Instituto Complutense de Análisis Económico
UNIVERSIDAD COMPLUTENSE
FACULTAD DE ECONOMICAS
Campus de Somosaguas
28223 MADRID

Teléfono 394 26 11 - FAX 294 26 13

Documento de trabajo

The Stochastic Bottleneck Linear Programming Problem

I.M. Stancu-Minasian
R. Caballero
E. Cerdá
M.M. Muñoz

No.9722 Diciembre 1997

Instituto Complutense de Análisis Económico
UNIVERSIDAD COMPLUTENSE

THE STOCHASTIC BOTTLENECK LINEAR PROGRAMMING PROBLEM

I.M. Stancu-Minasian*
Centre for Mathematical Statistics
The Romanian Academy.Bucharest. Romania.

R. Caballero
Departamento de Economía Aplicada (Matemáticas)
Universidad de Málaga. Spain.

E. Cerdá**
Departamento de Análisis Económico
Universidad Complutense de Madrid. Spain.

M.M. Múñoz
Departamento de Economía Aplicada (Matemáticas)
Universidad de Málaga. Spain.

ABSTRACT

In this paper we consider some stochastic bottleneck linear programming problems. In the case when the coefficients of the objective functions are simple randomized, the minimum-risk approach will be used for solving these problems. We prove that, under some positivity conditions, these stochastic problems are reduced to certain deterministic bottleneck linear problems. Applications of these problems to the bottleneck spanning tree problems and bottleneck investment allocation problems are given. A simple numerical example is presented.

RESUMEN

En este artículo se consideran algunos problemas de programación lineal estocástica "cuello de botella". Se utiliza la aproximación de mínimo-riesgo para el caso en que los coeficientes de las funciones objetivo de los problemas siguen aleatorización simple. Se demuestra que, bajo determinadas condiciones de positividad, estos problemas estocásticos se reducen a ciertos problemas lineales determinísticos "cuello de botella". Se dan dos aplicaciones de estos problemas y se presenta un ejemplo numérico.

Key Words: Stochastic Programming, Minimum-risk approach, Bottleneck problems, Spanning trees.

*The final version of this paper was written when I.M. Stancu-Minasian was visiting the Instituto Complutense de Análisis Económico, in the Universidad Complutense de Madrid, from October 1, 1997 to November 15, 1997 as invited researcher. He is grateful to the Institution.

**Correspondence to: Emilio Cerdá Tena. Departamento de Análisis Económico. Facultad de CC. Económicas y Empresariales. U.C.M. 28223 Somosaguas. Madrid. Spain. Ph. 34-1-394 23 39. Fax 34-1-394 26 13. E-mail ececo08@sis.ucm.es.

1 Introduction

In Frieze (1975) two algorithms (Primal and Threshold algorithm) are described for solving the bottleneck linear programming problem:

Problem BLP

$$\min\{z = \max\{c_j \mid j \in L(x)\}\}$$
$$\text{subject to} \ : \ Ax = b; x \geqslant 0.$$

where A,x,b and c=(c_1,...,c_n) are respectively an m×n matrix , an n-vector, a vector m-dimensional, and an n-vector and L(x) = $\{j \in I = \{1, ..., n\} \mid x_j > 0\}$.

Let

$$S = \{x \mid Ax = b; x \geqslant 0\}$$

the feasible set of problem BLP.

The problem BLP consists of finding a feasible solution which minimizes a bottleneck type objective.According to Bansal and Puri (1980,Theorem 1) the function z is a concave function , and the global minimum of z occurs at an extreme point of S (Corollary p.192).Moreover,any local optimal solution of Problem BLP is a global optimal solution of Problem BLP (Bansal and Puri (1980, Theorem 2),Seshan and Achary (1982, Theorem 2).

Garfinkel and Rao(1976) established a relationship between the problem BLP and a problem solvable by a "greedy algorithm" and developed two algorithms for solving the problem BLP.

The bottleneck problem was also studied by many authors. Bansal and Puri (1980) have given a procedure for ranking of solutions of the Problem BLP and also for finding its alternate k^{th} best (k$\geqslant$ 1) solutions.

Burkard and Rendl (1991) have studied the lexicographic bottleneck problem.

Minoux (1989) has given many applications of and studied the problem in which the objective function is the sum of a linear part and a bottleneck part.

Mathur et al. (1995) studied the bicriteria bottleneck linear programming problem

$$\min_{x \in S} \ (\mathrm{F(x)}\ ,\ \mathrm{T(x)})$$

where F and T are concave bottleneck functions and S is the non-empty feasible region defined by linear constraints.

