(\beta=\beta_{n+1})=p_{n+1}=0$. Note that this does not modify the distribution of β. ■

Proof of lemma 1. Given $q\geq 0$, we have $0=\beta_0 q\leq\beta_1 q\leq\ldots\leq\beta_n q\leq\beta_{n+1}q$, and also: $c-\beta_{n+1}q\leq c-\beta_n q\leq\ldots\leq c-\beta_1 q\leq c-\beta_0 q=c$. So, if $c-\beta_j\leq\tau<c-\beta_{j-1}q$, for $j\in\{1,\ldots,n+1\}$, then: $\text{Prob}(c_1=\tau)=1-h_0(j-1)$, and $\text{Prob}(c_1=c-\beta_i q)=p_i$ for $i\in\{0,\ldots,j-1\}$. Hence:

$$E[c_1/c,q] = \tau[1-h_0(j-1)]+\sum_{i=0}^{j-1}(c-\beta_i q)p_i = \tau[1-h_0(j-1)]+ch_0(j-1)-qh_1(j-1)$$

$$E[c_1^2/c,q] = \tau^2[1-h_0(j-1)]+\sum_{i=0}^{j-1}(c-\beta_i q)^2 p_i = \tau^2[1-h_0(j-1)]+c^2h_0(j-1)+q^2h_2(j-1)-2cqh_1(j-1)$$

■

Lemma 2. Consider the problem:

$$\begin{aligned}&\text{MAX } f(q) = (a-bq-c)q-2\lambda aKz_1(c,q)+\lambda Kz_2(c,q)\\ &q\geq 0\end{aligned}$$

with $\lambda>0$, $K\geq 0$, $a>c$; where $z_1(c,q)=\tau[1-h_0(i^*-1)]+ch_0(i^*-1)-qh_1(i^*-1)$; $z_2(c,q)=\tau^2[1-h_0(i^*-1)]+c^2h_0(i^*-1)+q^2h_2(i^*-1)-2cqh_1(i^*-1)$, and $i^*\in\{1,\ldots,n+1\}$.

If $b>\lambda Kh_2(i^*-1)$, then the optimal solution is: $q^*=\phi(a-c)$, where: $\phi=(1+2\lambda Kh_1(i^*-1))(2b-2\lambda Kh_2(i^*-1))^{-1}$. Also, $f(q^*)=K'(a-c)^2-\lambda K[a^2-(1-h_0(i^*-1))(a-\tau)^2]$; with: $K'=\lambda Kh_0(i^*-1)+½[1+2\lambda Kh_1(i^*-1)]\phi$.

■

Proof of lemma 2. Denoting by μ the multiplier associated with the constraint $q \geq 0$, and by q^* the solution, the Kuhn-Tucker conditions are: (i) $f'(q^*)+\mu=0$; (ii) $\mu \geq 0$; (iii) $q^* \geq 0$; (iv) $\mu q^*=0$. We have: $f'(q)=a-c-2bq+2\lambda aKh_1(i^*-1)+\lambda K[2h_2(i^*-1)q-2ch_1(i^*-1)]$. If $\mu>0$, then $q^*=0$ and so $f'(0)+\mu=0$. It is $f'(0)+\mu=(a-c)[1+2\lambda Kh_1(i^*-1)]+\mu$, and from the hypothesis of the proposition the first term is non-negative, and so it cannot be $f'(0)+\mu=0$ with $\mu>0$. Hence $\mu=0$, and, from (i), $f'(q^*)=0$ must hold. From the last equality, we have $q^*=\phi(a-c)$. The sufficient condition for maximum is: $f''(q)<0 \Leftrightarrow -2b+2\lambda Kh_2(i^*-1)<0 \Leftrightarrow b>\lambda Kh_2(i^*-1)$. By substitution of q^* in the objective function, we have: $f(q^*)=K'(a-c)^2-\lambda K[a^2-(1-h_0(i^*-1))(a-\tau)^2]$.

■

Proof of theorem 1. Let us consider the next auxiliary problem:

$$\underset{q(0),q(1)}{\text{MAX}} \quad E\{\sum_{t=0}^{1} \lambda^t(a-bq(t)-c(t))q(t) \ / \ c(0)\}$$

subject to: $c(1)=c(0)-\beta(0)q(0)$, $c(0)$ given and $q(t) \geq 0$ for $t=0,1$.

