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Technical Appendix to Cost Reduction and Consumer Search

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TECHNICAL APPENDIX TO

COST REDUCTION AND CONSUMER SEARCH

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Abstract: This text develops formally the dynamic model discussed in Pereira (98).

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1 Introduction

This text is a self-contained formal exposition of the dynamic model discussed in Pereira (98). I develop the model in section 2 and in section 3 I characterize equilibrium. In section 4 I do comparative statics.

2 The Model

In this section I present the model.

Consider a market for a non-storable homogeneous (search) good that opens for 2 periods.

Each of the game's 2 periods is composed of 3 stages. In period 1, first firms simultaneously choose investments; each firm observes only its cost realization. Second, firms simultaneously choose prices. And third, consumers simultaneously make their search and purchase decisions; then production and delivery take place instantaneously, and agents receive their period 1 payoffs. In period 2, first firms suffer a shock to their marginal production costs; each firm observes only its cost realization. Second, firms simultaneously choose prices; customers learn, free of charge, the current price of the firm they purchased from in period 1. And third, consumers simultaneously make their search and purchase decisions; then production and delivery take place instantaneously, agents receive their period 2 payoffs, and the market closes.

There is a continuum of consumers of unit measure. Consumers are identical and risk neutral. A consumer who buys at price p demands $x(p)$, where $x(.)$: $(0,+\infty) \longrightarrow (0,+\infty)$ is a twice differentiable, bounded function with a bounded inverse, decreasing and strictly concave. The surplus of a consumer who pays p is $S(p) := \int_p^\infty x(t)\mathrm{d}t$.

To obtain a price quote from a firm a consumer must pay a constant amount $\sigma \in (0,+\infty)$: the ***search cost.*** Within each period, search is instantaneous, a consumer may solicit any number of price quotes, and may at any time accept any offer received to date. Consumers learn in period 2, free of charge, the current price of the firm they bought from in period 1. This creates a ***switching cost***, equal to the expected search expenditure. I assume:

(A.1) Each consumer picks at random which firm to sample, from the set of firms whose price he does not know.

A consumer's ***information set*** just after his k-th search (or return) step in period t, H_{tk}, consists of all prices previously observed[1]. A consumer's ***strategy*** for stage 3 of period t is a stopping rule, $s_t(H_{tk})$, that for every

[1] In period 2, it includes the current price charged by the firm the consumer bought from in period 1.

possible search cost, and sequence of observations, says whether search should stop or continue. A consumer's ***payoff*** is the sum of expected period consumer surpluses, net of the search expenditure.

There is a continuum of firms of unit measure. Firms are risk neutral and may differ in marginal production costs.

A firm's ***cost reducing*** investment $a \in [0,+\infty)$ generates marginal cost level c_l with probability $\mu(a)$, and marginal cost level c_h with probability $1 - \mu(a)$, where $0 \leq c_l < c_h < +\infty$. This specification contains four assumptions: first, marginal production costs are constant; second, the cost type distribution is at most binary; third, each firm's probability of having a low cost depends only on its investment; and fourth, the support of the cost distribution is independent of the firms' individual and aggregate investments. I assume $\mu(.)$: $[0,+\infty) \longrightarrow [0,1]$ is a twice differentiable function, strictly increasing and concave: $\mu''(a) < 0 < \mu'(a)$, $\forall a \geq 0$; no investment induces a degenerate distribution at $c = c_h$: $\mu(0) = 0$; and firms can never get a low cost with probability 1: $\mu(a) < 1$, $\forall a < +\infty$, and $\lim_{a \to +\infty} \mu(a) = 1$. Later I will make an assumption that guarantees that in equilibrium firms make positive investments. Denote the realized value of variable y by $\bar{y}$.

