Consistency and Stability in Aggregation Operators: An Application to Missing Data Problems

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1 Introduction

An *aggregation operator* [1, 5, 7, 8, 9, 12] is usually defined as a real function A_n such that, from n data items $x_1,\ldots,x_n$ in $[0,1]$, produces an aggregated value $A_n(x_1,\ldots,x_n)$ in [0,1] [4]. This definition can be extended to consider the whole family of operators for any n instead of a single operator for an specific n. This has led to the current standard definition [4, 15] of a *family of aggregation operators* (*FAO*) as a set $\{A_n : [0,1]^n \to [0,1], n \in N\}$, providing instructions on how to aggregate collections of items of any dimension n. This sequence of aggregation functions $\{A_n\}_{n\in N}$ is also called *extended aggregation functions* (*EAF*) by other authors [15, 5].

In this work, we will deal with two different but related problems for *extended aggregation functions* or *family of aggregation operators*

On one hand, let us remark that in practice, it is frequent that some information can get lost, be deleted or added, and each time a cardinality change occurs a new aggregation operator A_m has to be used to aggregate the new collection of m elements. However, it is important to remark that a relation between $\{A_n\}$ and $\{A_m\}$ does not necessarily exist in a family of aggregation operators as defined in [4]. In this context, it seems natural to incorporate some properties to maintain the logical *consistency* between operators in a *FAO* when changes on the cardinality of the data occur, for which we need to be able to build up a definition of family of aggregation operators in terms of its logical consistency, and solve each problem of aggregation without knowing *apriori* the cardinality of the data. This is, the operators that compose a *FAO* have to be somehow related, so the aggregation process

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H. Bustince et al. (eds.), *Aggregation Functions in Theory and in Practise*,
Advances in Intelligent Systems and Computing 228,
DOI: 10.1007/978-3-642-39165-1_48,

remains *the same* throughout the possible changes in the dimension n of the data. Therefore, it seems logical to study properties giving sense to the sequences $A(2)$, $A(3), A(4), \ldots$, because otherwise we may have only a bunch of disconnected operators. With this aim, in [26, 27, 16] a notion of consistency based on the robustness of the aggregation process, i.e. *stability*, was studied. In this sense, the notion of *stability* for a family of aggregation operators is inspired in continuity, though our approach focuses in the cardinality of the data rather than in the data itself, so we can assure some robustness in the result of the aggregation process. Particularly, let $A_n(x_1, \ldots, x_n)$ be the aggregated value of the n-dimensional data $x_1, \ldots, x_n$. Now, let us suppose that a new element x_{n+1} has to be aggregated. If x_{n+1} is close to the aggregation result $A_n(x_1, \ldots, x_n)$ of the n-dimensional data $x_1, \ldots, x_n$, then the result of aggregating these $n+1$ elements should not differ too much with the result of aggregating such n items. Following the idea of stability for any mathematical tool, if $|x_{n+1} - A_n(x_1, \ldots, x_n)|$ is small, then $|A_{n+1}(x_1, \ldots, x_n, x_{n+1}) - A_n(x_1, \ldots, x_n)|$ should be also small. It is important to note that if the family $\{A_n\}$ is not symmetric (i.e. there exist a n for which the aggregation operator A_n is not symmetric), then the position of the new data is relevant to the final output of the aggregation process. From this observation, in [26, 27, 16] it some definitions of *stability* that extend the notion of self-identity defined in [29] were presented.

On the other hand, a problem that has not been received too much attention is how to obtain an aggregation when some of the variables to be aggregated are missing. If the aggregation operator function A_n present a clear definition for the case in which the dimension is lower, this problem is easily solved, but not always is a trivial task. Following the ideas of stability, in this paper we will deal with the problem of missing data for some well-known families of aggregation operators.

2 Consistency in Families of Aggregation Operators

As has been pointed out in the introduction, a *family of aggregation operators* (*FAO*) is a set of aggregation operators $\{A_n : [0,1]^n \rightarrow [0,1], n \in N\}$, providing instructions on how to aggregate collections of items of any dimension n. Few properties has been studied or defined to a *FAO* in general (see [27] for more details). In [15] it is shown that the aggregation functions of a family can be related by means of certain grouping properties. For example, continuity, symmetry or other well-known properties defined usually for aggregation functions can be defined in a general way for a family of aggregation operators imposing that these properties have to be satisfied for all n. Nevertheless, these kind of properties don't guarantee any consistency in the aggregation process since they don't establish any constraint among the different aggregation functions.

