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Abstract: In this strictly positional paper we propose a general approach to bipolar knowledge representation, where the meaning of concepts can be modelled by examining their decomposition into opposite and neutral categories. In particular, it is the semantic relationship between the opposite categories which suggests the *emergence* of a *paired structure* and its associated type of *neutrality*, being there three general types of neutral categories, namely *indeterminacy*, *ambivalence* and *conflict*. Hence, the key issue consists in identifying the semantic opposition characterizing the meaning of concepts and at the same time the type of neutrality rising in between opposites. Based on this first level of bipolar knowledge representation, paired structures in fact offer the means to characterize a specific bipolar valuation scale depending on the meaning of the concept that has to be verified. In this sense, a paired structure is a standard basic structure that allows learning and building different valuation scales, leading to linear or even more complex valuation scales.

Keywords: Bipolarity; Neutral concepts; Paired structures; Knowledge representation.

1 Introduction

Psychology and Neurology are providing relevant results for modelling human decision making. The human brain has specifically and successfully evolved to manage complex, uncertain, incomplete, and even apparently inconsistent information. For example, neurologists have shown that the part of the brain taking care of making up the last decision is different to the part of the brain in charge of the previous rational analysis of alternatives, being the first part associated to *emotions* [5], [6]. This is an extremely important result, as it suggests that different parts of our brain participate in our decision processes, each one following its own rules within a connected structure still under study. Among other key achievements, it has been recently shown the key role that concept representation plays in our knowledge process, along with the fact that the human brain manages positive information in a different way than negative information, suggesting some kind of *bipolarity* in the way that our brain handles information [11], [12]. In this way, positive and negative affects are not processed in the same region of the brain, as they are generated by clearly different neural processes [38].

The importance of bipolar reasoning in human activity was settled by Osgood, Suci and Tannenbaum in 1957 [35] -see also [26], [47]-, proposing a semantic theory based on the Semantic Differential scale for valuing the meaning of concepts. This theory became very popular for measuring attitudes in an easy way, where individuals are asked to use the Semantic Differential scale for valuing if a given object is perceived as being *positive*, *neutral* or *negative*. Thus, the object cannot be evaluated as being *positive* and *negative* at the same time. This led to some critics (see e.g. [12], [19], [26]) stating that the Semantic Differential scale does not consider other relevant attitudes like e.g. *ambivalence*, making it necessary to allow simultaneous positive and negative valuations. Thus, the *neutral* value that appears in a Semantic Differential scale can hardly be understood as a proper representation for certain attitudes that seem to escape the linear logic of such a scale.

Therefore, it can be stated that our internal decision making process is of a complex nature, implying previous differentiated knowledge acquisition and representation processes (see e.g. [31], [32]), and quite often implying multi-criteria arguments. In fact, a 1-dimensional scale is too poor for modelling most of our problems, not giving room to the true conflicts we perceive from reality. Hence, once such a complexity is acknowledged, our mathematical modelling must continuously balance precision and simplicity, just as our brain looks for relevant but at the same time manageable information. Within such Multicriteria Decision Making framework (see e.g. [24]), the Semantic Differential scale of Osgood [35] is commonly known as the *bipolar univariate model*, while a somehow modified Semantic Differential scale (see e.g. [26]) consisting of two unipolar scales joined together, respectively allowing simultaneous positive and negative measurements, is commonly known as the *unipolar bivariate model*. A well-known example for the bipolar univariate model in decision theory is Cumulative Prospect Theory [54] and its generalization by means of the Choquet integral with respect to bi-capacities [25].

Generally speaking, understanding concepts by their decomposition into *opposite poles* (*meaningful opposites* in the sense of Osgood [35]), enables us to capture the tension between somehow opposite arguments, a tension that is found in the way we understand most of the concepts we use. If an objective measure is not available, concepts in real life cannot be manipulated in an isolated manner, without taking into account their immediately related concepts. In some way, such simultaneous but somehow opposite views are unavoidable to start understanding the world, and indeed we need more complex knowledge structures to manage more than two views. But addressing two views is the minimum effort we should expect to acknowledge complexity.

The point of departure of this paper are the above considerations, focusing our attention on the construction of basic structures that deal with the *bipolar meaning of* *concepts* (see [39] for a previous attempt), together with the associated **neutrality** that emerges from the tension between *opposites* or *poles*. Notice here that **neutrality** in our sense should not be confused with the *neutral* value in the traditional sense (see [15], [16], [17], [24], [35], among others), given that we refer to a neutral category which does not entail linearity between opposite poles. Instead, we will stress the existence of different kinds of **neutrality** that allow the representation of a number of concepts directly generated from the semantic relation between poles.

Therefore, our proposal for examining bipolar knowledge representation is based on paired structures, understanding a paired structure as a basic structure allowing learning and building different valuation scales, not necessarily linear but more complex ones. In fact, what a paired structure represents is the semantic structure of its associated valuation scale. In this way, the measurement of the meaning of a concept, which occurs on the valuation scale, has proper sense only regarding its respective semantic structure. Hence, paired structures emerge from the semantic relation among opposites together with their characteristic neutrality, giving rise to a pertinent type of (bipolar) valuation scale.

From our point of view, the key issue at a first stage of knowledge is the structure associated to the semantic relation between poles, and how neutral concepts are generated. Again, as pointed out in [31], there are many different ways of being *in between* poles, but such *in-betweenness* does not necessarily entail a neutral situation; instead, it can refer to a particular *symmetry*, as an element of a scale with no conceptual meaning but that of being somehow equidistant from poles (being not able to choose among poles does not necessarily imply a concept behind). Particularly, we consider that symmetries are not *neutral* concepts, since they are generated outside a semantic argument.

The paper is organized as follows: in the next section we shall give an example to fix our intuition. This example will be completed throughout the paper. Our proposal will be formalized in Section 3, where we explain what we understand by paired concepts and paired structures. In Section 4 we shall analyse in detail the types of neutrality rising from paired structures, followed by an example on preference modelling. In Section 5 we present the standard procedure for building paired structures, ending with some final comments briefly clarifying our position.