To analyze ***idiosyncratic cost volatility*** denote the probability that a firm who had cost level c_τ in period 1 has cost level c_s in period 2 by $\nu_{\tau s}(\gamma)$ $(\tau,s = l,h)$; $\nu_{\tau s}$ is a differentiable function of $\gamma \in R$, a parameter which measures the shock's intensity. I assume: the shock is independent across all firms, and identically distributed across firms with the same cost type, that the probability a period 1 low cost firm remains low cost in period 2 is higher than the probability a period 1 high cost firm becomes low cost in period 2: $\nu_{ll} > \nu_{hl}$, and that a rise in γ increases the probability of a firm changing its cost type: $\nu'_{ll} < 0 < \nu'_{hl}$. The probability of a firm having a low cost in period 2 is $m(a,\gamma) := \mu(a)\nu_{ll}(\gamma) + [1 - \mu(a)]\nu_{hl}(\gamma)$.

Decompose the high cost as $c_h = c_l + \Delta$, $\Delta > 0$; c_l can be interpreted as a ***common cost*** component. I will use deterministic shifts[2] in c_l in period 2 to analyze the effects of ***industry wide cost volatility***.

The per consumer profit of a firm with cost c_τ who charges price p is $\pi(p;c_\tau) := (p - c_\tau)x(p)$ $(\tau = l,h)$. Let $p^m_\tau := \arg\max_p \pi(p;c_\tau)$. I call p^m_τ the cost c_τ firm's ***monopoly price***. By strict concavity of demand p^m_τ is unique and strictly increasing in c_τ. I assume only low cost firms can charge p^m_l without losing money, i.e., $p^m_l < c_h$. The expected period 1 consumer measure (or share) of a firm that charges price p in period 1 is $\varphi_1(p)$, and the expected period 2 consumer measure of a firm that charges price p' in period 2 and had consumer measure $\bar{\varphi}_1$ in period 1 is $\varphi_2(p';\bar{\varphi}_1)$. A firm's period t expected profit equals its period t per consumer profit, times the expected consumer measure: $\Pi(p;c_\tau,\varphi_t) := \pi(p;c_\tau)\varphi_t$. The sum of expected period profits is: $\Lambda(p,c_\tau,p',p'',c_s,\nu_{\tau\tau},\nu_{\tau s},\tilde{\varphi}_1) := \Pi(p;c_\tau,\varphi_1(p)) + [\nu_{\tau\tau}\,\Pi(p',c_\tau,\phi_2(p';\bar{\varphi}_1)) + \nu_{\tau s}\,\Pi(p'';c_s,\phi_{2s}(p'';\bar{\varphi}_1))]$ $(\tau,s = l,h)$.

2 A stochastic shock, perfectly correlated across firms, generates the same qualitative results.

A firm's ***information set*** in period t just before stage 2, H^τ_t, consists of the firm's investment level, cost realization and consumer measure realizations to date, and all prices it charged to date. A firm's stage 1 ***strategy*** is an investment level. A firm's stage 2 ***strategy*** is a pricing rule that for each possible history, says which price the firm should charge. A firm's ***payoff*** is the sum of expected period profits, net of the investment expenditure.

When a firm chooses to charge a price higher than the maximum consumers are willing to pay, and hence forgoes the chance of selling its product, I say the firm is ***inactive***; otherwise I say the firm is ***active***. I assume that consumers can only learn whether a firm is inactive through search.

The solution concept is a refinement of Nash equilibrium. First I restrict attention to symmetric pure strategies. Recall that consumers are identical, and that after uncertainty is resolved there are two firm types. Next I introduce the two remaining restrictions. Consumers do not know the prices charged by individual firms. However, they hold common beliefs about the price distribution across firms. I assume[3]:

(A.2) Consumers' search strategy satisfies ***sequential rationality***, i.e., consumers choose whether to search again to maximize net expected surplus, given the previously observed prices and their conjecture of the price distribution at the unsearched firms, conditional on any observed information.

(A.3) Consumers' beliefs about the price distribution satisfy the ***independent prices conjecture***, i.e., consumers believe firms choose prices independently and maintain this assumption throughout the search process.

The cumulative distribution function, $F_t(.;H_{tk})$, gives the consumers' beliefs about the (unconditional) market price distribution for period t;[4] the lowest and highest prices on its support are $\underline{p}_t$ and $\bar{p}_t$; $T(.\,|\,q)$ gives the consumers' beliefs about the price a firm charges in period 2, conditional on having charged price q in period 1. The price of a firm with cost c_τ in period t is $p_{t\tau}$.