In the aggregation operators' literature it is possible to find some properties for aggregation operators that can be understood as properties for the whole family establishing some relations among the different aggregation operators. Here we recall some of them.

An important notion that establish some relations among members of different dimensions in a FAO is the notion of recursivity. Recursivity was introduced in [8] in the context of OWA operators aggregation functions. Further, and following [8], in [1, 9, 12, 18] recursivity of a FAO was also studied in a more general way to establish some consistency in the aggregation process. In order to understand this notion of recursivity, first it necessary to defined the concept of *ordering rule*.

Definition 1 (see [1, 9, 12] for more details). Let us denote $\pi_n(x_1,\ldots,x_n) = (x_{\pi_n(1)},\ldots,x_{\pi_n(n)})$. An *ordering rule* π is a consistent family of permutations $\{\pi_n\}_{n\geq 2}$ such that for any possible finite collections of numbers, each extra item x_{n+1} is allocated keeping previous relative positions of items, i.e.

$\pi_{n+1}(x_1,\ldots,x_n,x_{n+1})$ equal to $(x_{\pi_n(1)},\ldots,x_{\pi_n(j-1)},x_{n+1},x_{\pi_n(j)},\ldots x_{\pi_n(n)})$ for some $j \in \{1,2,\ldots,n+1\}$.

Definition 2 (see [1, 9, 12] for more details). A family of aggregation operators $\{A_n : [0,1]^n \longrightarrow [0,1]\}_{n>1}$ is left-recursive if there exist a family of binary operators $\{L_n : [0,1]^2 \longrightarrow [0,1]\}_{n>1}$ verifying $A_2(x_1,x_2) = L_2(x_{\pi(1)};x_{\pi(2)})$ and

$A_n(x_1,x_2,\ldots,x_n)$ equal to $L_n(A_{n-1}(x_{\pi(1)},\ldots,x_{\pi(n-1)};x_{\pi(n)})$ for all $n \geq 2$,

where π is an ordering rule.

In a similar way, it is possible to define the right recursive rules.

Definition 3 (see [1, 9, 12] for more details). A family of aggregation operators $\{A_n : [0,1]^n \longrightarrow [0,1]\}_{n>1}$ is left-recursive if there exist a family of binary operators $\{R_n : [0,1]^2 \longrightarrow [0,1]\}_{n>1}$ verifying $A_2(x_1,x_2) = R_2(x_{\pi(1)};x_{\pi(2)})$ and

$A_n(x_1,x_2,\ldots,x_n)$ equal to $R_n(x_1,A_{n-1}(x_{\pi(2)},\ldots,x_{\pi(n-1)})$ for all $n \geq 2$,

where π is an ordering rule.

From previous two definitions, it is possible to introduce the concept of LR recursivity in the following way.

Definition 4 (see [1, 9, 12] for more details). A family of aggregation operators $\{A_n : [0,1]^n \longrightarrow [0,1]\}_{n>1}$ is left-right recursive if there exist two families of binary operators $\{R_n : [0,1]^2 \longrightarrow [0,1]\}_{n>1}$ and $\{L_n : [0,1]^2 \longrightarrow [0,1]\}_{n>1}$ verifying the left and the right recursive conditions simultaneously.

A particular case of previous definitions (when the binary aggregation is the same and the recursivity is from the left) can be founded in [4, 5], in which it is said that a FAO $\{A_n\}_{n\in N}$ will be *recursive* if it verifies

$$A_n(x_1,x_2,..,x_n) = A_2(A_{n-1}(x_1,x_2,..,x_{n-1}),x_n).$$

Let us observe that previous definitions guarantee certain consistency in the family $\{A_n\}$ since the A_n function is build taking into account the previous function A_{n-1}. Taking this definition into account, the previously described situation, in which the different operators A_n have no relation among them, cannot hold.