Let us finally remark in this introduction that this paper is not about formal logic or its interpretation, but rather it deals with knowledge and natural language representation by means of logical tools.

2 Preliminary Example: On the Representation and Measurement of Size

Let us try to illustrate our view through an example.

The meaning of *size* can be modelled based on some set of characteristic properties which can be somehow measured (subject to standard imprecision). Then, the measurement of a property occurs on a valuation scale which exists because there is a previous decomposition of the meaning of *size*, either as a 1-dimensional or *n*-dimensional concept. In this way, if *size* is represented as e.g. *size* = *height*, the verification of its occurrence can be valued within the real line. That is, by taking the *size* of a person as *height*, we can refer to how *tall* that person is. Let us examine this meaning of *size* = *height* in more detail.

Although we all know that height is measurable, we do not try to measure the height of each person we meet. Instead of saying “Paula’s height looks around 1.90 meters”, most people will talk about Paula as a *tall* person, i.e., in terms of the *tallness* concept (which can be regarded as a fuzzy context-dependent concept [55]). Indeed, a person’s *height* is usually judged in terms of the concepts *tall* and *short*, which constitute the reference or semantic landmarks for the evaluation of such a feature. We hardly use the notion of a person’s *height* without the landmarks provided by the poles *tall*/*short* or any other equivalent pair.

If our concept of *tallness* were crisp, the term “Paula is *tall*” would have a direct translation on the valuation scale in terms of *height*: for example, “Paula is *tall*” if and only if “Paula’s height is at least 1.70 meters”. As soon as we have this crisp definition, the concept of being *non-tall* is automatically created by the classical crisp negation: “Paula is *non-tall*” if and only if “Paula’s height is less than 1.70 meters”. That is, *tallness* is associated to the interval $[1.70,\infty)$ meanwhile *non-tallness* is associated to the interval $(0,1.70)$. In order to generate such paired concepts, we simply need to assume the existence of the crisp negation: a person x within a community X belongs to the set of *tall* people within X if and only if the height $h(x)$ of such a person is greater than or equal to 1.70. And a person x within the community X belongs to the set of *non-tall* people within X if and only if the height $h(x)$ of such a person is smaller than 1.70. The set of tall people is $Tall = \{x \in X \,/\, h(x) \geq 1.70\}$, and the set of non-tall people is $Non\text{-}tall = \{x \in X \,/\, h(x) < 1.70\}$. Within such a crisp context, no person can be *tall* and *non-tall* at the same time, and everybody will be either *tall* or *non-tall*. A simple paired structure to represent *height* has been built from only one concept (*tallness*) and its negation (*non-tallness*), and the characteristic functions of both crisp concepts

$$\mu_{Tall}(x) = 1 \iff h(x) \geq 1.70, \qquad \mu_{Tall}(x) = 0 \iff h(x) < 1.70$$

$$\mu_{Non-tall}(x) = 1 \iff h(x) < 1.70, \qquad \mu_{Non-tall}(x) = 0 \iff h(x) \geq 1.70$$

are defined in such a way that

$$\mu_{Tall}(x) = n(\mu_{Non-tall}(x)), \forall x \in X$$

$$\mu_{Non-tall}(x) = n(\mu_{Tall}(x)), \forall x \in X$$

being $n:\{0,1\} \rightarrow \{0,1\}$, such that $n(0) = 1$ and $n(1) = 0$, the only negation within the crisp $\{0,1\}$ framework (in fact, and within the crisp framework, the only one-to-one mapping from $\{0,1\}$ into itself different from the identity mapping).

Hence, *tallness* and *non-tallness* appear as paired concepts in a natural way within the crisp framework. But meanwhile a measurement of height is both available and precise, there is no room for any kind of *neutral* concept, although *borders* between a concept and its negation ($\{x \in X \,/\, h(x) = 1.70\}$ in the above example) might deserve specific attention, precisely because of the potential imprecision in measurement.

However, the introduction of *short* as the dual concept of *tall* allows different translations into measurable *height*, such that *tallness* and *shortness* exist now as paired concepts.

For example, and keeping the crisp approach, we can define that "Paula is *short*" if and only if "Paula's height is at most 1.60 meters". Again the concept of *non-short* is automatically created, in such a way that *Short* = $\{x \in X / h(x) \leq 1.60\}$, and the set of non-short people is *Non-short* = $\{x \in X / h(x) > 1.60\}$. In this way, all those people with height within the interval (1.60,1.70) are neither *tall* or *short*, leading to some kind of *indeterminacy* (we cannot assign any of the only two available concepts to some individuals). In this particular case we know that in order to solve this *indeterminacy* we can create an intermediate concept, like *Medium* = $\{x \in X / 1.60 < h(x) < 1.70\}$. But this is a different argument, conducted in a subsequent stage. At first, what we find from a semantic point of view is just that none of the two opposite, available concepts apply to some individuals. In general, *indeterminacy* represents this hesitant situation. That is, *indeterminacy* suggests the need of a new concept, which not necessarily will be an intermediate concept as *Medium* in the previous example. In fact, notice that the linearity between *Tall-Medium-Short* is due to the 1-dimensionality of *size* = *height*, which in no way is the only possible meaning for *size* (please be aware of the specificity of this example, where the linear representation behind is *a priori* known and the two opposite concepts we have chosen correspond to left and right tails).

On the other hand, we could have defined that "Paula is *short*" if and only if "Paula's height is at most 1.80 meters". In this case, *Short* = $\{x \in X / h(x) \leq 1.80\}$. Then all those people with height within the interval [1.70,1.80] will be both *tall* and *short*, leading to a certain kind of *ambivalence*. That is, now what we find from a purely semantic point of view is that both opposite concepts simultaneously apply. Clearly, in this specific example it is again suggested to intersperse an intermediate concept in between the poles, like *Medium* = $\{x \in X / 1.70 \leq h(x) \leq 1.80\}$, and then reshape *tallness* and *shortness* to avoid overlapping, for example, *Tall* = $\{x \in X / h(x) > 1.80\}$ and *Short* = $\{x \in X / h(x) < 1.70\}$. Once more, the construction of a linear valuation scale (or equivalently, the interpretation of such ambivalence as an intermediate concept *Medium*) depends on a different, subsequent argument relying on a previous interpretation of the semantics of the poles and their relationship.