An ***equilibrium*** is: a stopping rule for each period, consumer beliefs, a pricing rule for each period and cost type, and an investment level, $\left\{s^*_1(H_{1k}), s^*_2(H_{2k}), F^*_1(.;H_{1k}), F^*_2(.;H_{2k}), T^*(.\,|\,.), p^*_{1l}, p^*_{1h}, p^*_{2l}(H^l_2), p^*_{2h}(H^h_2), a^*\right\}$, such that:

3 I follow Bagwell & Ramey (96). See Pereira (98).

4 $F_t(p\,|\,H_t)$ gives the consumers' beliefs about the proportion of firms that charge a price no greater than p in period t.

(i) Given beliefs $F_1^*(.;H_{1k})$, $T^*(.|.)$, $F_2^*(.;H_{2k})$, and the search cost σ, consumers choose stopping rules $s_1^*(H_{1k})$ and $s_2^*(H_{2k})$ to maximize the net sum of the expected period surplus;

(ii) Given the stopping rules $s_1^*(H_{1k})$, $s_2^*(H_{2k})$, and the cost shock, firms choose pricing rules $p_{1\tau}^*$, $p_{2\tau}^*(H_2^\tau)$, and investment level a^*, to maximize the net sum of expected profits, i.e., to respectively solve the problems:

$$\max_p \Pi(p; c_\tau, \phi_1(p)),\ \tau = l,h$$

$$\max_p \Lambda(p_{1\tau}, c_\tau; p_{2\tau}, p_{2s}, c_s, \nu_{\tau\tau}, \nu_{\tau s}, \bar{\varphi}_1),\ \tau = l,h$$

$$\max_a \mu(a)\, \Lambda(p_{1l}, c_l; p_{2l}, p_{2h}, c_h, \nu_{ll}, \nu_{lh}, \bar{\varphi}_1(p_{1l})) + [1 - \mu(a)]\, \Lambda(p_{1h}, c_h; p_{2l}, p_{2h}, c_l, \nu_{hh}, \nu_{hl}, \bar{\varphi}_1(p_{1h})) - a$$

(iii) Beliefs $F_1^*(.;H_{1k})$, $T(.|.)$, and $F_2^*(.;H_{2k})$ agree with the price distributions induced by the cost shock, investment level a^*, and pricing rules $p_{2\tau}{}^*(H_2^\tau)$, $p_{1\tau}^*$.

3 Characterization of Equilibrium

In this section I construct the equilibrium by working backwards. The consumers' equilibrium behavior consists of holding reservation prices. Low cost firms are always active and charge their monopoly price. High cost firms, for either period, are sometimes active, others inactive, which allows for four types of equilibria. When high cost firms are active they charge the minimum of the reservation price and their monopoly price.

3.1 Second Period

3.1.1 Third Stage: The Search Game

In this sub-section I characterize the consumers' period 2 search equilibrium.

Given (A.2) consumers optimize with respect to beliefs, which given (A.3) do not depend of the prices observed. Thus, the consumer's search problem can be solved using dynamic programming. Under my assumptions sequential search is optimal (Morgan & Manning (85), proposition 3).

The period 2 maximum expected surplus, net of the search expenditure, of a consumer who's best available offer is p and behaves optimally is $V_0(p)$. After receiving an offer a consumer must choose between one of two actions: accept the best available offer and terminate search, the value of which is $S(p)$, or, draw a new price at cost σ, and subsequently behave optimally, the expected value of which is $K_0(\sigma) := -\sigma + \int V_0(p^*)\, dF_2(p')$, where $p^* = \min\{p,p'\}$. Search should stop when a sufficiently attractive price is observed. The Bellman equation of the consumer's problem is:

$$V_0(p) = \max\{S(p), K_0\} \tag{1}$$

Given that demand is bounded and (A.1), (1) has a well defined and finite optimal value function and optimal search terminates in a finite number of steps with probability 1 (De Groot: Lemma 1, p. 350, Th. 1, p. 347).