Other properties that establish some conditions among the different members of the whole family are the following:

Definition 5. Decomposability. [see [4, 5] for more details]

A family of aggregation operators $\{A_n\}_{n\in N}$ satisfies the *decomposability* property if $\forall n,m = 1,2,...$, $\forall x \in [0,1]^m$ and $\forall y \in [0,1]^n$ the following holds:

$$A_{m+n}(x_1,x_2,..,x_m,y_1,y_2,..,y_n) =$$
$$A_{m+n}(\underbrace{A_m(x_1,x_2,..,x_m),...,A_m(x_1,x_2,..,x_m)}_{\text{m times}},y_1,y_2,..,y_n)$$

Definition 6. Bisymmetry [see [4, 5] for more details]

A family of aggregation operators $\{A_n\}_{n\in N}$ satisfies the *bisymmetry* property if $\forall n,m = 1,2,...$ and $\forall x \in [0,1]^{mn}$ the following holds:

$$\begin{aligned} A_{mn}(x_1,x_2,..,x_{mn}) &= A_m(A_n(x_{11},x_{12},..,x_{1n}),...,A_n(x_{m1},x_{m2},..,x_{mn})) \\ &= A_n(A_m(x_{11},x_{21},..,x_{m1}),...,A_m(x_{1n},x_{2m},..,x_{mn})) \end{aligned}$$

Although previous definitions impose some stability or consistency to a family of aggregation operators, these ones are more focused on the way in which it is possible to build the operator aggregation function of dimension n from aggregation operators of lower dimensions than on a general idea of stability or consistency. Moreover/However, pursuing the idea of consistency of a family of aggregation operators and based on the self-identity definition given by Yager in [29], in [26, 27, 16] the notion of *strict stability* of a FAO was defined in three different levels. The idea is simple: in a family of aggregation operators, A_n and A_{n+1} should be closely related, in the sense that if a new item has to be aggregated and such a new item is the result of the aggregation of the previous n items, then the result of the aggregation of these $n+1$ items should be close to the aggregation of the n previous ones. Otherwise, the aggregation operator function A_{n+1} would differ too much from the aggregation operator function A_n, producing and *unstable* family $\{A_n\}_{n\in N}$. Taking into account that in general FAOs are not necessarily symmetric, two possibilities (left and right stability) were analyzed in the definition of strict stability.

Definition 7. Let $\{A_n : [0,1]^n \to [0,1], n \in N\}$ be a family of aggregation operators. Then, it is said that:

1. $\{A_n\}_n$ is a R-strictly stable family if

   $$A_n(x_1,x_2,...x_{n-1},A_{n-1}(x_1,x_2...,x_{n-1})) = A_{n-1}(x_1,x_2,...,x_{n-1})$$

   holds $\forall n \geq 3$ and $\forall \{x_n\}_{n\in N}$ in $[0,1]$

2. $\{A_n\}_n$ is a L-strictly stable family if

   $$A_n(A_{n-1}(x_1,x_2,...,x_{n-1}),x_1,x_2,...x_{n-1}) = A_{n-1}(x_1,x_2,...,x_{n-1})$$

   holds $\forall n \geq 3$ and $\forall \{x_n\}_{n\in N}$ in $[0,1]$

Although previous definitions can be relaxed from an asymptotic and probabilistic point of view (see [27]), in this work we are going to focus on the strict stability conditions just exposed.

3 On j-L and i-R Stability

Previous definitions impose that the information that has to be aggregated appears in the last or in the first position. Obviously, this assumption could be relaxed. Taking into account that the stability concept presented in [27] could be relaxed, in [2], it is introduced the notion $j-L$ stability, imposing now that the new datum enters in the i-th position from the right. And similarly, we can define the $i-R$ strictly stability imposing that the new datum enters in the j-th position. Obviously, the relaxed versions of strict stability from an asymptotic and probabilistic point of view could be defined in a similar way.

Definition 8. Let $\{A_n : [0,1]^n \to [0,1], n \in N\}$ be a family of aggregation operators. Then, it is said that:

1. $\{A_n\}_n$ is a i-R-strictly stable family if

$$A_n(x_1,x_2,...x_{n-i},A_{n-1}(x_1,x_2...,x_{n-1}),\ldots,x_{n-1}) = A_{n-1}(x_1,x_2,...,x_{n-1})$$

   holds $\forall n \geq 3$ and $\forall \{x_n\}_{n\in N}$ in $[0,1]$.