If the optimal policy of this auxiliary problem satisfies the additional constraint $c(1) \geq \tau$, then it is also the optimal policy of **problem (P)**. The proof has two steps. In the first step, we solve the auxiliary problem, and we show that the optimal solution is the one given in part i) of the theorem. In the second step, we prove that, under the optimal policy of the auxiliary problem and the conditions given in the hypothesis of the theorem, $c(1)>\tau$ holds with probability 1.

First step: for the auxiliary problem, we denote by $\pi_a^*(0)=\{q_a^*(0),q_a^*(1)\}$ to the optimal policy, and by $V_a^*(c(t),t)$ for $t=0,1$ the value function in period t. The functional equation of Bellman, associated with the auxiliary problem is:

$$V_a^*(c(t),t) = \underset{q(t)\geq 0}{\text{Max}} \quad \{[a-bq(t)-c(t)]q(t)+\lambda E[V_a^*(c(t+1),t+1)]\}$$

where $c(1)=c(0)-\beta(0)q(0)$ and we take $V_a^*(c(2),2)=0$. Now we prove that, under the hypothesis, we have: $V_a^*(c(t),t)=V^*(c(t),t)$ and $q_a^*(t)=q^*(t)$ for $t=0,1$. In effect, for $t=1$, the problem to be solved is static and deterministic, in fact it is the problem of **lemma 2** taking $c=c(1)$, $q=q(1)$, $K=0$ and i^* arbitrary. From that lemma we have: $q_a^*(1)= \phi(1)(a-c(1))$, and the sufficient condition for maximum holds since $b>0$. We also have: $V_a^*(c(1),1)=K(1)(a-c(1))^2$. Now, let $c(0)$ be given, since: $V_a^*(c(1),1)=K(1)a^2-2aK(1)c(1)+K(1)c(1)^2$, we have:

$$V_a^*(c(0),0) = \underset{q(0)\geq 0}{\text{Max}} \quad \{[a-bq(0)-c(0)]q(0)-2\lambda aK(1)E[c(1)/c(0),q(0)]+\lambda K(1)E[c(1)^2/c(0),q(0)]\}+\lambda K(1)a^2$$

where, according to **lemma 1**, it is $E[c(1)/c(0),q(0)]=\tau[1-h_0(n)]+c(0)h_0(n)-q(0)h_1(n)$, and $E[c(1)^2/c(0),q(0)]=\tau^2[1-h_0(n)]+c(0)^2h_0(n)+q(0)^2h_2(n)-2c(0)q(0)h_1(n)$. Hence, to solve this problem we can apply **lemma2** with $c=c(0)$, $q=q(0)$, $i^*=n+1$, and add to the objective function the constant $\lambda K(1)a^2$. We have: $q_a^*(0)= \phi_{n+1}(0)(a-c(0))$, $V_a^*(c(0),0)=K_{n+1}(0)(a-c(0))^2$, and this concludes the first step.

Second step: We prove that, under $\pi_a^*(0)$, it is $\text{Prob}(c(1)=\tau)=0$. In effect, $\text{Prob}(c(1)=\tau)=0 \Leftrightarrow c(0)-\beta_n q_a^*(0)>\tau \Leftrightarrow c(0)-\beta_n\phi_{n+1}(0)(a-c(0))>\tau \Leftrightarrow c(0)>(\tau+\beta_n\phi_{n+1}(0)a)(1+\beta_n\phi_{n+1}(0))^{-1}$, and this concludes the second step.

■

Proof of proposition 1. If $\text{Prob}(c(1)>\tau)=0$ under $\pi^*(0)$, then the **problem (P)** is analogous to the auxiliary problem given in the proof of **theorem 1**. Under the optimal policy of that auxiliary problem, $\text{Prob}(c(1)>\tau)=0 \Leftrightarrow c(0)>(\tau+\beta_n\phi_{n+1}(0)a)(1+\beta_n\phi_{n+1}(0))^{-1}$, as has been proved in the second step of **theorem 1**, and it does not happen by hypothesis of the proposition, so it must be $\text{Prob}(c(1)>\tau)>0$ under $\pi^*(0)$, and the lowest positive probability for that is p_n, when: $c(0)-\beta_n q^*(0) \leq \tau < c(0)-\beta_{n+1}q^*(0)$.

■

Proof of theorem 2. Let us consider the next auxiliary problem:

$$\underset{q(0),q(1)}{\text{MAX}} \quad E\{\sum_{t=0}^{1} \lambda^t(a-bq(t)-c(t))q(t) \ / \ c(0)\}$$

with $q(t) \geq 0$ for $t=0,1$ and where $c(1)$ conditioned on $c(0)$ and $q(0)$ has the probability distribution: $\text{Prob}(c(1)=\tau/c(0),q(0))=1-h_0(j-1)$; $\text{Prob}(c(1)=c(0)-\beta_i q(0)/c(0),q(0))=p_i$ for $i=1,\dots,j-1$.