Emelichev et al. (1995) studied a class of discrete vector problems with one linear criterion and several "bottleneck" criteria, and proved that any efficient solution can be obtained by solving a single-criterion problem with an aggregated criterion which is a linear function.

Some real-world problems can be modeled as bottleneck problems, as for instance some problems in plant layout (Francis and White (1974)),the political districting problem (Garfinkel and Nemhauser (1970)), and the m-center plant location problem (Garfinkel, Neebe and Rao (1974)).

The problem BLP is a generalization of two well-known bottleneck problems:The bottleneck assignment (Yechiali (1968),Garfinkel (1971),Gross (1959),Geetha and Vartak (1994)) and the bottleneck transportation (Yechiali (1971),Stancu-Minasian and Tigan (1985)).In what follows we refer to these two problems.

The bottleneck transportation problem has the form

Problem BTP

$$\min\{z = \max\{t_{ij} \mid (i,j) \in L^*(X)\}\}$$

$$\text{subject to} \;:\; \sum_{j=1}^{n} x_{ij} = a_i, i = 1, ..., m$$

$$\sum_{i=1}^{m} x_{ij} = b_j, j = 1, ..., n$$

$$x_{ij} \geqslant 0, i = 1, ..., m; j = 1, ..., n$$

where

a_i is the available amount at the i^{th} supply point,

b_j is the requirement at the j^{th} demand point,

t_{ij} is the transportation time (independent of the amount of commodity shipped) from i^{th} supply point to the j^{th} demand point,

$X=(x_{ij})$ is an element of the set of all feasible solutions of the classical transportation problem which we denote by S^*,

$L^*(X)=\{(i,j) \in \{1,...,m\} \times \{1,...,n\} \mid x_{ij} > 0\}$ is the positive graph of X.

The problem BTP is to find a transportation plan which makes the most time-consuming trip as short as possible.

If we consider the problem BTP in a production context,instead of the time t_{ij} we have the rate of production R_{ij} (on a production line) of a man

belonging to group (origin) i when he is assigned to job (destination) j. Here a_i´s are interpreted as the number of men available in the i^{th} group and b_j men required for the j^{th} job.

We have the problem

Problem PP

$$\max\{R = \min\{R_{ij} \mid (i,j) \in L^*(X)\}\}$$

Considering that R_{ij} is a continuous nonnegative random variable with distribution function $F_{ij}(.)$,Yechiali (1971) reduces the solving of the Problem PP to maximization of the expected rate of production of the line

$$\max_X \left\{E(R) = E(\min_{(i,j)\in L^*(X)} R_{ij})\right\} \tag{1.1}$$

He shows that for the family of Weibull distributions (and in particular Exponential distributions) with scale parameters λ_{ij} and shape parameter β,Problem (1.1) is reduced to solving a deterministic fixed charge transportation problem with nonlinear costs and with "set-up" cost matrix $((\lambda_{ij}))$

$$\min\left\{\sum_{(i,j)\in L^*(X)} \lambda_{ij}\right\} \tag{1.2}$$

In the particular case when m=n and $a_i = b_j = 1$ for all i and j then the problems (1.1) and (1.2) are transformed,respectively,into the stochastic bottleneck assignment problem and the assignment problem as considered by Yechiali (1968).

Unlike the method from Yechiali (1968 and 1971),Tigan and Stancu-Minasian (1985) use the minimum-risk approach for solving the Problem BTP in the case in which t_{ij} are random variables.

This approach consists of finding the optimal solution of the following problem

$$v(z) = \max\; P\{w \mid \max\{t_{ij} \mid (i,j) \in L^*(X)\} \leqq z\}$$

Tigan and Stancu-Minasian (1985) show that this problem is equivalent ,under certain hypotheses,to a deterministic bottleneck transportation problem.

In this paper we use the minimum-risk approach to obtain the solution of the stochastic bottleneck linear programming problem.It is shown that this problem can be reduced,under certain hypotheses,to a deterministic bottleneck linear problem.In the particular case of bottleneck transportation problem we rediscover the results presented by Tigan and Stancu-Minasian (1985).

The paper is organized in the following way.Section 2 is a main part of this paper,we expound our approach to solve the stochastic bottleneck linear programming problem.In Section 3 we consider a generalization of Problem BLP i.e. the minimax problems and in Section 4 we consider applications to the bottleneck spanning tree problems and investment allocation problems.In Section 5 we give a simple numerical example.

2 The minimum-risk approach

Now let c_j assume random values with simple randomization,i.e. they are of the form:

$$c_j = c_j' + t(\omega)c_j'' \text{ , } \forall j \in I \tag{2.1}$$

where c_j' and c_j'' are constants and $t(\omega)$ is a random variable in a probability space (Ω,K,P),with the continuous strictly increasing distribution function $T(.)$.