2. $\{A_n\}_n$ is a j-L-strictly stable family if

$$A_n(x_1,\ldots,x_{j-1},A_{n-1}(x_1,x_2,...,x_{n-1}),x_j,\ldots,x_{n-1}) = A_{n-1}(x_1,x_2,...,x_{n-1})$$

   holds $\forall n \geq 3$ and $\forall \{x_n\}_{n\in N}$ in $[0,1]$.

Let us observe that the $i-L$ and $j-R$ strict stability conditions previously defined are equivalent (for any i and/or j) when the FAO is symmetric. But, in general, it is very difficult that a non-symmetric FAO satisfies simultaneously more than one condition (see [27] for more details). In our opinion, the conditions that a general FAO should satisfy to be strictly stable shouldtake into account the structure of the data that has to be aggregated (and of course also the way in which this family is defined).

In a similar way as symmetric FAOs impose indirectly that the structure of the data hasn't effect in the aggregation result (since the order in which the information is aggregated is not relevant), non-symmetric families of aggregation operators makes the assumption that the data has an inherent structure and thus the position of the data items in the aggregation process is relevant. Strict stability or consistency of an aggregation process (among other properties) should also take into account that the data may present some structure. In the following section, we will present some possible definitions of stability for non-symmetric FAO that will be dependent of the structure of the data that is aggregated.

4 Dealing with Weights and Missing Data: An Application of Stability

To illustrate an interesting application of the concept of stability let us introduce a very simple example. Suppose a multi-criteria decision problem having four criteria C_1, C_2, C_3, C_4. A jury, after some deliberations, evaluates the different alternatives on the four criteria and then uses a weighted mean operator as aggregation rule. Then, what happens if for one alternative some information related with the criteria C_4 has been lost or deleted? What should be the aggregation of the remaining information? As has been pointed out in the introduction, this dimensional problem has not received too much attention in the aggregation literature. This problem is related with the following question: what should be the relations between aggregation operators of different dimensions to be consistent? In this section, we will analyze the stability of some well-known families trying to deal with this problem.

Let us recall again that our aim is not to decide how the vector of weights $w^4 = (w_1^4, w_2^4, w_3^4, w_4^4)$ should be, but to guarantee some stability or consistence in the aggregation process. For example, it would seem rather inconsistent to choose $w^4 = (1/8, 4/8, 1/8, 2/8)$ if data is available regarding the four mentioned criteria, but also choosing $w^3 = (0.8, 0.2, 0)$ in case the criteria C_4 presents a missing value for one of the alternatives. From the point of view of consistency, this jury would not be stable.

We first focus our attention in the weighted mean aggregation family. This family, $\{W_n, n \in N\}$, is defined through a vector of weights $w^n = (w_1^n, ..., w_n^n) \in [0,1]^n$ in such a way that $W_n(x_1, ..., x_n) = \sum_{i=1}^{n} w_i^n x_i$, where $\sum_{i=1}^{n} w_i^n = 1$ and $(x_1, ..., x_n) \in [0,1]^n \;\; \forall n$. The stability of this family was studied from a *LR* point of view in [27]. Nevertheless, as we will see below, this study can not be directly applicable to the missing value problem in aggregation problems. In the $\{W_n\}_n$ FAO, the weights associated to the elements being aggregated represent the *importance* of each one of these elements in the aggregation process. For this reason, the weighted mean surely is one of the most relevant and used aggregation operators in many different areas (e.g. statistics, knowledge representation problems, fuzzy logic, multiple criteria decision making, group decision making, etc.), and one of the most studied problems in all these areas is how to determine these *importance* weights.

A missing data problem appears when for a specific object $x = (x_1, \ldots, x_n)$ one of its values is missing. In the previous example, $n = 4$, the information regarding an alternative is aggregated through $W_4(x_1, \ldots, x_4) = \sum_{i=1,4} w_i^4 x_i$, and the importance of the four criteria has been established by means of the four dimensional vector $w^4 = \left(w_1^4, w_2^4, w_3^4, w_4^4\right) = (1/8, 2/4, 1/8, 1/4)$. Now, consider an alternative x that presents the values $x = (0.3, 1, 1, \textit{not evaluable})$. What should be the aggregation? What should be the aggregation operator A_3?