If we already know that $\text{Prob}(c(1)=\tau) \geq 1-h_0(j)$ under $\pi_a^*(0)$ then, if the optimal policy of this auxiliary problem satisfies the additional constraint $c(0)-\beta_j q_a^*(0) \leq \tau < c(0)-\beta_{j-1}q_a^*(0)$, then it is also the optimal policy of **problem (P)**. Now the proof has two steps. In the first step, we solve the auxiliary problem, and we show that the optimal solution is the one given in part i) of the theorem. In the second step, we prove that, under the optimal policy of the auxiliary problem and the conditions given in the hypothesis of the theorem, $c(1)=\tau \Leftrightarrow \beta(0)$ takes value in $\{\beta_j,\dots,\beta_n\}$ that is, $\text{Prob}(c(1)=\tau)=1-h_0(j-1)$.

First step: for the auxiliary problem, we denote by $\pi_a^*(0)=\{q_a^*(0),q_a^*(1)\}$ the optimal policy, and by $V_a^*(c(t),t)$ for $t=0,1$ the value function in period t. The functional equation of Bellman, associated with the auxiliary problem is:

$$V_a^*(c(t),t) = \underset{q(t)\geq 0}{\text{Max}} \quad \{[a-bq(t)-c(t)]q(t)+\lambda E[V_a^*(c(t+1),t+1)]\}$$

subject to the probability distribution of $c(1)$ conditioned on $c(0)$ and $q(0)$ given for the auxiliary problem, and taking $V_a^*(c(2),2)=0$. Now we prove that, under the hypothesis, $V_a^*(c(t),t)=V^*(c(t),t)$ and $q_a^*(t)=q^*(t)$ for $t=0,1$ holds. In effect, for $t=1$, the problem to be solved is the problem of **lemma 2** taking $c=c(1)$, $q=q(1)$, $K=0$ and i^* arbitrary. From that lemma we have: $q_a^*(1)= \phi(1)(a-c(1))$, and the sufficient condition for maximum holds since $b>0$. Furthermore: $V_a^*(c(1),1)=K(1)(a-c(1))^2$. Now, let $c(0)$ be given, we have:

$$V_a^*(c(0),0) = \underset{q(0)\geq 0}{\text{Max}} \quad \{[a-bq(0)-c(0)]q(0)-2\lambda aK(1)E[c(1)/c(0),q(0)]+\lambda K(1)E[c(1)^2/c(0),q(0)]\}+\lambda K(1)a^2$$

where, according to **lemma 1**, it is $E[c(1)/c(0),q(0)]=\tau[1-h_0(j-1)]+c(0)h_0(j-1)-q(0)h_1(j-1)$, and $E[c(1)^2/c(0),q(0)]=\tau^2[1-h_0(j-1)]+c(0)^2h_0(j-1)+q(0)^2h_2(j-1)-2c(0)q(0)h_1(j-1)$. So, to solve this problem we apply **lemma2** with $c=c(0)$, $q=q(0)$, $i^*=j$, and add to the objective function the constant $\lambda K(1)a^2$. We have: $q_a^*(0)=\phi_j(0)(a-c(0))$, $V_a^*(c(0),0)=K_j(0)(a-c(0))^2+K_{j,0}(0)$, and this concludes the first step.

Second step: We prove that, under $\pi_a^*(0)$, it is $\text{Prob}(c(1)=\tau)=1-h_0(j-1)$. In effect,
$\text{Prob}(c(1)=\tau)=1-h_0(j-1)\Leftrightarrow c(0)-\beta_j q_a^*(0)\leq\tau<c(0)-\beta_{j-1}q_a^*(0)\Leftrightarrow$
$c(0)-\beta_j\phi_j(0)(a-c(0))\leq\tau<c(0)-\beta_{j-1}\phi_j(0)(a-c(0))\Leftrightarrow$
$(\tau+\beta_{j-1}\phi_j(0)a)(1+\beta_{j-1}\phi_j(0))^{-1}<c(0)\leq(\tau+\beta_j\phi_j(0)a)(1+\beta_j\phi_j(0))^{-1}$, and this concludes the second step.