The minimum-risk problem corresponding to level z,associated with the bottleneck linear programming problem (Problem BLP) consists of finding the optimal solution of the following programming problem.

Find

$$v(z) = \max_{x\in S} P\left\{\omega \mid \max_{j\in L(x)} (c_j' + t(\omega)c_j'') \leqslant z\right\} \tag{2.2}$$

We assume that

$$c_j'' \geqslant 0, \forall j \in I \text{ and there exists at least } j \in I, \text{ such that } c_j'' > 0 \tag{2.3}$$

In what follows,we shall show that ,under assumption (2.3),the minimum-risk problem (2.2) can be solved by a deterministic bottleneck linear problem which does not depend on the distribution function of the random variable $t(\omega)$.

Theorem 1 *If assumption (2.3) holds and if the distribution function T is continuous and strictly increasing, then the minimum-risk solution of problem (2.2) does not depend on T and it can be obtained by solving the following bottleneck linear problem:*

$$\max_{x\in S} \min_{j\in L(X)} g_j$$

where

$$g_j = \begin{cases} (z-c_j')/c_j'', & if\ c_j'' \neq 0 \\ +\infty, & if\ z-c_j' \geqslant 0, and\ c_j''=0 \\ -\infty, & if\ z-c_j' < 0,\ and\ c_j''=0 \end{cases}$$

<u>Proof</u>:
From (2.1) we get :

$$F(x,z)=P\left\{\omega \mid \max_{j\in L(x)} (c_j' + t(\omega)c_j'') \leqslant z\right\} = P\left\{\omega \mid c_j' + t(\omega)c_j'' \leqslant z, \forall j \in L(x)\right\}$$

Hence,according to (2.3),we have :

$$F(x,z) = P\left\{\omega \mid t(\omega) \leqslant g_j, \forall j \in L(x)\right\} = P\left\{\omega \mid t(\omega) \leqslant \min_{j\in L(x)} g_j\right\} = T(\min_{j\in L(x)} g_j)$$

The problem (2.2) becomes

$$\max_{x\in S} F(x,z)=\max_{x\in S} T(\min_{j\in L(x)} g_j).$$

Hence,by virtue of the assumption that $T(.)$ is continuous and strictly increasing,we get:

$$v(z) = \max_{x\in S} F(x,z) = T(\max_{x\in S} \min_{j\in L(x)} g_j)$$

Thus,the theorem is proven.

We assume now that

$$c_j = c_j' + t_j(\omega)c_j'', \forall j \in I \tag{2.4}$$

where $t_j (j \in I)$ are independent random variables with continuous and strictly increasing distribution functions $T_j(.)$.

Also,we assume that

$$c_j'' > 0, \forall j \in I. \tag{2.5}$$

In this case,the minimum-risk solution of problem (2.2) depends on $T_j(.)$. Indeed,as in the previous case,we have:

$$\begin{aligned} F(x,z) &= P\left\{\omega \mid \max_{j\in L(x)}(c'_j + t_j(\omega)c''_j) \leqslant z\right\} = \\ &= P\left\{\omega \mid (c'_j + t_j(\omega)c''_j) \leqslant z, \forall j \in L(x)\right\} = \\ &= P\left\{\omega \mid t_j(\omega) \leqslant g_j, \forall j \in L(x)\right\} = \Pi_{j\in L(x)} T_j(g_j) \text{ ,where } g_j = \frac{z - c'_j}{c''_j} \end{aligned}$$

Hence,

$$\max_{x\in S} F(x,z) = \max_{x\in S} \Pi_{j\in L(x)} T_j(g_j) \qquad (2.6)$$

This problem is equivalent to the following optimization problem:

$$\max_{x\in S} \ln\left(\Pi_{j\in L(x)} T_j(g_j)\right) = \max_{x\in S} \sum_{j\in L(x)} \ln\left(T_j(g_j)\right) \qquad (2.7)$$

Hence,we have:

Theorem 2 *If the assumption (2.5) holds and the distribution functions T_j of $t_j(\omega)$ are continuous and strictly increasing,then the minimum-risk solution of the problem*

$$\max_{x\in S} P\left\{\omega \mid \max_{j\in L(x)}(c'_j + t_j(\omega)c''_j) \leqslant z\right\}$$

can be obtained by solving the problem (2.6) or,equivalently,the problem (2.7).

We remark that the problem (2.7) is a fixed charge problem,which however depends on the distribution functions T_j of $t_j(\omega)$.