If we decide to use the weighted mean aggregation function for $n = 3$ (i.e. $A_3 = W_3$), the problem here is to determine the weights vector w^3. A possibility is to impose that W_3 and W_4 satisfy the strict stability conditions. Nevertheless, asstudied in [27], for non-symmetric FAOs, it is very difficult that more than one stability condition is satisfied simultaneously. Let us observe that the different strict stability conditions (L, R, $i-L$ or $j-R$ for different i and j positions) will give us different possibilities and solutions for the vector w^3. So, what stability condition should we choose? Taking into account that the 4-th value x_4 is the one missing, it seems reasonable to impose the R (or equivalently the 4-L or 1-R) strict stability condition, i.e.

$$W_4(x_1,x_2,x_3,W_3(x_1,x_2,x_3)) = W_3(x_1,x_2,x_3)$$

for any x_1, x_2 and x_3 in $[0,1]$ Concerning our example, this condition holds if and only if

$$\frac{1}{8}x_1+\frac{4}{8}x_2+\frac{1}{8}x_3+2/8(w_1^3x_1+w_2^3x_2+w_3^3x_3) = w_1^3x_1+w_2^3x_2+w_3^3x_3 \ \ \forall x_1,x_2,x_3 \in [0,1],$$

which is equivalent to say that $w^3 = (\frac{1}{6},\frac{4}{6},\frac{1}{6})$. Let us observe that this vector maintains the relative proportions between the original weights for the non- missing values in the positions 1, 2 and 3.

In the previous example, the fourth value of the alternative $x = (0.3, 1, 1, \textit{not evaluable})$ is missing. But what should be the aggregation if the missing value is the second one? In general, and for non-symmetric FAOs where the position in which data appear is relevant, if there is some information $x = (x_1,\dots,x_n)$ that has to be aggregated and we have a missing value x_j, we should impose strict $j-L$ stability or equivalently $(n-(j+1))-R$ strict stability to find the relations that should exist between the aggregation functions A_n and A_{n-1} in the whole family. In the following proposition it is established a condition that guarantees the strict j-L stability of the family $\{W_n\}_n$.

Proposition 1. *(see also [2]) Let* $w^n = (w_1^n,...,w_n^n) \in [0,1]^n, n \in N$*, be a sequence of weights of a weighted mean family* $\{W_n\}_{n\in N}$ *such that* $\sum_{i=1}^{n} w_i^n = 1$ *holds* $\forall n \geq 2$. *Then, the family* $\{W_n\}_{n\in N}$ *is a j-L-strict stable family if and only if the sequence of weights satisfies*

$$\begin{cases} w_k^n = (1-w_j^n)\cdot(w_k^{n-1}) & for \ \ k = 1,\dots,j-1 \\ w_{k+1}^n = (1-w_j^n)\cdot(w_k^{n-1}) & for \ \ k = j,\dots,n-1 \end{cases} \quad \forall n \in N.$$

Proof.

Note that for a generic weighted mean FAO $\{W_n\}_{n\in N}$ with weights w^n, $n \in N$, the j-L-strict stability property can be restated as

$$0 = |A_n(x_1,\dots,x_{j-1},A_{n-1}(x_1,x_2,...,x_{n-1}),x_j,\dots,x_{n-1}) - A_{n-1}(x_1,x_2,...,x_{n-1})|$$

which is equivalent to

$$\sum_{i=1}^{j-1}(w_i^n-(1-w_j^n)w_i^{n-1})x_i+\sum_{i=j}^{n-1}(w_{i+}^n-(1-w_j^n)w_i^{n-1})x_i=0\forall x_1,\ldots x_{n-1}\in[0,1].$$

From previous equation it is straightforward to conclude that the proposition holds.

In order to extend the previous properties to a more general class of FAO, we will analyze the j-L strict stability for transformations of the original FAO. But, let us first introduce the following notations and definitions.