■

Proof of proposition 2. If $\text{Prob}(c(1)>\tau)=1-h_0(j-1)$ under $\pi^*(0)$, then the **problem (P)** is analogous to the auxiliary problem given in the proof of **theorem 2**. Under the optimal policy of that auxiliary problem we have:
$\text{Prob}(c(1)=\tau)=1-h_0(j-1)\Leftrightarrow(\tau+\beta_{j-1}\phi_j(0)a)(1+\beta_{j-1}\phi_j(0))^{-1}<c(0)\leq(\tau+\beta_j\phi_j(0)a)(1+\beta_j\phi_j(0))^{-1}$
as has been proved in the second step of **theorem 2**, and it does not happen by hypothesis of the proposition. Since we already know that $\text{Prob}(c(1)=\tau)\geq 1-h_0(j-1)$ under $\pi^*(0)$, then it must be $\text{Prob}(c(1)=\tau)>1-h_0(j-1)$ that is, $\text{Prob}(c(1)=\tau)\geq 1-h_0(j-2)$, when $c(0)-\beta_{j-1}q^*(0)\leq\tau<c(0)-\beta_{j-2}q^*(0)$.

■

Proof of theorem 3. Let us consider the next auxiliary problem:

$$\underset{q(0),q(1)}{\text{MAX}} \quad E\{\sum_{t=0}^{1}\lambda^t(a-bq(t)-c(t))q(t)\}$$

subject to: $c(1)=\tau$, $c(0)$ given and $q(t)\geq 0$ for $t=0,1$.

If the optimal policy of this auxiliary problem satisfies the additional constraint $c(0)-\beta_1 q_a^*(0)\leq\tau$, then it is also the optimal policy of **problem (P)**.

For the auxiliary problem, we denote by $\pi_a^*(0)=\{q_a^*(0),q_a^*(1)\}$ the optimal policy, and by $V_a^*(c(t),t)$ for $t=0,1$ the value function in period t. The functional equation of Bellman, associated with the auxiliary problem is:

$$V_a^*(c(t),t) = \underset{q(t)\geq 0}{\text{Max}} \quad \{[a-bq(t)-c(t)]q(t)+\lambda E[V_a^*(c(t+1),t+1)]\}$$

where $c(1)=\tau$ and $V_a^*(c(2),2)=0$. Now we prove that, under the hypothesis, we have: $V_a^*(c(t),t)=V^*(c(t),t)$ and $q_a^*(t)=q^*(t)$ for $t=0,1$. In effect, for $t=1$, the problem to be solved is the one of **lemma 2** taking $c=c(1)$, $q=q(1)$, $K=0$ and i^* arbitrary. From that lemma we have: $q_a^*(1)= \phi(1)(a-c(1))$, and the sufficient condition for maximum holds since $b>0$. We also have: $V_a^*(c(1),1)=K(1)(a-c(1))^2$. Now, let $c(0)$ be given, we have:

$$V_a^*(c(0),0) = \underset{q(0)\geq 0}{\text{Max}} \quad \{[a-bq(0)-c(0)]q(0)-2\lambda aK(1)E[c(1)/c(0),q(0)]+\lambda K(1)E[c(1)^2/c(0),q(0)]\}+\lambda K(1)a^2$$

where, according to **lemma 1**, it is $E[c(1)/c(0),q(0)]=\tau$, and $E[c(1)^2/c(0),q(0)]=\tau^2$. Hence, to solve this problem we can apply **lemma2** with $c=c(0)$, $q=q(0)$, $i^*=1$, and add to the objective function the constant $\lambda K(1)a^2$. We have: $q_a^*(0)= \phi_1(0)(a-c(0))$, $V_a^*(c(0),0)=K_1(0)(a-c(0))^2+K_{1,0}(0)$.

On other hand, we have that, under $\pi_a^*(0)$, $\text{Prob}(c(1)=\tau)=1\Leftrightarrow c(0)-\beta_1 q_a^*(0)\leq\tau\Leftrightarrow$ $c(0)-\beta_1\phi_1(0)(a-c(0))\leq\tau\Leftrightarrow c(0)\leq(\tau+\beta_1\phi_1(0)a)(1+\beta_1\phi_1(0))^{-1}$.