Now, it is important to realize that *tallness* and *shortness* exist as paired concepts no matter if they can be translated into a measurable *height*.

Our main argument is based upon the above observation: two opposite crisp concepts that refer to the same property, and depending on their semantics, can generate two neutral concepts (*indeterminacy* and *ambivalence*, or both). A more careful analysis of *indeterminacy* and *ambivalence* might suggest specific scales (see e.g. [27]) by modifying the definition of the two basic opposite concepts and/or introducing new intermediate concepts. But these valuation scales and their corresponding semantics can only be properly understood by firstly addressing the particular semantic relationship between the correspondent poles. And that is the role we give to paired structures, as well as the reason why we find it important to define and study them.

In addition to *indeterminacy* and *ambivalence*, there is a third standard neutrality that can appear in more complex situations. In the above example we have based our arguments upon the existence of a unique, linearly-based property for understanding *size*, given by *height*. In practice, however, most of our concepts are complex in the sense that they can be decomposed into simpler concepts (see e.g. [23]). In this case, our evaluation proceeds through an (perhaps non conscious) aggregation process.

For example, when talking about the *size* of a person we can define two opposite categories like *big* and *small*. But being a *big* or *small* person might depend on *height* (*tall* versus *short*) and *weight* (*fat* versus *slim*). Of course it may be the case that a person is neither *big* nor *small*, being there *indeterminacy*, and another person can be both *big* and *small*, being there *ambivalence*. But it can also happen that we cannot choose among the two concepts *big* and *small* because there is a *conflict* behind, i.e. both opposite concepts hold in a conflictive manner. It is not the same to say that a person is both *big* and *small* because it is simultaneously both *tall* and *short* and both *fat* and *slim*, than to say that such a person is *big* and *small* because it is *tall* but *slim*, or *fat* but *short*. In this more complex framework we can find *conflict* as a third kind of neutrality associated to opposite concepts, together with *indeterminacy* and *ambivalence*.

To conclude this example, let us briefly illustrate the notion of *point of symmetry*. To this end, let us assume again that the meaning of *size* is interpreted solely in terms of the measurable characteristic *height*, and that our references are given by the predicates *Tall* and *Not-tall*, but let us now consider these as fuzzy (i.e. non-crisp) concepts. That is, we now allow both notions to be evaluated on the interval [0,1] rather than on the binary set {0,1}. Then, for any $x \in X$, for instance we may set the meaning of *Tall* and its complement to be represented by the fuzzy sets

$$\mu_{Tall}(x) = \begin{cases} 0 & \text{if } h(x) < 1.70 \\ \dfrac{h(x) - 1.70}{1.80 - 1.70} & \text{if } 1.70 \le h(x) \le 1.80 \\ 1 & \text{if } h(x) > 1.80 \end{cases} \quad , \quad \mu_{Not-tall}(x) = 1 - \mu_{Tall}(x).$$

As usually admitted, this kind of representation enables both *Tall* and its complement to exhibit a somehow imprecise semantics, avoiding the boundary problems associated to crisp predicates. But, apart from that, we still have just a predicate and its complement as poles, which therefore fully explain the whole universe of discourse *X*, again leaving no room for any neutral concept. However, contrarily to the previous crisp context, now it is possible for an object to be equally associated to a pole and its complement, as now $\mu_{Tall}(x) = \mu_{Not-tall}(x)$ can hold for a given object $x \in X$ (i.e., whenever $h(x) = 1.75$). We refer to this situation by saying that, in a fuzzy context, a reference predicate and its complement admit a *point of symmetry* between them.

It is important to stress that such point of symmetry does not represent a concept different than poles (either a neutral or an intermediate one): as exposed above, there are no other available options beside poles as these are complementary notions. Rather, in this situation we hesitate between both poles as a consequence of their imprecise, fuzzy semantics, which allows an object to be considered as e.g. *half tall* and *half non-tall*. This hesitation is clearly different from a semantic point of view to those associated to a situation of overlap (ambivalence), lack of covering (indeterminacy) or conflict between the poles. Particularly, a point of symmetry should not be confused with an intermediate concept between the poles: though both notions are somehow related to linear scales, the latter indeed represents a different valuation option from the poles, while the former represents just an equilibrium between them, only available due to the imprecise representation of these references.

3 Semantic opposition and paired structures

Let us study the relationships that can arise between a pair of concepts when these two concepts are semantically related, constituting the reference landmarks for a certain linguistic representation of reality. We will refer to those pairs of concepts fulfilling such relationship as *paired concepts*. We will use the term *pole* to refer to any of the two concepts being paired. Some concepts that constitute paired concepts are, as pointed out in the previous section, *tall*/*short*, *fat/slim*, *big/small*, but it is easy to find other examples such as *cheap*/*expensive* or *good/bad* (see e.g. [40]).

3.1 Paired concepts

Paired concepts are not simply a couple of concepts. These two concepts must somehow define a specific structure. Hence, our first objective is to clarify what we mean by *paired*. In fact, as pointed out above, our mind is able to represent complex situations, related to interests and emotions, by using a pair of landmarks or poles that constitute the references for evaluation. Such reference concepts or poles allow configuring the evaluation framework in which information can be assessed. In other words, they constitute the referential context in which pieces of information are understood. As shown in our preliminary example, only if *size* is assigned the meaning of *height* as *tallness*, and such meaning is translated into a valuation scale with its specific measure, only then we can be confused in thinking that *tallness* can be understood without *shortness*. Otherwise, if such a measure has not yet been provided, even the concept of *tallness* requires the concept of *shortness* in order to be understood.

Hence, poles appear in pairs. We cannot understand most concepts without also knowing the meaning of those other concepts that define their limits. In this sense, two concepts have to be related in some specific way to effectively configure an appropriate referential context, i.e., in order to properly constitute a pair of reference landmarks. The previous arguments suggest the existence of a certain structure, which emerges from a pair of somehow *opposite* concepts, and constitutes the abovementioned evaluation framework arising from these reference poles.