Definition 9. Let $f:[0,1]\to A$ be a continuous and injective function, and let $\{\phi_n : A\to A,\ n\in N\}$ be a family of aggregation operators defined in the domain A. Then, the transformed aggregation operator family $\{M_f^{\phi_n}\}_{n\in N}$ is defined as:

$$M_f^{\phi_n}(x_1,\ldots,x_n)=f^{-1}\left(\phi_n\left(f(x_1),\ldots,f(x_n)\right)\right)$$

Let us observe that if f is the identity function, then the transformation family coincides with the original family. If $\{\phi_n\}_{n\in N}$ is the mean or the weighted mean then $M_f^{\phi_n}$ is called quasi-arithmetic mean or weighted quasi-arithmetic mean. The quasi-arithmetic mean functions are very important in many aggregation analysis. Some well-known quasi-arithmetic aggregation families are: the geometric mean (when $f(x)=log(x)$), the harmonic mean (when $f(x)=1/x$) and the power mean (when $f(x)=x^p$), among others. It is important to remark that some of the most usual aggregation operators families (as for example the productory $\{P_n\}_{n\in N}$), can not be transformed or extended directly. For example if $f(x)=5x$, then $A=[0,5]$, but we can not guarantee that for all $n\in N$, $P_n\left(f(x_1),\ldots,f(x_n)\right)=\prod_{i=1}^n f(x_i)$ belongs to the interval $[0,5]$.

In the following proposition, we show that j-L-strict stability remains after transformation.

Proposition 2. *Let $\{\phi_n\}_{n\in N}$ and $\{M_f^{\phi_n}\}_{n\in N}$ be a family of aggregation operators and its extension or transformed aggregation. Then:*

$\{M_f^{\phi_n}\}_{n\in N}$ is a j-L-strictly stable family if and only if $\{\phi_n\}_{n\in N}$ is a j-L -strictly stable family in the domain A.

Proof:

Taking into account that $M_f^{\phi_n}\left(x_1,\ldots,x_{j-1},M_f^{\phi_n}(x_1,\ldots,x_{n-1}),\ldots,x_n\right)$ can be rewritten as

$$f^{-1}\left(\phi_n\left(f(x_1),\ldots,f(x_{j-1}),\phi_{n-1}(f(x_1),\ldots,f(x_{n-1})),\ldots,f(x_n)\right)\right),$$

the j-L strict stability condition for $\{M_f^{\phi_n}\}_{n\in N}$ can be formulated as

$$f^{-1}\left(\phi_n\left(f(x_1),\ldots,f(x_{j-1}),\phi_{n-1}(f(x_1),\ldots,f(x_{n-1})),\ldots,f(x_n)\right)\right) =$$
$$f^{-1}\left(\phi_{n-1}(f(x_1),\ldots,f(x_{n-1}))\right).$$

Hence, since f is a continuous and injective function, such a condition holds if and only if $\{\phi_n\}_n$ is an strictly stable family in A. And thus, the proposition holds.

Corolary The weighted quasi-arithmetic aggregation operators family is a j-L-strict stable family if and only if the sequence of weights satisfies

$$\begin{cases} w_k^n = (1-w_j^n)\cdot(w_k^{n-1}) \;\; for \;\; k=1,\ldots,j-1 \\ w_{k+1}^n = (1-w_j^n)\cdot(w_k^{n-1}) \;\; for \;\; k=j,\ldots,n-1 \end{cases} \quad \forall n \in N.$$

To conclude the study of the missing values in aggregation operators from a stability point of view, we will try to extend the previous analysis to a situation in which more than one value could be missing. Let us suppose that we have two missing values in the positions $r < s$. So we have $x = (x_1,\ldots,x_{r-1}, missing, \ldots, mising, x_{s+1},\ldots,x_n)$. Let us observe that the j-L strict stability condition can be stated in a more general way by imposing conditions between the aggregation functions A_n and A_{n-2} for this purpose.