Given that $S_p < 0$, that $S(+\infty) \le K_0(+\infty) < K_0(\sigma) < S(\underline{p}_2)$, for $\sigma \in (0,+\infty]$, and that the value of search is decreasing in σ, it follows from the intermediate value theorem that for every $\sigma \in (0, \bar{\sigma}_2]$, $\bar{\sigma}_2 \le +\infty$, equation:

$$S(\rho_2) = -\sigma + \int_{\underline{p}_2}^{\bar{p}_2} V_0(p^*)\, dF_2(p) \tag{2}$$

has a unique solution $\rho_2 \in (0,+\infty]$. Given ρ_2, and using (2) and $S_p < 0$, it follows that:

$$V_0(p) = \begin{cases} S(p) & \Leftarrow \quad p < \rho_2 \\ S(\rho_2) & \Leftarrow \quad p \ge \rho_2 \end{cases} \tag{3}$$

, and that the optimal period 2 search strategy consists of holding a reservation price, ρ_2. Using (3) on (2):

$$\int_{\underline{p}_2}^{\rho_2} [S(p) - S(\rho_2)]\, dF_2(p) = \sigma \tag{4}$$

From (4) it follows that for every strictly positive search cost, the period 2 reservation price is strictly bigger that the lowest price charged in period 2: $\forall \sigma > 0$, $\underline{p}_2 < \rho_2$.

3.1.2 Second Stage: The Pricing Game

In this sub-section I characterize the prices charged in equilibrium in period 2.

If a firm charges a price higher than the reservation price, $p > \rho_2$, it makes no sales. If a firm charges a price no bigger than the reservation price, $p \le \rho_2$, it keeps its period 1 customers[5], and in addition, given (A.1) and that there is a continuum of consumers and firms, it gets an expected consumer measure equal to the measure of consumers searching in period 2 (consumers that in period 1 bought from a firm that is inactive in period 2) divided by the measure of active firms. The expected consumer measure of a firm that charges p is:

$$\phi_2(p, \rho_2, \bar{\varphi}_1) = \begin{cases} 0 & \Leftarrow \quad p > \rho_2 \\ \bar{\varphi}_1 + \dfrac{\Delta C}{n_2} & \Leftarrow \quad p \le \rho_2 \end{cases}$$

where n_t is the measure of firms that in period t charge a price no bigger than the reservation price (i.e., active firms), and ΔC is the measure of consumers searching in period 2 (I omit n_2 in ϕ_2). Since $\underline{p}_2 < \rho_2$, $n_2 > 0$.

[5] Recall that there is a switching cost.

Lemma 1: In equilibrium, in period 2: **(i)** Price is non-decreasing in the cost level: $\underline{p}_2 = p_{2l} \le p_{2h} = \tilde{p}_2$; **(ii)** The low cost firms' price is strictly lower than the reservation price: $p_{2l} < \rho_2$; **(iii)** Low cost firms charge their monopoly price: $p_{2l} = p_l^m$; **(iv)** When the reservation price is no smaller than the high cost level, high cost firms charge the minimum of the reservation price and their monopoly price; otherwise, they are inactive:

$$p_{2h} = \begin{cases} \min(\rho_2, p_l^m) & \Leftarrow \quad c_h \le \rho_2 \\ \hat{p}_2 \in (\rho_2, +\infty) & \Leftarrow \quad c_h > \rho_2 \end{cases}$$

Proof: See Pereira (98). ¶

Using **Lemma 1**, when the reservation price is no smaller than the high cost level, all firms are active; otherwise, only low cost firms are active:

$$n_2 = \begin{cases} 1 & \Leftarrow \quad \rho_2 \in [c_h, +\infty] \\ \tilde{m} & \Leftarrow \quad \rho \in (p_l^m, c_h) \end{cases}$$

Furthermore, the period 2 market cumulative distribution function of prices is:

$$\text{Prob}[P_2 \le p \mid \tilde{m}] = \begin{cases} 0 & \Leftarrow \quad p < p_{2l} \\ \tilde{m} & \Leftarrow \quad p_{2l} \le p < p_{2h} \\ 1 & \Leftarrow \quad p \ge p_{2h} \end{cases} \qquad (5)$$

3.2 First Period

3.2.1 Third Stage: The Search Game

In this sub-section I characterize the consumers' period 2 search equilibrium. Now the consumers' problem is more complicated since the reward function may not be monotonic[6] on price. Nevertheless, it can be solved using dynamic programming and sequential search is optimal.