3 Minimax problems

In this Section we consider a generalization of the Problem BLP.So we consider the following problem:

Problem GBLP

$$\min_{x\in S} \{z = \max\{c_j(x_j) \mid x_j > 0\}\}$$

where each $c_j(x_j)$ is a piecewise constant increasing function and S is the feasible set defined by linear constraints.

Here the coefficient $c_j(x_j)$ depends on x_j while in Problem BLP $c_j(x_j)$ is independent of x_j and is equal to c_j as long as $x_j > 0$.

For the solving of this problem , Achary et al. (1982) presented four algorithms : a)threshold algorithm, b) an upper bounding technique , c) a primal dual approach and d) a branch and bound algorithm.

In what follows,we consider that $c_j(x_j) = c_j x_j$ such that the problem GBLP becomes

Problem GBLP1

$$\min_{x\in S} \{z = \max\{c_j x_j \mid x_j > 0\}\}$$

This problem is referred to in the literature as one with a minimax objective function.

Minimax problems of this type arise in various contexts and have been studied by many authors.Kaplan (1974) considered a maximin problem and suggested a simple procedure when the problem possesses an optimum ray solution x, i.e. ,$c_j x_j = t, \forall j = 1, ..., n$.The general form of Kaplan's problem is solved by Ahuja(1985), which developed two algorithms:a parametric algorithm and a primal-dual algorithm.However these algorithms are presented for a minimax linear programming problem,but can be easily adapted to solve the maximin linear programming problem.

Yang and Shen (1988) give an algorithm which requires $0(n^2)$ operations to solve the problem.

$$\max\{z = \min\{c_j x_j\} \mid x_j > 0\}$$
$$\text{subject to: } \sum_j b_j x_j \leqslant m$$

where b_j, c_j, x_j, m are positive integer and $\sum_j b_j \leqslant m$.

In what follows,we shall assume that c_j are random with simple randomization, i.e.

$$c_j = c'_j + t(\omega)c''_j$$

where c'_j, c''_j (j=1,...,n) are constant scalars and $t(\omega)$ is a random variable in a probability space (Ω, K, P) with the continuous strictly increasing distribution function T(.).

We assume that

$$c''_j x_j \neq 0 \text{ for all } x_j \text{ and } j \in \{1, ..., n\} \tag{3.1}$$

The minimum-risk approach to the minimax problem GBLP1 consists of finding the optimal solution of the following programming problem

$$v(z) = \max_{x \in S} P\left\{\omega \mid \max_j c'_j(\omega) x_j \leqslant z\right\} \tag{3.2}$$

But the problem (3.2) is a Chebyshev problem in which the functions are of particular form

$$z_j(x) = c_j x_j$$

According to Stancu-Minasian (1984)(see,also, Tigan and Stancu-Minasian (1983)) the following theorem is immediate.

Theorem 3 *If assumption (3.1) holds and the distribution function $T(z)$ of $t(\omega)$ is continuous and strictly increasing,then the minimum-risk solution of problem (3.2) does not depend on $T(z)$ and can be obtained by solving the deterministic piecewise linear fractional programming problem*

$$\max_{x \in S} \min_j \frac{z - c'_j x_j}{c''_j x_j}, \text{ if } c''_j x_j > 0 \tag{3.3}$$

or

$$\min_{x \in S} \max_j \frac{z - c'_j x_j}{c''_j x_j}, \text{if } c''_j x_j < 0 \tag{3.4}$$

We remark that these problems can be solved by use of a parametric algorithm similar to Dinkelbach's algorithm for fractional programming.

Remark 1 *In the case of transportation problems,the form of Problem GBLP is*

$$\min_{X \in S^*} \{z = \max\{c_{ij}(x_{ij}) \mid x_{ij} > 0\}\}$$

where $c_{ij}(x_{ij})$ is the transportation cost from i^{th} supply point to the j^{th} demand point.

A particular case is that in which the transportation cost $c_{ij}(x_{ij})$ is directly proportional to the amount of commodity shipped,i.e.,

$$c_{ij}(x_{ij}) = c_{ij} x_{ij}$$

The problem GBLP1 becomes

$$\min_{X \in S^*} \{z = \max\{c_{ij} x_{ij} \mid x_{ij} > 0\}\} \tag{3.5}$$

This problem was studied by Achary and Seshan (1981).

In the case of simple randomization of the coefficients c_{ij}, a similar result to theorem 3 can be stated and proved for the minimax problem (3.5).