Definition 10. Let $\{A_n : [0,1]^n \to [0,1], n \in N\}$ be a family of aggregation operators. Then, it is said that $\{A_n\}_n$ is a $r-s$-L-strictly stable family if

$$A_n(x_1,\ldots,x_{r-1},A_{n-2}(x_1,x_2,...,x_{n-2}),x_r,\ldots,A_{n-2}(x_1,x_2,...,x_{n-2}),x_{s-1},\ldots,x_{n-2})$$

coincides with $A_{n-2}(x_1,x_2,...,x_{n-2})$ $\forall n \geq 3$ and $\forall \{x_n\}_{n\in N}$ in $[0,1]$

Following this equation of the $r-s$-L strict stability, it is possible to build the aggregation operator A_{n-2} from A_n for a given n. Let us continue with the example of the four criteria. If now for one alternative the values associated with the criteria 2 and 3 are missing, and we decide to use the weighted mean aggregation function for $n=2$ (i.e. $A_2 = W_2$), the problem is to determine the weights vector w^2 from w^4 (which is the available information). Then, it seems reasonable to impose the $2-3$-L stability condition to find the weights associated with the aggregation operator W_2 i.e:

$$W_4(x_1, W_2(x_1,x_2), W_2(x_1,x_2), x_2) = W_2(x_1,x_2)$$

for any x_1, x_2 in $[0,1]$.

For notational convenience, we have denoted by x_1 the value for the first variable and by x_2 the value for the fourth variable. So it is $x = (x_1, missing, missing, x_2)$. Then, the condition above holds if and only if

$$\frac{1}{8}x_1 + \frac{4}{8}(w_1^2x_1 + w_2^2x_2) + \frac{1}{8}(w_1^2x_1 + w_2^2x_2) + 2/8(x_2) = w_1^2x_1 + w_2^2x_2 \;\; \forall x_1,x_2,x_3 \in [0,1],$$

which is equivalent to say that $w^2 = (\frac{1}{3}, \frac{2}{3})$. Let us observe that this vector maintains the relative proportions between the original weights for the non-missing values in the positions 1 and 4.

We would like to conclude this section pointing out that it is possible to define strict stability for a sequence of positions $r_1, \ldots r_k$ in a similar way as done above for two positions, allowing us to establish consistency conditions between the aggregation functions A_n and A_{n-k}.

5 Final Comments

In this work, we have continued with the key issue of the relationship that should hold between the operators in a family $\{A\}_n$ in order to understand they properly define a *consistent*. The basic concepts of consistency addressed as stability of a family of aggregation operators was presented in [27, 16, 26] in which it is defined the L and R strict stability in different levels. In this work we have extended some of these previous definitions into a more general framework defining the $i-L$ and $j-R$ strict stability for a family of aggregation operators and some of its analysis to the weighted quasi-arithmetic means families. In addition, we present an interesting application of the strict stability conditions to deal with missing data problems in an aggregation operator framework.

References

[1] Amo, A., Montero, J., Molina, E.: Representation of consistent recursive rules. European Journal of Operational Research 130, 29–53 (2001)

[2] Beliakov, G., James, S.: Stability of weighted penalty-based aggregation functions. Fuzzy Sets and Systems, doi:10.1016/j.fss.2013.01.007

[3] Beliakov, G., Pradera, A., Calvo, T.: Aggregation Functions, a Guide to Practitioners. STUDFUZZ, vol. 221. Springer, Heidelberg (2007)

[4] Calvo, T., Kolesarova, A., Komornikova, M., Mesiar, R.: Aggregation operators, properties, classes and construction methods. In: Calvo, T., et al. (eds.) Aggregation Operators New Trends and Aplications, pp. 3–104. Physica-Verlag, Heidelberg (2002)

[5] Calvo, T., Mayor, G., Torrens, J., Suñer, J., Mas, M., Carbonell, M.: Generation of weighting triangles associated with aggregation fuctions. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 8(4), 417–451 (2000)

[6] Carlsson, C.H., Fuller, R.: Fuzzy reaasoning in decision making and optimization. Springfield-Verlag, Heidelberg (2002)

[7] Cutello, V., Montero, J.: Hierarchical aggregation of OWA operators: basic measures and related computational problems. Uncertainty, Fuzzinesss and Knowledge-Based Systems 3, 17–26 (1995)

[8] Cutello, V., Montero, J.: Recursive families of OWA operators. In: Proceedings FUZZ-IEEE Conference, pp. 1137–1141. IEEE Press, Piscataway (1994)