These ideas suggest that we should focus on the structural/semantic opposition between paired concepts. However, such opposition does not have a unique possible representation or definition, and the different opposition relationships between poles will in fact generate different structures.

For example, *very tall* and *very short* are opposite concepts, and *more or less tall* and *more or less short* are also opposite concepts, but they indeed suggest very different spaces *in between* them (the first pair of poles cannot hold at the same time, while the second pair can). In this way, we refer to *duality* to capture in a general sense such a semantic opposition.

3.2 Duality: negation, antonym and antagonism

In this paper, we focus on three particular forms of such a duality relation, specifically *negation*, *antonym* and *antagonism* (see e.g. [48], [49], but also [41]).

Before formalizing the meaning of these duality relationships, it is important to make explicit that we assume that any concept or predicate *P* (and thus particularly reference poles) can be represented as a fuzzy set μ_P over a particular universe of discourse *X*, in such a way that $\mu_P(x) \in [0,1]$ denotes the degree up to which an object $x \in X$ verifies predicate *P*. Nonetheless, notice that this assumption is not really necessary, as we would only need the predicate logic framework of classical or *crisp* sets (defined through a binary {0,1} scale instead of a continuous [0,1] scale). But as crisp sets are special instances of fuzzy sets, we study paired concepts as paired fuzzy sets for examining bipolar knowledge representation in a more general framework.

In a first approach, we will also assume reference concepts or poles to be *simple*, in the sense of referring to a characteristic depending on just a single criterion or dimension (like *tall* and *short* refer to *size* = *height*), i.e. not admitting a further decomposition in a set of underlying criteria or sub-concepts. Later on we will remove this assumption and analyze also *complex* multidimensional reference concepts, as could be *big/small* or *good/bad*, which usually require a further decomposition and aggregation processes in a set of underlying criteria.

Now, let us recall that a *negation* within the fuzzy context (see again [48], [49]) is usually understood as a non-increasing function $n:[0,1]\rightarrow[0,1]$ such that $n(0) = 1$ and $n(1) = 0$. A strictly decreasing, continuous negation being also *involutive* (i.e. such that $n(n(v)) = v$ for all v in [0,1]) is called a *strong* negation. If $F(X)$ denotes the set of all fuzzy sets (i.e. predicates) over a given universe *X*, then any strong negation *n* determines a *complement* operator $N: F(X)\rightarrow F(X)$ such that $N(\mu)(x) = n(\mu(x))$ for any predicate $\mu \in F(X)$ and any object $x \in X$.

On the other hand, an *antonym* operator was defined [50] as a mapping $A: F(X)\rightarrow F(X)$ verifying,

A1) $A^2 = Id$ (i.e. A is involutive);

A2) $\mu(x) \le \mu(y) \Rightarrow A(\mu)(y) \le A(\mu)(x)$ for all $\mu \in F(X)$ and $x, y \in X$;

A3) $A \le N$ for a given complement operator *N* (i.e. *A* is sub-additive with respect to the complement operator *N* or, equivalently, with respect to the negation *n* that defines *N*).

Finally, for the purposes of this paper we will refer to an *antagonism* or *dissimilarity* operator as a mapping $D: F(X)\rightarrow F(X)$ that fulfils,

D1) $D^2 = Id$ (i.e. *D* is involutive);

D2) $\mu(x) \le \mu(y) \Rightarrow D(\mu)(y) \le D(\mu)(x)$ for any $\mu \in F(X)$ and $x, y \in X$.

It is important to note that the main difference among antonym and antagonism operators lies on that antonyms are always sub-additive, in the sense of A3. In other words, the antonym always lies behind the complement, while antagonistic predicates could be over-additive, i.e. it could be that $D \geq N$ for a certain complement N. As we shall see, the distinction between sub-additive and over-additive duality operators is relevant as it reflects two essentially different semantic relationships between poles, not necessarily incompatible (two concepts can overlap while at the same time not fully explaining reality).

Observe also that complement is a special case of antonym, as well as that antagonism somehow suggests a generalization for antonyms. Finally, note that all the considered duality operators (negation, antonym, dissimilarity) are involutive, i.e., they naturally define pairs of concepts related through duality.

3.3 Paired structures

Paired structures belong to a first level of (bipolar) knowledge representation. They stand as the subjacent structure that allows making sense of (bipolar) valuation scales, where neutrality is not simply a symmetry point but a concept in itself that requires further examination. With the purpose of exploring the nature of such neutrality, we propose to go further into the roots of bipolarity and go beyond standard valuation scales by dealing with *paired concepts* and *paired structures*.

In a general sense, we consider that two concepts P,Q are *paired* if and only if $P = D(Q)$, and thus also $Q = D(P)$, holds for a certain semantic duality operator D. Then, we assume that two paired concepts provide semantic references for a linguistic assessment of a characteristic, in such a way that a semantic structure emerges from these two referential notions. We refer to these semantic valuation structures as *paired structures*. Our point is that, besides the original pair of opposite references, a paired structure emerges together with its neutral concepts, as a consequence of the specific semantic relationship holding among opposite poles.

In this paper we are mainly concerned with analysing the features of these paired structures, and particularly (as it will be shown throughout this paper) the manner in which the semantic relationship between paired concepts determines both the semantic valuation structure arising from them and its representative power.

A first observation is that negation (in the sense of complement) *not-P* of a concept P cannot be viewed as a different concept from P, as it happens with antonyms and antagonists. Not only it does not add qualitatively different information, but it is kind of unnatural to estimate *not-P* independently from P (moreover, we should also remind that intuition works with positively defined concepts, and negation itself is not positively defined, see e.g. [14], but also [31]). This is a key issue whenever we look for the possibility that two opposite or dual concepts generate additional neutral concepts from their semantics. Still, it should be noticed that we refer to a *semantic argument* to define a paired structure. Therefore, although in our opinion antonym and antagonistic relationships represent the proper framework for paired structures, it is true that negation or complement appears as a very specific case of semantic opposition (see e.g. [52]). In this sense, a concept and its negation indeed constitute a paired structure that implies a particular empty space between poles, and in fact, duality operators different from negation allow generating additional neutral concepts *in between* poles.