■

Proof of theorem 4. Since $0<\beta_1<\ldots<\beta_n$, we have, under **(14)** and **(19)**, $\phi_j(0)>0$ for $j=1,\ldots,n+1$. Then $(\tau+\beta_1\phi_1(0)a)(1+\beta_1\phi_1(0))^{-1}$ is a strict convex linear combination of τ and a and so (i) holds. To show (ii) note that it can be rewritten as:
$(1+\beta_{j-1}\phi_j(0))^{-1}\tau+(1-(1+\beta_{j-1}\phi_j(0))^{-1})a<(1+\beta_j\phi_j(0))^{-1}\tau+(1-(1+\beta_j\phi_j(0))^{-1})a$. From this inequality it is clear that both sides are strict convex linear combinations of τ and a, so the inequality holds if and

only if $(1+\beta_{j-1}\phi_j(0))^{-1}>(1+\beta_j\phi_j(0))^{-1}$, or equivalently $\beta_j>\beta_{j-1}$ which is true. To show (iii) note that it can be rewritten as: $(1+\beta_j\phi_j(0))^{-1}\tau+(1-(1+\beta_j\phi_j(0))^{-1})a<(1+\beta_j\phi_{j+1}(0))^{-1}\tau+(1-(1+\beta_j\phi_{j+1}(0))^{-1})a$. From this inequality it is clear that both sides are strict convex linear combinations of τ and a, so the inequality holds if and only if $(1+\beta_j\phi_j(0))^{-1}>(1+\beta_j\phi_{j+1}(0))^{-1}$, or equivalently $\phi_{j+1}(0)>\phi_j(0)$, which follows inmediately from the definitions of $\phi_j(0)$ and $\phi_{j+1}(0)$. To prove (iv) it is enough to note that $(\tau+\beta_n\phi_{n+1}(0)a)(1+\beta_n\phi_{n+1}(0))^{-1}$ is a strict convex linear combination of τ and a.

■

Proof of theorem 5. Note first that $q^*(1)$ and $V^*(c(1),1)$ are obtained as in **theorems 1, 2** and **3**. This remarks the fact that the optimal output and the value function for the second period are always the same function of the cost for that period. Now we calculate $q^*(0)$ and $V^*(c(0),0)$. From **proposition 2**, and the fact that $c(0)\leq(\tau+\beta_j\phi_{j+1}(0)a)(1+\beta_j\phi_{j+1}(0))^{-1}$, we have that $\text{Prob}(c(1)=\tau)\geq 1-h_0(j-1)$, and since $\text{Prob}(c(1)=\tau)\geq 1-h_0(j-1)\Leftrightarrow c(0)-\beta_j q(0)\leq\tau\Leftrightarrow q(0)\geq\beta_j^{-1}(c(0)-\tau)$ then, the optimal policy for **problem (P)** must verify the constraint $q(0)\geq\beta_j^{-1}(c(0)-\tau)$. In the auxiliary problem of the **theorems 2** (with $j=2,\ldots,n$) and **3** (with $j=1$), we consider the **problem (P)** with the lowest possible probability for $c(1)=\tau$[1] without taking into account the constraint $q(0)\geq\beta_j^{-1}(c(0)-\tau)$, and when doing so, the constraint is verified if and only if: $c(0)\in((\tau+\beta_{j-1}\phi_j(0)a)(1+\beta_{j-1}\phi_j(0))^{-1},(\tau+\beta_j\phi_j(0)a)(1+\beta_j\phi_j(0))^{-1}]$, which does not hold under the hypothesis of the theorem, since $c(0)>(\tau+\beta_j\phi_j(0)a)(1+\beta_j\phi_j(0))^{-1}$. Given the hypothesis of the theorem, the objective function, given $c(0)$, is concave, and so the best q(0) is the one which satisfies exactly the constraint, that is $q^*(0)=\beta_j^{-1}(c(0)-\tau)$. Given this value for $q^*(0)$, we have: $V^*(c(0),0)=K_{Fj,0}(0)+K_{Fj,1}(0)c(0)+K_{Fj,2}(0)c(0)^2$, and also $\text{Prob}(c(1)=\tau)=1-h_0(j-1)$.