Remark 2 *In the particular case of a linear bottleneck assignment problem, the problem (3.5) was studied by Pfreschy (1995 and 1996).He proved that the expected value of the optimal solution tends towards the lower end of the range of cost coefficients for any distribution function as long as the upper end of the cost range is bounded.He also derives functions in n as explicit upper and lower bounds for the expected optimal value in the case of uniformly distributed (0 , 1) cost data.*

Remark 3 *Similar results can be obtained for a more general form of Problem GBLP1 i.e.*

$$\min_{x \in S} \{z = \max\{c_j x_j \mid j \in S_i; x_j > 0, i = 1, ..., p\}\}$$

where $(S_1, S_2, ..., S_p)$ is a partition of the index set $I=\{1, 2, ..., n\}$.

The deterministic case of this problem was studied by Gupta and Punnen (1989), who proposed two algorithms.

4 Applications

4.1 Bottleneck spanning tree problems

In this subsection we apply the results of the previous sections to study the bottleneck spanning tree problem.

Let $G = (N, E)$ be an undirect graph with n vertices $N = \{v_1, v_2, ..., v_n\}$ and m edges $E = \{e_1, e_2, ..., e_m\}$.

For each edge $e_j \in E$ there is an associated cost c_j. A Spanning tree $T = T(N, S)$ of G is a partial graph satisfying the following conditions:

a) T has the same vertex set as G,

b) $S \subseteq E, |S| = n-1$, where $|S|$ denotes the cardinality of set S,

c) T is connected.

Denote by $\Im$ the set of all spanning trees of the graph G.

A spanning tree T in $G = (N, E)$ can be represented by a vector of 0-1 variables $X = (x_1, ..., x_m)$,where

$$x_i = \begin{cases} 1, & if\ e_i \in T \\ 0, & otherwise \end{cases}$$

Conversely,if $\{e_i \mid x_i = 1\}$ becomes a spanning tree on G with vertex set N , $X = (x_1, ..., x_m)$ is also called spanning tree.Let f be the total cost (weight) associated with the spanning tree,

$$f(T) = \sum_{e \in S} w_e.$$

The minimal spanning tree problem is

Problem STP

$$\min \left\{ f(X) = \sum_{j=1}^{m} c_j x_j \mid X : \text{spanning tree} \right\}$$

The linear programming formulation of the minimal weight spanning tree problem is (Andersen et al. (1996))

$$\min\ f(X) = \sum_{j=1}^{m} c_j x_j$$

$$\text{subject to:} \quad \sum_{e \in E} x_e = n - 1 \tag{4.1}$$

$$\sum_{e \in E_p} x_e = |P| - 1, \forall P \subset N, P \neq \emptyset \tag{4.2}$$

$$x_e \geqslant 0 \text{ and integer}, \forall e \in E \tag{4.3}$$

where $E_p \{e \in E \mid e = (i, j)$ where i,j $\in P\}$.

Constraint (4.1) ensures that exactly n-1 edges are used,and (4.2) ensures that there are no cycles.

The bottleneck spanning tree problem is

Problem BSTP

$$\min\ \max\ \{c_j \mid x_j = 1\}$$

or equivalently

$$\min\ \max\ \{c_j \mid e_j \in T\ ,\ \text{T : spanning tree}\}$$

The minimal spanning tree problem is well-known and efficient algorithms for solving it exist (Chandrasekaran (1977), Cheritan and Tarjan (1976), Ford and Fulkerson (1962), Yao (1975), Andersen et al. (1996)). The stochastic spanning tree problem is considered by Ishii et al. (1981 and 1995) , Geetha and Nair (1993) and Mohd (1994). Ishii and Nishida (1983) consider a stochastic version of bottleneck spanning tree problem in the edge of which costs are random variables.

$$\min\ f$$

$$\text{subject to} \ : \ P\{\max\{c_j \mid e_j \in T\} \leqslant f\} \geqslant \alpha\ ,\ \text{T : spanning tree}$$

where each c_j is assumed to be distributed according to the normal distribution $N(\mu_j, \sigma_j^2)$, with mean μ_j and variance σ_j^2 , and they are mutually independent.

They show that, under reasonable restrictions,the problem can be reduced to a minimal bottleneck spanning tree problem in a deterministic case.Mohd (1994) introduced several modifications to the algorithm of Ishii et al. (1981), including theorems which concern the proxy problem.

Unlike of them , we consider now the minimum-risk approach for solving the stochastic bottleneck spanning tree problem.