As suggested in the preliminary example, neutrality may arise in different manners, whenever an object fails to be fully explained from the two reference concepts, denoting a situation that in turn is related to different types of hesitation regarding the poles. Neutrality will in fact represent a different concept from the poles, another relevant available option for evaluating objects. And depending on the circumstances, such neutrality can suggest to search for an alternative *symmetrical* category (like in the *tall/short* example a *medium* category was suggested from the existence of ambivalence or indeterminacy). Particularly, different paired structures (arising from different semantic relationships between the poles) are able to represent different types of neutrality in our knowledge about reality. Again, we stress our key idea that a paired structure requires two opposite or dual concepts, together with a semantic building procedure leading to the emergence of new neutral valuation concepts. The existence of such neutrality, which is a consequence of the semantic relationship between opposite concepts, will sometimes be definitive in order to identify the particular paired structure we are dealing with.

4 Neutrality

Neutral concepts are generated from two opposite concepts whenever an object cannot be properly explained in terms of the poles. From a fuzzy approach, the point here is that the two poles may not form a *fuzzy partition* of the universe of discourse, see [44] or any of its generalizations based upon any alternative logic (e.g. [13], but particularly [1], [2]). In this context, the term *neutral* means that besides the character or semantics of both poles, their relationship implies the existence of an additional concept that applies to those objects that cannot be properly explained from poles. But as suggested in [31], there are different *types of neutrality* or neutral categories.

4.1 Types of neutrality

Examining the bipolar opposition holding among paired concepts, it may be on the one hand that both poles somehow overlap [8], suggesting that both poles may be reshaped to create some middle concept between the new poles. On the other hand, it may be that objects cannot be fully explained solely in terms of the given poles, suggesting the search for an additional concept [10]. Moreover, poles might show some conflictive behavior. The particular neutrality we find depends on the semantic relationship between our paired references (in the next section we shall describe a building procedure to obtain our paired structures and their associated neutral concepts). Let us now concentrate in describing some of the different ways of being *in between* poles and the types of neutral concepts that can appear within paired structures.

Let us start by analysing the case of a paired structure given by a predicate P and its complement $Q = N(P) \equiv$ *not-P*. As discussed above, the complement *not-P* of a concept P is not logically independent of P, and thus the former does not actually define a different category nor provides different information from that of the original concept *P*. That is, *complementation* constitutes a very specific semantic relationship, in which the references just allude to the verification or lack of verification of a single concept or pole. In this way, when for example we conceive *size* = *height* as expressed in terms of two complementary references (e.g., *tall* and *non-tall*), we are in fact just measuring a single notion, since any verification degree of *tall* corresponds to an inverse degree of verification of *non-tall*.

Hence, *neutral* concepts can only arise when we really deal with two logically independent poles. As already shown, in a binary {0,1} setting, if the valuation 1 is assigned to e.g. *tall* people, then *non-tall* people obtain the valuation 0. Thus, there is no room for anything else, since both references are understood as crisp, precise notions that complement each other to cover the whole universe of discourse.

Notice that, in case we admit a pair of complementary references to be modelled as fuzzy (instead of crisp) predicates, basically the same situation remains to hold. That is, in a fuzzy context a predicate and its negation are still not logically independent, and also they are still able to explain the entire universe of discourse. Therefore, again no neutral concepts can arise in this setting.

However, a fuzzy representation enables the modelling of a certain kind of linguistic uncertainty, usually associated to imprecise predicates. And, as pointed out at the end of Section 2, this linguistic imprecision now allows a predicate and its negation to simultaneously hold, up to a certain degree, for a given object. In this sense, for instance, an individual can be assessed as being both *half tall* and *half not-tall*. This leads to a particular hesitation, as we find difficulties to choose between both references, although such hesitation does not suggest a third, different concept to come into play. Rather, the individual is adequately explained in terms of the available pair of references, but their imprecise nature admits the emergence of a *symmetry point* expressing an equilibrium or balance between them. Thus, in accordance with the discussion at the beginning of this section, symmetry points neither represent a concept (or valuation alternative) different from the poles (as intermediate concepts or symmetries do represent) nor they are to be considered a specific kind of neutrality (in the sense given in the introduction of this paper).

More formally, given a strong negation n and assuming that a membership function is associated to complementary poles, in such a way that $\mu_{N(P)}(x) = n(\mu_P(x))\ \forall x \in X$, a (fuzzy) symmetry point can be found when $\mu_{N(P)}(x) = \mu_P(x)$ for a certain $x \in X$. Notice that, since n is a negation, the latter equality can only take a single value restricted to the interval (0,1), and in fact it takes the value 0.5 in case we use the standard negation $n(x) = 1 - x$.

A similar scenario is reached if we allow probabilistic uncertainty instead of linguistic uncertainty, i.e., if we consider the poles as events with an associated probability for any object in consideration. As long as both poles are considered as complementary notions, they cover the whole sample space, leaving no room for other alternative events. Then, a (probabilistic) symmetry point is found whenever the probabilities of both poles are equal for a certain object (in which case they are to be equal to 0.5).

Thus, in case of complementary references, neither fuzzy nor probabilistic uncertainty by themselves can lead to the apparition of neutral concepts or symmetries constituting valuation alternatives different from the poles. That is, the crucial point in order to enable the emergence of neutral concepts is not whether the modelling of our references admits a fuzzy or probabilistic representation of uncertainty, but whether our references are complementary or not.

