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Remarks on the Weak–Polynomial Convergence on a Banach Space

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We shall be concerned in this note with some questions posed by Carne, Cole and Gamelin in [3], involving the weak–polynomial convergence and its relation to the tightness of certain algebras of analytic functions on a Banach space.

Let X be a (real or complex) Banach space. In [3], a sequence $(x_j) \subset X$ is said to be weak–polynomial convergent to $x \in X$ if $P(x_j) \longrightarrow P(x)$ for all continuous polynomial P on X; and the space X is defined to be a Λ–space if, whenever (x_j) is a sequence in X which is weak–polynomial convergent to 0, then $\|x_j\| \longrightarrow 0$. It is shown in [3] that ℓ_p is a Λ–space for $1 \leqslant p < \infty$; it is also proved that $L_p(\mu)$ is a Λ–space for $2 \leqslant p < \infty$ and $L_1[0,1]$ is not a Λ–space, and the question is posed as to whether $L_p(\mu)$ is a Λ–space for $1 < p < 2$. Our next result will provide an affirmative answer to this question.

First, we recall that super–reflexive Banach spaces can be defined as those spaces which admit an equivalent uniformly convex norm. In particular, spaces $L_p(\mu)$ are super–reflexive for $1 < p < \infty$ and any measure μ (see, e.g. [6, Chap.3]).

THEOREM 1. *Every super–reflexive Banach space is a Λ–space.*

Remark. In [4], a Banach space X is defined to be in the class W_p $(1 < p < \infty)$ when each bounded sequence in X admits a weakly–p–convergent subsequence. Along the lines of Theorem 1, it can be shown that if X^* is in the class W_p for some p $(1 < p < \infty)$ then X is a Λ–space. In particular, it follows from [4] and [5] that the dual Tsirelson space T and the spaces $(\oplus \ell_\infty^n)_p$ and $(\oplus T)_p$ $(1 < p < \infty)$ are Λ–spaces. The authors like to thank Jesús F. Castillo for providing this remark (and other useful comments).

The notion of Λ–space was introduced in [3], in relation to the tightness of certain algebras of analytic functions on a (complex) Banach space. We recall that a uniform algebra A on a compact space K is said to be tight on K if, for all $g \in C(K)$, the Hankel–type operator $S_g \colon A \longrightarrow C(K)/A$ defined by $S_g(f) = fg + A$ is weakly compact. Now let Z be a complex dual Banach space, with open unit ball B, and let $A(B)$ be the algebra generated by the weak*–continuous linear functionals on the closed unit ball $\overline{B}$ (regarded as functions on the weak*–compact set $\overline{B}$). It is proved in [3] that if $A(B)$ is tight on $\overline{B}$, then Z is reflexive. It is also proved that if Z is an infinite–dimensional Λ–space with the metric approximation property, then $A(B)$ is not tight. We shall give an extension of this last

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result.

First, we define a Banach space X to be a κ–space if there exists a weakly null sequence in X which is not weak–polynomial convergent to 0. In other words, X is a κ–space if, and only if, there exists a continuous polynomial P on X which is not weakly sequentially continuous; it is clear that P can be chosen to be m–homogeneous, for some m. Concerning κ–spaces, we have the following

PROPOSITION. *Let X be a Banach space.*

1) If X is a reflexive, infinite–dimensional Λ–space, then X is a κ–space.

2) If X is reflexive and a quotient of X is a κ–space, then X is a κ–space.

3) If X has a weakly null normalized Schauder basis (a_n) and there exists a continuous linear operator $T: X \longrightarrow \ell_p$ $(1<p<\infty)$ such that (Ta_n) is the canonical basis of ℓ_p, then X is a κ–space.

4) If a complemented subspace of X is a κ–space, then X is a κ–space.

Remark. In Proposition above, *3)* applies whenever X is a Banach space of finite cotype with a weakly null unconditional basis. Arguments of this kind have been also used in [2] and [1] to find a continuous polynomial which is not weakly sequentially continuous on the quasi–reflexive James space, J, and the dual Tsirelson space, T, respectively. On the other hand, *4)* covers a wide class of operator spaces defined on a κ–space. For example, the spaces $L(X)$ and $K(X)$, of bounded linear and compact linear operators on X are κ–spaces if X is.

Finally, it follows from [7] or [3, 7.1] that any infinite–dimensional Banach space with the Dunford–Pettis property is not a κ–space.

THEOREM 2. *Let Z be a complex dual Banach space. Suppose that Z is a κ–space with the approximation property. Then $A(B)$ is not tight on $\overline{B}$.*

Remark. The following examples may be interesting:

(A) The original Tsirelson space, T^*, is a reflexive space with an unconditional basis, which does not have any quotient isomorphic to ℓ_p $(1<p<\infty)$ and which is not a κ–space (it is shown in [1] that every continuous polynomial on T^* is weakly sequentially continuous). Therefore T^* is a Banach space for which [3, 9.3] and [3, 9.4] and our theorem 2 cannot be applied. Hence, the tightness of $A(B)$, for T^*, remains open.

(B) If Z is a reflexive κ–space with the approximation property, then $E = T^* \times Z$ is also a reflexive κ–space with the approximation property. So, E provides examples of Banach spaces satisfying our theorem 2, which are not Λ–spaces.

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