The period 1 maximum expected surplus, net of the search expenditure, of a consumer who's acceptance set is A_1 and behaves optimally is $V_1(A_1)$. The expected value in period 1 of drawing a new price at cost σ, and subsequently behaving optimally is $K_1(\sigma) := -\sigma + \int V_1(A_1)\, dF_1$. The period 2 net maximum expected surplus, of a consumer who's best available offer in period 1 is p and behaves optimally is $G(p) = \int V_0(u)\, dH(u \mid p)$. The value of accepting the best available offer p and terminating search in period 1 is $S(p) + G(p)$. The Bellman equation of the consumers' problem is:

[6] When this occurs, the optimal set of acceptable prices may be disconnected and the reservation price property lost.

$$V_1(A_1) = \max\{S(p) + G(p), K_1\}$$

As before, the consumer's problem has a well defined and finite optimal value function, and optimal search terminates in a finite number of steps with probability 1.

If consumers search the expression for $V_1(A_1)$ can be written more explicitly as:

$$V_1(A_1) = -\sigma + V_1(A_1)\left[1 - \int_{A_1} dF_1(p)\right] + \int_{A_1} [S(p) + G(p)]\, dF_1(p)$$

The optimal acceptance set is A_1^*. Some manipulation gives:

$$\int_{A_1^*} [S(p) + G(p) - V_1(A_1^*)]\, dF_1(p) = \sigma \qquad (6)$$

which defines A_1^*. Following Hey (79), let $A_1^+ := \{p \mid S(p) + G(p) - V_1^* > 0\}$, $A_1^0 := \{p \mid S(p) + G(p) - V_1^* = 0\}$, and $A_1^- := \{p \mid S(p) + G(p) - V_1^* < 0\}$. The sets A_1^+ and A_1^- are non-empty given that $S(+\infty) + G(+\infty) \le K_1(+\infty) < K_1(\sigma) < S(\underline{p}_1) + G(\underline{p}_1)$, for $\sigma \in (0, +\infty)$, and that the value of the search problem is decreasing in σ; and A_1^0 is then non-empty given the continuity of $S(.) + G(.)$. Then, $A_1^* = A_1^+ \cup A_1^{00}$, where A_1^{00} is any subset of A_1^0, i.e., a consumer's optimal strategy is to: stop to search when he observes a price on A_1^*, otherwise continue to search.

3.2.2 Second Stage: The Pricing Game

In this sub-section I characterize the firms' optimal pricing strategy, and I further characterize the consumers' period 2 optimal search strategy.

By a previous argument, the expected consumer share of a firm that charges p is:

$$\phi_1(p, \rho_1) = \begin{cases} 0 & \Leftarrow \quad p \in A_1^* \\ \dfrac{1}{n_1} & \Leftarrow \quad p \notin A_1^* \end{cases}$$

As before $n_1 > 0$.

Lemma 2: In equilibrium, in period 1: **(0)** Low cost firms are always active: $p_{1l} \in A_1^*$; **(i)** The price of active firms is non-decreasing in the cost level: $p_{1\tau} \in A_1^*, \forall \tau \Rightarrow \underline{p}_1 = p_{1l} \le p_{1h} = \tilde{p}_1$; **(ii)** If price is decreasing in the cost level, then high cost firms are inactive: $p_{1h} < p_{1l} \Rightarrow p_{1h} \notin A_1^*$; **(iii)** The low cost firms' price is strictly lower than the highest acceptable price: $p_{1l} < \sup A_1^*$; **(iv)** Low cost firms charge their monopoly price: $p_{1l} = p_l^m$.

Proof: (0) Notice first that at least one type of firms must be active, otherwise the left-hand side of (6) is zero while the search cost is strictly positive. Now suppose $p_{1h} \in A_1^*$ and $p_{1l} \notin A_1^*$. If it is optimal for high cost firms to be active, then it must be the case that charging p_{1h} earns them a non-negative profit. Thus, if a low cost firm deviates and charges p_{1h} it will also make a strictly positive profit, contradicting the optimality of p_{1l}.