We consider that the costs are linear functions of the same random variable t(ω) (the simple randomization case) having a distribution function T(z) continuous and strictly increasing i.e.

$$c_j = c_j' + t(\omega) c_j''\ (j = 1, ..., m)$$

The minimum-risk problem corresponding to level z,associated with the bottleneck spanning tree problem (Problem BSTP) consists of finding the optimal solution of the following programming problem.

$$\text{v(z)} \ = \max\ \text{P} \left\{ \omega \mid \max_j\ (c_j' + t(\omega) c_j''\ ) \leq z\ ;\ e_j \in T\ ,\ T : \text{spanning tree} \right\} \tag{4.4}$$

The following theorem can be proved by an analogous argument, like the one in the proof of Theorem 1.

We assume that $c''_j \neq 0$ for all j = 1,...,n. (4.5)

Theorem 4 *If assumption (4.5) holds and if the distribution function T is continuous and strictly increasing,then the minimum-risk solution of problem (4.4) does not depend on T and can be obtained by solving the following bottleneck spanning tree problem*

$$\max \ \min \left\{ \frac{z - c'_j}{c''_j} \mid e_j \in T \ , T : \textit{spanning tree} \right\} \ , \textit{if } c''_j > 0$$

or

$$\min \ \max \left\{ \frac{z - c'_j}{c''_j} \mid e_j \in T, T : \textit{spanning tree} \right\}, \textit{if } c''_j < 0$$

Remark 4 *Although the results presented here are restricted to the bottleneck spanning tree problem,the methods can be adapted and applied to various other problems in graphs.In particular,it is possible to derive similar results for bottleneck shortest path problems or bottleneck Steiner trees (for a deterministic case see,Sarrafzadeh and Wong (1992) and Ganley and Salowe (1996)).In the first case $\mathcal{F}$ represents path sets between two nodes and in the second case $\mathcal{F}$ represents the set of all Steiner trees.Given a set of vertices in which each vertex is labeled as demand or Steiner, a Steiner tree is a tree connecting all demand points and some (or none or all) Steiner points.Thus, the Steiner tree problem is more general than the spanning tree problem.*

Remark 5 *A similar approach can be applied to the stochastic bottleneck graph partition (BGP) problem.The BGP problem is to partition the nodes of a graph into two equally sized sets,so that the maximum edge weight in the cut separating the two sets is minimum (Hochbaum and Pathria (1996)).*

4.2 Bottleneck investments allocation problems

Now we shall present a different approach to investment allocation problems.These problems are classical and can be stated as follows:An investor wishes to invest in n production activities (or in n securities) a certain

amount of money.If we denote by x_i the percentage of the fund which is going to be invested in activity i (or in security i),then the vector $x = (x_1, ..., x_n)$ satisfies the constraints

$$\sum_{i=1}^{n} x_i = 1 \ , \ x_i \geqslant 0 \ ,$$

to which one can add other constraints based on economic considerations. The income corresponding to the investment strategy x is

$$V(x) = \sum_{i=1}^{n} \xi_i x_i$$

where ξ_i constitutes the income obtained when it is invested the whole in activity i.

The optimal selection problem for an investment portfolio is

$$\max \ V(x) = \sum_{i=1}^{n} \xi_i x_i$$

$$\text{subject to:} \qquad x \in S' = \left\{ x \mid \sum_{i=1}^{n} x_i = 1 \ , \ x_i \geqslant 0 \ , \mathrm{x} \in S \right\}$$

where S results from other economic constraints.

The problem becomes more complicated since ξ_i are not constants,but random variables. In the classical approach ,considering that ξ_i are normal variables,the problem is reduced to a nonlinear fractional programming problem (Stancu-Minasian (1997)).

Now we shall present a variant of this problem , different from the classic one.We suppose that we want to find a solution so as to maximize the minimum of the income ξ_i. The following bottleneck programming model arises:

$$\max_{x \in S'} \ \min \{ \xi_i \mid x_i > 0 \}$$

This problem can be approached by the method of the previous Section,i.e. the minimum-risk approach , and we obtain the following problem

$$\max_{x \in S'} \ P \{ \omega \mid \ \min \{ \xi_i \mid x_i > 0 \} \geqslant u \}$$

In the simple randomization case of the random variable ξ_i ($\xi_i = \xi'_i + t(\omega)\xi''_i$) in which $t(\omega)$ has a distribution function T which is continuous and strictly increasing, a similar result to Theorem 1 can be derived.