■

Proof of theorem 6. Since **(14)** and **(19)** hold, the optimal output for the first period is given by **theorems 1** to **3** and **5**. It suffices to show continuity in the points of $c(0)$ which satisfy with equality **(18)**, **(23)** or **(27)**. From **theorem 4**, these points are: i) $(\tau+\beta_j\phi_{j+1}(0)a)(1+\beta_j\phi_{j+1}(0))^{-1}$ for $j=1,\ldots,n$ and ii) $(\tau+\beta_j\phi_j(0)a)(1+\beta_j\phi_j(0))^{-1}$ for $j=1,\ldots,n$. In the points given in i), the optimal output from the left is $\beta_j^{-1}(c(0)-\tau)$, and from the right it is $\phi_j(0)(a-c(0))$, for $j=1,\ldots,n$, so we must have: $\beta_j^{-1}((\tau+\beta_j\phi_{j+1}(0)a)(1+\beta_j\phi_{j+1}(0))^{-1}-\tau)=\phi_j(0)(a-(\tau+\beta_j\phi_{j+1}(0)a)(1+\beta_j\phi_{j+1}(0))^{-1})$, and both sides of this equality can be rewriten as $\phi_{j+1}(0)(a-\tau)(1+\beta_j\phi_{j+1}(0))^{-1}$. In the points given in ii), the optimal output from the left is $\phi_j(0)(a-c(0))$, and from the right it is $\beta_j^{-1}(c(0)-\tau)$, so we must have: $\beta_j^{-1}((\tau+\beta_j\phi_j(0)a)(1+\beta_j\phi_j(0))^{-1}-\tau)=\phi_j(0)(a-(\tau+\beta_j\phi_j(0)a)(1+\beta_j\phi_j(0))^{-1})$, and both sides of this equality can be rewritten as $\phi_j(0)(a-\tau)(1+\beta_j\phi_j(0))^{-1}$.

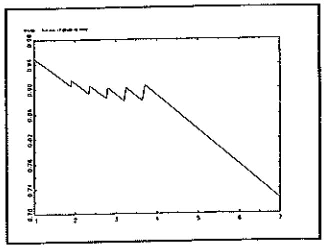
■

[1] When applying these theorems, we already know that $\text{Prob}(c(1)=\tau)\geq 1-h_0(j-1)$ must hold under $\pi^*(0)$, so we consider: $\text{Prob}(c(1)=\tau)=1-h_0(j-1)$.

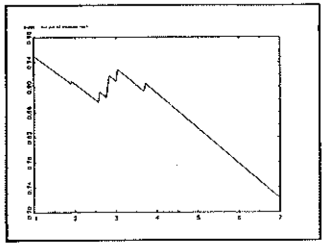
Appendix III

Table 1

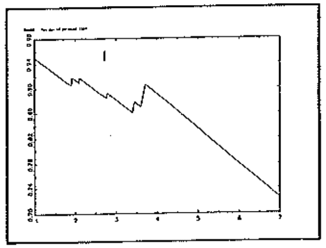
Example 1					
Value	1	1.5	2	2.5	3
Prob	0.2	0.2	0.2	0.2	0.2
Example 2					
Value	1	1.8	2	2.2	3
Prob	0.1	0.2	0.4	0.2	0.1
Example 3					
Value	1	1.2	2	2.8	3
Prob	0.3	0.15	0.1	0.15	0.3
Example 4					
Value	0.5	1.5	2	2.5	3.5
Prob	0.3	0.15	0.1	0.15	0.3
Example 5					
Value	1.6	1.8	2	2.2	2.4
Prob	0.1	0.2	0.4	0.2	0.1
Example 6					
Value	1/2	2	7/3	8/3	3
Prob	36/130	1/10	27/130	27/130	27/130
Example 7					
Value	1	4/3	5/3	2	3
Prob	27/130	27/130	27/130	1/10	36/130



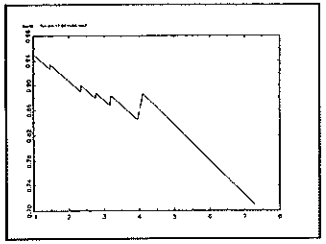
Ex 1



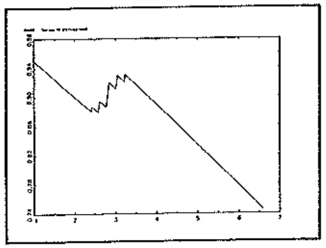
Ex. 2



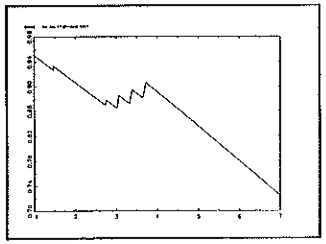
Ex 3



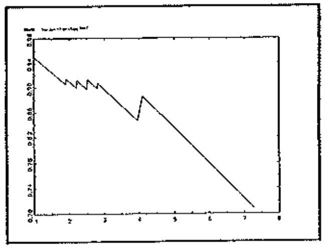
Ex. 4



Ex 5



Ex. 6



Ex. 7