Before dropping the assumption of complementary references, let us briefly consider an aspect in relation with the estimation of exact membership or probability degrees. Such degrees are usually introduced in order to enable the modelling of certain kind of uncertainty, either linguistic or probabilistic. But at the moment of estimating these degrees, we may easily have to face a different kind of uncertainty, related to the difficulty of choosing an exact value for them. To some extent, depending on the context and the specific problem being addressed, we may be forced to admit that our valuations or degrees are subject to some imprecision regarding its estimation. Thus, such an *estimation imprecision* represents a different kind of uncertainty from those usually associated to imprecise predicates or uncertain events, in fact a kind of uncertainty related to the way we express other uncertainties in our models. Particularly, it is possible to allow a range of imprecision levels for our estimations, in such a way that *absolute imprecision* may be taken to represent an alternative case of symmetry between poles, expressing ignorance or lack of knowledge. Instances of such absolute imprecision can be found, for example, when we can only state that the actual probability value is simply between [0,1] (see [46]), or that the actual degree of membership is simply between [0,1] (see [45]). In both situations we find a certain hesitation to choose among the poles, in this case due to a lack of knowledge. But such an absolute estimation imprecision should not be considered as a *neutral* concept, as it is not created from the semantic tension between the poles.

However, as stated above, the situation changes qualitatively when we move to the case of a pair of poles (P,Q) related through a non-trivial duality operator D, such that $P = D(Q)$ and $Q = D(P)$ are not complementary poles.

Let us start by considering the poles as related through an antonym or sub-additive duality operator A, such that $Q = A(P)$. If each pole is contained in the complement of the other pole we can assume that they are logically independent. That is, we cannot obtain one pole from the other by means of a logical operation, but necessarily as a consequence of a *semantic operation*. Thus, each pole provides different information, allowing potential neutral concepts within such a paired structure. As a consequence of the sub-additive nature of the involved semantic operator, both poles are somehow *separated* (in the sense of not covering the whole universe of discourse), leaving *space* for some objects that could not verify either one pole or the other. This is not a situation in which we hesitate about whether an object verifies a pole or not, but one in which we are quite sure that reference concepts cannot fully explain an object. This type of neutrality, which we call *indeterminacy*, is therefore different to the imprecision symmetry, and it is not necessarily associated to a fuzzy representation framework, but to an idea of separation between poles.

For example if we consider the semantic references given by the poles *very tall* and *very short*, it is quite likely that, when applied to any appropriate universe of discourse (for instance the students in a classroom), there can be some individual who cannot be considered as either *very tall* or *very short*. This produces an *indeterminacy* which should not be confused with a degree of half-verification of both poles. Instead, *indeterminacy* strongly suggests the search for more information, perhaps in order to introduce a new category, maybe *half-a-way* in a linear order where poles stand as extreme values. But *indeterminacy* does not necessarily lead to such a linear scale.

Consider now that the poles are related through an antagonistic over-additive duality operator D, such that $Q = D(P)$. Thus, each logically independent pole provides different information, and as a consequence of the over-additive nature of the semantic operator, both poles are somehow *overlapping* (in the sense of redundantly covering the universe of discourse and leaving *no space* in between poles). As previously stated, this is not a situation in which we know that reference concepts cannot fully explain an object, but one in which we are not sure which one of those concepts fully explains the object. Then, this type of neutrality, which we call *ambivalence*, is also different to the imprecision symmetry, not necessarily associated to a fuzzy representation framework, but to an idea of redundancy in between poles.

As an instance, consider now the references *not very short* and *not very tall* for understanding the *size = height* of an individual. Some *extremely short* and *extremely tall* students can fail to verify either pole (*indeterminacy*). But certainly some students can fulfil both opposite references, being *not very tall* and *not very short*, at the same time, thus producing *ambivalence* between poles. In a fuzzy classification context (see again [1], [2]), the appearance of *ambivalence* would signal overlapping classes, suggesting the search for more restricted references or classes that exclude the overlap.

In the previous examples we started from the opposition between a pole and its negation, in the simplest case, or between two dual poles, but in both cases we are assuming that these references are modelled based on a simple, 1-dimensional underlying criterion or characteristic. In many contexts, however, the poles are rather *complex* dual concepts (as *good/bad or big/small*, for example), that show a multi-dimensional nature and suggest decomposition in terms of simpler reference concepts. This situation can be associated to a multi-criteria framework, in which the verification of a pair of poles is necessarily obtained through the aggregation of several criteria. It is in this context where *conflict* can naturally appear as another type of neutrality. Such *conflict* should be naturally expected within multi-criteria paired structures, whenever serious arguments for both poles are simultaneously found in different, independent criteria (like when in the *big/small* example we find that someone is simultaneously *very tall* but *very slim*). This situation suggests that complex poles can show a kind of conflictive relation, different to the ambivalent overlapping associated to redundant poles over a simple, 1-dimensional characteristic. This type of neutrality relates to another different hesitation, associated to a disagreement or collision between arguments that refer to different criteria. In fact, this *conflict* should not be expected when dealing with paired structures on a 1-dimensional argument. Such colliding arguments get a clear meaning only within multi-criteria paired structures, once poles are understood as complex positively-defined dual concepts that harvest families of underlying independent arguments. Of course, different kinds of *conflict* can be acknowledged in higher multi-dimensional problems besides the above conflict between two underlying criteria.

4.2 Example: preference representation

To conclude this section, let us introduce now a brief example in which the previous ideas are illustrated in the context of preference modelling and representation. In our opinion, preference models constitute a particularly adequate framework to illustrate some issues of paired structures in relation with bipolar knowledge representation models and how all those neutralities can simultaneously appear in practice.

Let us remind that preference models are typically based on the pairwise comparison of decision alternatives in relation with a family of preference predicates. For instance, we may analyse two possible holyday trips, let us denote them by x and y, in terms of whether we consider x to be clearly preferred to y. The implicit comparison predicate in this case is *more preferred than*, which is usually referred as the *strict preference* predicate. Note that we can carry out the comparison the other way round, applying the same strict preference predicate to the same two alternatives, but switching the order of x and y, so in this case we would be analysing whether y is strictly preferred to x. Such reversed predicate is also referred as the *inverse* strict preference, and in practice we need at least both predicates (a preference and its inverse) in order to devise the preference relationship between x and y. This allows understanding preference models in terms of a bipolar representation framework, since a preference and its inverse in fact play the role of opposite references (poles), from which other predicates (like indifference or incomparability) could be defined, as usually are, in order to enable capturing a wider spectrum of preference attitudes and produce more realistic representation models. The set of all considered or allowed predicates constitutes then a *preference structure*, in which some relations among the different predicates should hold (in order to acquire an actual structural performance, see again [31]), and which represents the fundamental valuation structure or scale for preference representation.