5 Numerical example

To illustrate our approach,we consider the following problem:

$$\begin{aligned} & \min\max\{c_j \mid x_j > 0\} \\ \text{subject to } : \quad & x_1 - x_4 - 6x_5 + x_6 + x_7 = 1 \\ & x_2 - x_4 - 4x_5 - x_6 + x_7 = 2 \\ & x_3 + 2x_4 - x_5 + x_6 = 0 \\ & x_i \geqslant 0, i = 1, ..., 7 \end{aligned}$$

where the coefficients $(c_1, c_2, ..., c_7)$ are random variables which depend linearly on the same random variable t ,whose distribution function is continuous and strictly increasing,as follows:

$$c_1 = 21 + 3t \ ; \ c_2 = 24 + 3t \ ; \ c_3 = 23 + 2t \ ;$$

$$c_4 = 10 + 5t \ ; \ c_5 = 18 + 3t \ ; \ c_6 = 21 + 2t \ ;$$

$$c_7 = 9 + 3t.$$

We choose the level z=15 and denote the feasible set by S.

According to Theorem 1, the solution of our problem can be obtained by solving the following deterministic bottleneck linear problem:

$$\max_{x \in S} \ \min\{c_j \mid x_j > 0\}$$

where $(c_1, c_2, ..., c_7) = (-2, -3, -4, 1, -1, -3, 2)$.

Applying a modified version of the algorithm given by Bansal and Puri (1980), we obtain that the optimal solution of this problem and hence of our initial problem is

$$x = (0; 0; 0; 1/4; 1/2; 0; 17/4).$$

References

Achary,K.K. and C.R.Seshan (1981). A time minimising transportation problem with quantity dependent time. *European J. Oper. Res.* **7**(3), 290-298.

Achary,K.K.,C.R.Seshan and V.G.Tikekar (1982).A generalized bottleneck linear programming problem.*Indian J.Technol.* **20**, 5-9.

Ahuja, R.K. (1985). Min max linear programming problem. *Operations Research Letters* **4**, 131-134.

Akgül, M. (1984). On a minimax problem.*Opsearch* **21** (1), 30-37.

Andersen, K.A., K.Jörnsten and M.Lind (1996). On bicriterion minimal spanning trees: An approximation. *Computers and Operations Research* **23** (12), 1171-1182.

Bansal, S. and M.C.Puri (1980). A min max problem. *Z. Operations Res.* **24** (5), 191-200.

Burkard, R.E. and F.Rendl (1991). Lexicographic bottleneck problems. *Operations Research Letters* **10**, 303-308.

Chandrasekaran, R. (1977). Minimal ratio spanning trees. *Networks* **7** (4), 335-342.

Cheriton, D. and R.E.Tarjan (1976).Finding minimum spanning trees.*SIAM J.Comput.*, **5**(4), 724-742.

Emelichev, V.A., M.K.Kravtsov and O. A. Yanushkevich (1995). Conditions for Pareto optimality in a discrete vector problem on a system of subsets (Russian).*Zh. Vychisl. Mat. i, Mat. Fiz.* **35** (11), 1641-1652.

Ford, L.R. and D.R. Fulkerson (1962). *Flows in Networks.* Princeton Univ. Press, New Jersey.

Francis, R.L. and J.A.White (1974).*Facility Layout and Location:An Analytical Approach.*Prentice Hall, Englewood Cliffs, N.J.

Frieze, A.M. (1975). Bottleneck linear programming. *Opl. Res. Q.* **26** (4), 871-874.

Gabow, H.N. and R.E. Tarjan (1988). Algorithm for two bottleneck optimization problems. *J. Algorithms* **9** (3), 411-417.

Ganley, J.L. and J.S. Salome (1996). Optimal and approximate bottleneck Steiner trees.*Operations Research Letters* **19** (5), 217-224.

Garfinkel, R.S.(1971).An improved algorithm for the bottleneck assignment problem. *Operations Research* **19**, 1747-1751.

Garfinkel, R.S., A.W. Neebe and M.R. Rao (1974). The m-center problem:bottleneck facility location. Working Paper Series No. 7414, Graduate School of Management, University of Rochester, Rochester, New York.

Garfinkel,R.S. and G.L. Nemhauser (1970). Optimal political districting by implicit enumeration techniques. *Management Sci.* **16B**,495-508.

Garfinkel, R.S. and Rao,M. (1976). Bottleneck linear programming. *Math. Programming* **11** (3), 291-298.

Geetha, S. and K.P.K. Nair (1993). On stochastic spanning tree problem. *Networks* **23** (8), 675-679.

Geetha,S. and M.N. Vartak (1994). The three-dimensional bottleneck assignment problem with capacity constraints. *European J. Oper. Res.* **73**, 562-568.

Gross, O. (1959). The bottleneck assignment problem. Paper P-1630, The RAND Corporation, Santa Monica, CA.

Gupta, S.K. and A.K. Mittal (1982). A min-max problem as a linear programming problem. *Opsearch* **19** (1), 49-53.