(i) Noting that the expected consumer measure is constant with respect to price when $p_{1\tau} \in A_1^*$, for all τ, the argument in Pereira (96) applies.

(ii) (i) implies that if the price is decreasing in the cost level, then both types of firms cannot be active. Using (0) the result follows.

(iii) The case $p_{1l} < p_{2l}$ is obvious. If $p_{1h} < p_{2l}$, then $p_{1l} < sup\ A_1^*$ must hold otherwise the left-hand side of (6) is zero while the search cost is strictly positive.

(iv) As in Pereira (98). ¶

Given Lemmas 1 and 2, and the definition of $v_{\tau\varsigma}$, the distribution of the price a firm charges in period 2, conditional on charging price q in period 1 is:

$$\text{Prob}[P_2 \le p \mid P_1 = q] = \begin{cases} \begin{cases} 0 & \Leftarrow \; p < \underline{p}_2 \\ v_{ll} & \Leftarrow \; \underline{p}_2 \le p < \overline{p}_2 \\ 1 & \Leftarrow \; p \ge \overline{p}_2 \end{cases} & \Leftarrow \; q = p_{1l} \\ \begin{cases} 0 & \Leftarrow \; p < \underline{p}_2 \\ v_{hl} & \Leftarrow \; \underline{p}_2 \le p < \overline{p}_2 \\ 1 & \Leftarrow \; p \ge \overline{p}_2 \end{cases} & \Leftarrow \; q \ne p_{1l} \end{cases}$$

Next I will further characterize the consumers' optimal search strategy. Given **Lemmas 1 and 2**, and the expression above, $G(p_{1l}) = v_{ll}S(p_l^m) + (1-v_{ll})S(\rho_2)$, and assuming the equilibrium refinement that when consumers observe a firm charging a price different from p_l^m they infer that the firm has a high cost $G(p') = v_{hl}S(p_l^m) + (1-v_{hl})S(\rho_2)$. Hence, $G(p) - G(p') = (v_{ll} - v_{hl})[S(p_l^m) - S(\rho_2)] > 0$. Hence, for $p_l^m < p'$, $S(.) + G(.)$ is decreasing, and the set $A_1^0 \cap \{p|p \ge p_l^m\}$ is a singleton; denote its value by ρ_1. Furthermore, $A_1^* \cap \{p|p \ge p_l^m\} = [p_l^m, \rho_1]$, which allows after some manipulation allows one to write equation (6) as

$$\int_{\underline{p}_1}^{\rho_1} [S(p) - S(\rho_1)]\, dF_1(p) + \int_{\underline{p}_1}^{\rho_1} [G(p) - G(\rho_1)]\, dF_1(p) = \sigma \tag{7}$$

Thus, the optimal period 1 strategy consists of holding a reservation price, ρ_1. Equation (7) holds for $\sigma \in (0, S(p_h^m)]$, and $S(p_h^m) < \overline{\sigma}_2$, so from now on restrict attention to $\sigma \in (0, S(p_h^m)]$.

Next I will characterize the high cost firms' optimal pricing strategy. I assume:

(A.4) $\pi(p_l^m, c_h) + v_{ll}p(p_l^m, c_l) + (1 - v_{ll})\pi(p_h^m, c_h) < 0$

(A.4) ensures that high cost firms become inactive in period 1, when the period 1 reservation price becomes *sufficiently* low[7]. Next I introduce notation. The period 1 reservation price value, r_1, that makes high cost firms indifferent between being active and inactive in period 1, when the period 1 reservation is below the high cost level and period 2 reservation price no lower than the high cost level, $\rho_1 < c_h \le \rho_2$, is defined by:

$$\frac{1}{n_1}\left[\pi(r_1, c_h) + v_{hl}\pi(p_l^m, c_l) + (1 - v_{hl})\pi(p_{2h}, c_h)\right] = 0$$