Then, our point is that a preference structure cannot be understood unless we consider both the semantic relationship between the basic preference predicates from which the structure is built, and how the different preference predicates are related to, and obtained from, the configuration of those basic references. That is, a preference structure constitutes an instance of second-level valuation structure, in which the semantics of its components is based on a particular interpretation of the first-level structure arising from the opposing references. In other words, preference structures rely on an underlying paired structure, from which the semantics of the different preference predicates is developed in terms of their relationship with the references and the different neutralities arising from their semantic opposition.

For instance, in preference representation it is usually admitted (see e.g. the classical book of Von Neumann and Morgenstern [34], as well as [21]) that a strict preference and its inverse can simultaneously fail to hold for some pairs of objects. Then, since neither of the poles are verified, a neutral indeterminacy is reached, which is usually interpreted as a preference predicate of *indifference*, constituting an additional intermediate concept in preference structures. However, it is important to note that assigning the meaning of *indifference* to such indeterminacy does not belong to the semantic relation between poles. In fact, at a first semantic level such indeterminacy is just saying that x is not strictly preferred to y, and that neither y is preferred to x. It could be that in fact x and y are not comparable, or that we have not enough information or knowledge to state any preference judgement. Thus, such a meaning of *indifference* is only acquired through a set of assumptions taken to produce a specific second level interpretation of the underlying paired structure.

On the other hand, *indifference* can also hold if we are considering weak preferences instead of strict preferences as opposite poles (remind that a weak preference, represented by the predicate *at least as preferred as*, is just the negation of the inverse strict preference). In fact, it is usually admitted (see for instance [21]) that a weak preference and its inverse weak preference can overlap, thus defining a different *ambivalent* category available for the first-level valuation of pairs of alternatives. However, assigning the meaning of *indifference* to such ambivalence again depends on a subsequent, second-level argument taken to interpret the neutralities of the underlying paired structure.

Another issue of paired structures that preference models help to illustrate is that of the *simultaneous* appearance of the different types of neutrality. In the previous examples related to the height of a group of students, we explicitly assumed that the involved duality operator was sub-additive (over-additive). This enabled the emergence of indeterminacy (ambivalence) as the specific type of neutrality associated to such duality operator, but at the same time it excluded the emergence of ambivalence (indeterminacy). However, general duality operators are not constrained to be either sub-additive or over-additive, thus allowing the different types of neutrality to appear together in the same paired structure and be assigned to different objects, in such a way that an object may be associated to indeterminacy while another one may verify ambivalence, or any other kind of neutrality. That is, the different types of neutrality are not mutually exclusive, but they can appear simultaneously in a paired structure as a consequence of the particular semantic relationship between the references. This is particularly usual and relevant in preference models, in which different objects (i.e., pairs of alternatives) may be associated to different preference predicates, in turn arising from different first-level neutralities.

For example, within preference representation (see [21], [43], but also [33]), we may use the following terms when comparing two alternatives x and y:

- x is preferred to y (strict preference $x>y$);
- y is preferred to x (strict preference $y>x$);
- x outranks y (weak preference $x\geq y$);
- y outranks x (weak preference $y\geq x$);
- x is equivalent to y (indifference $x\sim y$);
- x is in conflict with y (incomparability $x\|y$);
- Ignorance (lack of knowledge regarding the preference status of x and y).

Notice that all these preference predicates can appear and be applied within the same decision problem, and that they may in turn be associated to different available valuations of the underlying paired structure. For instance, when dealing with strict preferences as poles, as stated above, indeterminacy is usually interpreted as *indifference*, and the overlap of these references is usually associated to a *conflict* (arising from a collision on different underlying criteria), in turn commonly interpreted as *incomparability*. Note also that while the first six relations are standard in preference representation (see again [21], [33], [43]), the last one should be introduced to prevent lack of information, or simply to model the initial ignorance stage when we have not enough information to discriminate between alternatives [31]. To some extent, this last category may be associated to a state of absolute imprecision in our estimation of the basic preference degrees (as we do not have any idea of what degrees to assign).

In summary, let us stress that different types of neutrality may appear depending on the semantic duality relating the poles. But it is the existing semantic relationship between poles determines the particular meaning of the different types of neutrality that may arise, and at the same time, of the whole paired structure. All these kinds of neutrality are quite often confused (and labelled under the same word) since they all provoke hesitation. However, the point here is that behind each one of those hesitancies or neutralities underlies a different informative status.

5 Building paired structures

Not stressing the structural issue, as pointed out in [31], implies certain potential doubts about basic issues, starting from the concept of *pole* itself. The concept of *pole* should be precisely described; otherwise it will not be possible to distinguish any arbitrary couple of concepts from a paired approach. Moreover, the role and meaning of the neutral element should be also clarified. If structural issues are not properly addressed in our mathematical model, we might be easily confused between different approaches simply because the proposed mathematical models appear as isomorphic. But such one-to-one correspondence is only due to the fact that the existing relations between elements have not been made explicit. This was the main issue raised in [31], in order to explain the *intuitionistic* discussion underlying Dubois, Gottwald, Hajek, Kacprzyk and Prade [14] and Atanassov [4]. Our mathematical models should focus on capturing all the essential aspects of reality. Listing elements should be accompanied of their associated structure, by describing the relation between those elements. It is the structural difference what justifies a denomination, not the other way round. Structural performance of a set of concepts does not come with a set of isolated objects or names. If these elements suggest a structure, such a structure should be formally stated.

We offer now a standard procedure to build up paired structures:

1) As already stressed, we start from a concept and its negation (only one option in the crisp case). Such negation is a must in our model, since it defines what’s inside and outside our original concept. But a concept and its negation are paired concepts that cannot generate any additional concept, since both contain the same information.

2) Then, we need a dual concept, different from such a negation. In this way we obtain two paired concepts whose semantic relation will generate additional and specific neutral concepts.

3) In case our original concept and its dual concept do not overlap (dual concept implies negation), *indeterminacy* arises (both paired concepts do not fully explain

reality, and it is suggested a search for additional information, perhaps a new intermediate concept or symmetry).