Gupta, S.K. and A.P. Punnen (1983). Minimax linear programmes with grouped variables. *Opsearch* **26** (3), 177-186.

Hochbaum, D.S. and Pathria, A. (1996). The bottleneck graph partition problem. *Networks*,**28** (4), 221-225.

Ishii, H. and T. Matsutomi (1995). Confidence regional method of stochastic spanning tree problem. *Math. Comput. Modelling* **22** (10-12), 77-82.

Ishii, H. and T. Nishida (1983). Stochastic bottleneck spanning tree problem. *Networks* **13**, 443-449.

Ishii, H., S. Shiode, T. Nishida and Y.Namasuya (1981). Stochastic spanning tree problem. *Discrete Applied Mathematics* **3**, 263-273.

Ishii, H., S. Shiode and T. Nishida (1981). Chance constrained spanning tree problem. *J. Oper. Res. Soc. Japan* **24** (2), 147-158.

Kaplan, S. (1974). Applications of programs with maximin objective functions to problems of optimal resource allocation. *Operations Research* **22** (4), 802-807.

Mathur, K., S. Bansal and M.C. Puri (1993). Bicriteria bottleneck linear programming problem. *Optimization* **28** (2), 165-170.

Mathur, K., M.C. Puri and S. Bansal (1995). On ranking of feasible solutions of a bottleneck linear programming problem. *Top* **3** (2), 265-283.

Minoux, M. (1989). Solving combinatorial problems with min-max-minsum objective and applications. *Math. Programming* **45** (2), 361-372.

Mohd, I. B. (1994). Interval elimination method for stochastic spanning tree problem. *Applied Mathematics and Computation* **66** (2-3), 325-341.

Pferschy, U.(1995). The random linear bottleneck assignment problem. *Integer programming and combinatorial optimization,* 145-156. Lecture Notes in Comput. Sci., 920, Springer, Berlin.

Pferschy, U. (1996). The random linear bottleneck assignment problem. *RAIRO Rech. Opér.* **30** (2), 127-142.

Sarrafzadeh, M. and C.K. Wong (1992). Bottleneck Steiner trees in the plane.*IEEE Transactions on Computers* **41**(3), 370-374.

Seshan, C.R. and K.K. Achary (1982). On the bottleneck linear programming problem. *European J. Oper. Res.* **9**, 347-352.

Stancu-Minasian, I.M.(1984). *Stochastic Programming with Multiple Objective Functions.* Ed. Academiei, Bucureşti, România and D. Reidel Publishing Company, Dordrecht, Boston, Lancaster.

Stancu-Minasian, I.M. (1997). *Fractional Programming: Theory,Methods and Applications.* Kluwer Academic Publishers,Dordrecht.

Stancu-Minasian, I.M. and St. Tigan (1985). The minimum-risk approach to the bottleneck transportation problem. Itinerant Seminar on Functional Equations, Approximation and Convexity (Cluj-Napoca,1985), 203-208. Preprint 85-6, Univ. "Babeş-Bolyai", Cluj-Napoca.

Stancu-Minasian, I.M. and St. Tigan (1990). On some fractional programming models ocurring in minimum-risk problems.In: Generalized Convexity and Fractional Programming with Economic Applications. A. Cambini, E.Castagnoli, L. Martein, P.Mazzoleni and S. Schaible (eds.).*Proceedings of the International Workshop on "Generalized Concavity,Fractional Programming and Economic Applications"*, held at the University of Pisa, Italy, May 30-June 1,1988.Lecture Notes in Economics and Mathematical Systems, 345, Springer-Verlag, 295-324.

Tigan, St. and Stancu-Minasian,I.M. (1983). Criteriul riscului minim pentru problema Cebîşev. *Lucrările celui de-al IV-lea Simpozion "-Modelarea cibernetică a proceselor de producţie"* 26-28 mai 1983. ASE-Bucureşti, Vol. I, 338-342.

Tigan, St. and I.M. Stancu-Minasian (1985). The stochastic bottleneck transportation problem. *Anal. Numér. Théor. Approximation* **14** (2), 153-158.

Yang, Y.L. and S.W. Shen (1988). A generalization of a bottleneck problem (Chinese). *J. Numer. Methods Comput. Appl.* **9** (4), 214-218.

Yao, A.C. (1975). An 0(|E| log log$|V|$) algorithm for finding minimum spanning trees. *Information Processing Letters* **4**, 21-23.

Yechiali, U. (1968). A stochastic bottleneck assignment problem. *Management Sci.* **14** (11), 732-734.

Yechiali, U. (1971). A note on a stochastic production-maximizing transportation problem. *Naval Res. Logist. Quart.* **18** (3), 429-431.