The period 1 reservation price value, r_2, that makes high cost firms indifferent between being active or inactive in period 1, when both reservation prices are below the high cost level, $\rho_1, \rho_2 < c_h$, is defined by:

$$\frac{1}{n_1}\left[\pi(r_2, c_h) + v_{hl}\pi(p_l^m, c_l)\right] = 0$$

Lemma 3: In equilibrium, in period 1:**(i)** The value r_1 is a continuous decreasing function of the minimum of the high cost firms' monopoly price and period 2 reservation price, and the high cost firms' transition probability, $r_1(v_{hl}, min(\rho_2, p_h^m))$, such that r_1: $(0, v_{ll}) \times [c_h, p_h^m] \longrightarrow (\underline{r}_1, \overline{r}_1)$, where $\overline{r}_1 := r_1(0, c_h) = c_h$ and $\underline{r}_1 := r_1(v_{ll}, p_h^m)$. When the period 1 reservation price is below the high cost level and period 2 reservation price no lower than the high cost level, the set of parameter values for which high cost firms are active in period 1 is non-empty, and the set of values for which high cost firms are inactive in period 1 is also non-empty. The value r_2 is a continuous decreasing function of the high cost firms' transition probability, $r_2(v_{hl})$, such that r_2: $(0, v_{ll}) \longrightarrow (\underline{r}_2, \overline{r}_2)$, where $\overline{r}_2 := r_2(0) = c_h$ and $\underline{r}_2 := r_2(v_{ll})$. When both reservation prices are below the high cost level, the set of parameter values for which high cost firms are active in period 1 is non-empty, and the set of values for which high cost firms are inactive in period 1 is also non-empty. **(ii)** When the reservation price is no smaller than the high cost level, high cost firms charge the minimum of the reservation price and their monopoly price; when the reservation price is lower than the high cost level, but not *too low* high cost firms charge the reservation price; otherwise high cost firms are inactive, i.e.,

[7] The set of parameters values for which high cost firms are inactive in period 1 is non-empty.

$$p_{1h} = \begin{cases} \min\{\rho_1, p_l^m\} & \Leftarrow \quad c_h \le \rho_1 \\ \rho_1 & \Leftarrow \quad (r_1 \le \rho_1 < c_h \le \rho_2) \text{ or } (r_2 \le \rho_1 < c_h, \rho_2 < c_h) \\ \hat{p}_1 \in (\rho_1, +\infty) & \Leftarrow \quad (p_l^m \le \rho_1 < r_1 < c_h \le \rho_2) \text{ or } (p_l^m \le \rho_1 < r_2 < c_h, \rho_2 < c_h) \end{cases}$$

Proof: (i) When $\rho_2 \ge c_h$, if a firm a high cost firm charges $p_{1h} = \rho_1$ its payoff is

$$\pi(\rho_1, c_h)\frac{1}{n_1} + \left[v_{hl}\pi(p_l^m, c_l) + (1 - v_{hl})\pi(p_{2h}, c_h)\right]\left(\frac{1}{n_1} + \frac{\Delta C}{n_2}\right) \tag{8}$$

and if charges $p_{hi} < \rho_1$ its payoff is:

$$\left[v_{hl}\pi(p_l^m, c_l) + (1 - v_{hl})\pi(p_{2h}, c_h)\right]\left(\frac{\Delta C}{n_2}\right) \tag{9}$$

Equating (8) and (9) gives $(1/n_1)\left[\pi(r_1, c_h) + v_{hl}\pi(p_l^m, c_l) + (1 - v_{hl})\pi(p_{2h}, c_h)\right] = 0$. Define $\psi_1(r_1; v_{hl}, p_{2h}) := (1/n_1)\left[\pi(r_1, c_h) + v_{hl}\pi(p_l^m, c_l) + (1 - v_{hl})\pi(p_{2h}, c_h)\right]$. It is straightforward that $0 < \psi_1(c_h; v_{hl}, p_{2h}), \forall(v_{hl}, p_{2h})$; condition **(A.4)** implies that $\psi_1(p_l^m; v_{hl}, p_{2h}) < 0, \forall(v_{hl}, p_{2h})$; and since ψ_1 is monotonic on all its arguments, it follows from the intermediate value theorem that for every (v_{hl}, p_{2h}) on $(0, v_{ll}) \times (c_h, p_h^m)$, there is one and only one r_1 on (p_l^m, c_h). The implicit function theorem implies that $r_1 = r_1(v_{hl}, p_{2h})$ with $r_1 : (0, v_{ll}) \times [c_h, p_h^m] \longrightarrow [\underline{r}_1, \bar{r}_1]$. Since $\bar{r}_1 < r_1(0, c_h) = c_h$, there is a ρ_l on (r_1, c_h), for every v_{hl} and p_{2h}. **(A.4)** implies that $p_l^m < r_1(v_{ll}, p_h^m) < \underline{r}_1$, hence there is a ρ_1 on (p_l^m, r_1), for every v_h and p_{2h}. Case $\rho_2 < c_h$ is similar.