[9] Cutello, V., Montero, J.: Recursive connective rules. International Journal of Intelligent Systems 14, 3–20 (1999)
[10] Dubois, D., Gottwald, S., Hajek, P., Kacprzyk, J., Prade, H.: Terminological difficulties in fuzzy set theory - the case of intuitionistic fuzzy sets. Fuzzy Sets and Systems 156, 485–491 (2005)
[11] Fung, L.W., Fu, K.S.: An axiomatic approach to rational decision making in a fuzzy environment. In: Zadeh, L.A., et al. (eds.) Fuzzy Sets and their Applications to Cognitive and Decision Processes, pp. 227–256. Academic Press, New York (1975)
[12] Gómez, D., Montero, J.: A discussion of aggregation functions. Kybernetika 40, 107–120 (2004)
[13] Gómez, D., Montero, J., Yáñez, J., Poidomani, C.: A graph coloring algorithm approach for image segmentation. Omega 35, 173–183 (2007)
[14] Gómez, D., Montero, J., Yánez, J.: A coloring algorithm for image classification. Information Sciences 176, 3645–3657 (2006)
[15] Grabisch, M., Marichal, J., Mesiar, R., Pap, E.: Aggregation Functions. Encyclopedia of Mathematics and its Applications (2009)
[16] Gómez, D., Montero, J., Tinguaro Rodríguez, J., Rojas, K.: Stability in aggregation operators. In: Greco, S., Bouchon-Meunier, B., Coletti, G., Fedrizzi, M., Matarazzo, B., Yager, R.R. (eds.) IPMU 2012, Part III. CCIS, vol. 299, pp. 317–325. Springer, Heidelberg (2012)
[17] Bustince, H., Barrenechea, E., Fernández, J., Pagola, M., Montero, J., Guerra, C.: Aggregation of neighbourhood information by means of interval type 2 fuzzy relations
[18] Bustince, H., de Baets, B., Fernández, J., Mesiar, R., Montero, J.: A generalization of the migrativity property of aggregation functions. Information Sciences 191, 76–85 (2012)
[19] Kolesárová, A.: Sequential aggregation. In: González, M., et al. (eds.) Proceedings of the Fifth International Summer School on Aggregation Operators, AGOP, Universitat de les Illes Balears, Palma de Mallorca, pp. 183–187 (2009)
[20] López, V., Garmendia, L., Montero, J., Resconi, G.: Specification and computing states in fuzzy algorithms. Uncertainty, Fuzziness and Knowledge-Based Systems 16, 301–336 (2008)
[21] López, V., Montero, J.: Software engineering specification under fuzziness. Multiple-Valued Logic and Soft Computing 15, 209–228 (2009)
[22] López, V., Montero, J., Rodriguez, J.T.: Formal Specification and implementation of computational aggregation Functions. In: Computational Intelligence, Foundations and Applications Proceedings of the 9th International FLINS Conference, pp. 523–528 (2010)
[23] Montero, J.: A note on Fung-Fu‘s theorem. Fuzzy Sets and Systems 13, 259–269 (1985)
[24] Montero, J., Gómez, D., Bustince, H.: On the relevance of some families of fuzzy sets. Fuzzy Sets and Systems 158, 2429–2442 (2007)
[25] Montero, J., López, V., Gómez, D.: The role of fuzziness in decision making. In: Wang, P.P., Ruan, D., Kerre, E.E. (eds.) Fuzzy Logic. STUDFUZZ, vol. 215, pp. 337–349. Springer, Heidelberg (2007)
[26] Rojas, K., Gómez, D., Rodríguez, J.T., Montero, J.: Some properties of consistency in the families of aggregation operators. In: Melo-Pinto, P., Couto, P., Serôdio, C., Fodor, J., De Baets, B. (eds.) Eurofuse 2011. AISC, vol. 107, pp. 169–176. Springer, Heidelberg (2011)

[27] Rojas, K., Gómez, D., Rodríguez, J.T., Montero, J.: Strict Stability in Aggregation Operators. Fuzzy sets and Systems (2013), doi:10.1016/j.fss.2012.12.010
[28] Yager, R.R.: On ordered weighted averaging aggregation operators in multi-criteria decision making. IEEE Transactions on Systems, Man and Cybernetics 18, 183–190 (1988)
[29] Yager, R.R., Rybalov, A.: Nonconmutative self-identity aggregation. Fuzzy Sets and Systems 85, 73–82 (1997)
[30] Yager, R.R.: Prioritized aggregation operators. International Journal of Approximate Reasoning 48, 263–274 (2008)