4) In case our original concept and its dual concept overlap, *ambivalence* arises (the existence of a new concept associated to such overlapping is suggested together with a reshaping of the poles into more precise concepts).

5) Of course *indeterminacy* and *ambivalence* can appear simultaneously (overlapping in some objects might suggest *ambivalence* and lack of fulfilment in some other objects might suggest *indeterminacy*).

6) We can also detect *conflict* if our paired concepts are viewed in our mind as complex concepts that can be decomposed into and aggregated from simpler concepts.

7) Each one of those simpler concepts is subject to the previous arguments (a pole, its negation, its dual concept and potential *indeterminacy, ambivalence, conflict*, …)

As previously discussed, a fuzzy (or probabilistic) representation is not actually needed in order to define a paired structure and enable the emergence of the abovementioned types of neutrality. In fact, the main difference between a crisp and a fuzzy approach is the increased complexity of estimating fuzzy membership functions for all the involved concepts, both the references and the neutral ones. That is, our original concept here comes with a membership function, that should be estimated, and according to a particular fuzzy negation, we shall obtain the estimated membership function of the complementary concept. But the degree of membership to our dual concept should be also estimated. Then, some objects might suggest indeterminacy, other objects might suggest ambivalence, and others might suggest conflict. Membership functions of all these neutral concepts should be also estimated, since it is natural to assume that in such a fuzzy context these neutral concepts will also be gradable. Symmetry points may arise in this context whenever the degrees of membership to a concept and its negation are the same. However, symmetry points are associated to equilibrium between imprecise references, and thus they do not properly define new intermediate concepts or valuation alternatives different from the poles. All these steps also apply in case of a probabilistic representation, although now we have to estimate the probability distributions of all the involved concepts instead of their membership functions.

Both in case we deal with probabilities or membership functions (no matter if they are defined in the unit interval or in any other scale) we are subject to an estimation imprecision problem. If we cannot estimate exact values, it may be important to represent such uncertainty by means of a more complex formalism. A standard solution is to consider some kind of type II probabilities or fuzziness (see [28] and the book edited by Bustince, Herrera and Montero [9]). But an easy approach to imprecision is to associate an interval to each imprecisely estimated value. The wider such an interval is, the more imprecise our estimation. Maximum imprecision will be then associated to the complete interval, which would mean that we simply do not have any useful information about such estimated value (see [46] for a complete approach within a probabilistic framework, and [45] for the seminal approach within a fuzzy one). However, as previously discussed, this symmetric situation should not be associated to a concept generated from the semantic relationship between the poles.

6 Final comments

In this paper we have presented a systematic approach to different types of paired sets, to be considered as an alternative to the notion of bipolarity proposed by Zhang and Zhang [56] and particularly by Dubois and Prade [15], [16], [17], both deeply related to Atanassov's intuitionistic fuzzy sets [3]. All these models are somehow based upon two basic opposite concepts (but see [31]). The main aim of this strictly positional paper is to bring some light into this discussion, stressing the constructive argument towards a more general structure that should particularly stress the role of neutral concepts. We postulate that paired structures represent the basic model for most learning processes, which quite often starts from two opposite concepts (see also [20], [30]). Our approach introduces semantics as a key aspect to be taken into account. It is particularly stressed that if the semantic structure of the model is not properly specified, we might be confused by different structures that share equivalent underlying lattices but not a common semantics. In fact, it has been shown that each particular semantic relationship between the poles refers to a specific structure that generates characteristic neutral concepts.

We therefore claim that the term *paired* should be mainly associated to the existence of two different but dual concepts whose relation can create specific *neutral* concepts. In this sense,

- If properties associated to both poles are non-overlapping antonymous concepts, they cannot simultaneously hold, and they produce a neutral element to represent *indeterminacy* (which should not be confused with an intermediate category within a linear scale). This should be in our opinion the right allocation of Atanassov's Intuitionistic Fuzzy Sets [3] (avoiding the "non-membership" as the opposite pole to "membership", as pointed out in [31]).
- If properties associated to both poles imply overlapping antagonistic concepts, they allow a neutral element to represent *ambivalence* (which in certain contexts might be acknowledged as an intermediate value within a linear scale).
- If properties associated to both poles are conceived as complex, they may allow a neutral element to represent *conflict.*

Hence, we find at least three different neutral aspects that paired concepts can create: *indeterminacy* (which can be later on justified in terms of lack of information or poorness of the system defined by the poles if viewed as a classification system), *ambivalence* and *conflict*. It is the specific semantic tension (opposition) between our two basic poles the key aspect to be analysed. Of course these three different neutralities can appear in the same problem, together with other representations of *uncertainty* and *imprecision*. How we can simultaneously manage all these parameters becomes a suggesting and necessary objective for future research.

As a consequence, we see three main types of paired fuzzy sets:

- Those "basic" paired concepts based upon the negation of both poles, with no additional neutral concept being allowed.
- Those "dual" paired concepts based upon a 1-dimensional duality, subject to antonym or antagonistic components, but allowing *indeterminacy* and *ambivalence* neutralities.
- And those "complex" paired concepts based upon multidimensional dualities, where in addition to *indeterminacy* and *ambivalence* we can find different levels of *conflict*.

A possible drawback of the general case of paired fuzzy structures is that they can be considered too complex for some applications, since it might imply the direct estimation of quite a number of degrees of verification, for each object. An alternative approach is to consider that some degrees of membership or membership functions can be obtained from a smaller subset of estimations, by means of appropriate operators previously defined. Hence, the whole system can be fully described from a few basic estimations. Notice that, to some extent, this is similar to what is usually done in preference modelling: see e.g. [21], [33] and particularly [29], or the extensions into Belnap's logic [7] in [36], [51]. It is also interesting to see the similarities between fuzzy preference structures in [21], [22], and the continuous extension proposed in [36] and [37], further analysed in [42] and [53]. In all these examples, aggregation plays a crucial role in constructing and constraining the semantics of different notions from that of the reference concepts (see e.g. [18]).

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