(ii) Follows from (i) and **Lemma 1.** ¶

Using **Lemmas 2** and **3** the measure of active firms in period 1 is:

$$n_1 = \begin{cases} 1 & \Leftarrow \quad \rho_1 \in [c_h, +\infty) \\ \tilde{\mu} & \Leftarrow \quad \rho_1 \in (p_l^m, c_h) \end{cases}$$

the measure of consumers searching in period 2 is:

$$\Delta C = \begin{cases} 0 & \Leftarrow \quad (c_h \le \rho_1, \rho_2) \text{ or } (\rho_1 < c_h < \rho_2) \\ 1 - \tilde{m} & \Leftarrow \quad r_2 \le \rho_1, \rho_2 < c_h \\ v_l & \Leftarrow \quad p_l^m \le \rho_1 < r_2, \rho_2 < c_h \end{cases}$$

the period 1 market cumulative distribution function of prices is:

$$\text{Prob}[P_1 \le p \mid \tilde{\mu}] = \begin{cases} 0 & \Leftarrow \quad p < p_{1l} \\ \tilde{\mu} & \Leftarrow \quad p_{1l} \le p < p_{1h} \\ 1 & \Leftarrow \quad p \ge p_{1h} \end{cases} \tag{10}$$

and, $G(p_{1l}) = v_{ll}S(p_l^m) + (1 - v_{ll})S(\bar{p}_2)$, $G(p \neq p_l^m) = v_{hl}S(p_l^m) + (1 - v_{hl})S(\bar{p}_2)$, thus

$$G(p_{1l}) - G(p) = (v_{ll} - v_{hl})[S(p_l^m) - S(\bar{p}_2)] \tag{11}$$

3.2.3 First Stage: The Investment Game

In this sub-section I characterize the investment equilibrium.

Assume that $\mu'(0)$ is big enough[8] to guarantee that it is never optimal to set investment to zero. The necessary condition for the investment problem is:

$$\mu'(a^*)[\Pi(p_{1l}; c_l, \phi_1(p_{1l})) - \Pi(p_{1h}; c_h, \phi_1(p_{1h}))] + m_a(a^*, \gamma)\{\Pi(p_{2l}; c_l, \phi_2(p_{2l}; \phi_1(p_{1l}))) - \Pi(p_{2h}; c_h, \phi_2(p_{2h}; \phi_1(p_{1h})))\} - 1 = 0 \tag{12}$$

3.4 Equilibrium of the Whole Game: Existence and Stability

In this sub-section I show that equilibrium exists and discuss stability.

Given **Lemmas 1-3** there can be four types of price equilibria depending on whether high cost firms choose to be active or inactive in each of the of the two periods. The characterization of the investment and search best response functions is a straightforward application of the implicit function theorem (see appendix).

Using **Lemmas 1-3**, equation (12) defines the firms' investment best response function:

$$a = A(\rho_1, \rho_2; c_l, \gamma) \tag{13}$$

which is of the form: $A(.) : [p_l^m, +\infty) \times [p_l^m, +\infty)) \times [0, +\infty) \times (0, +\infty) \longrightarrow [\underline{a}, \bar{a}]$, where $\underline{a} := A(p_h^m, p_h^m, 0, +\infty)$, $\bar{a} := A(r_2 - \varepsilon, c_h - \varepsilon, +\infty, -\infty)$, $\varepsilon > 0$, and is differentiable, except at $\rho_1 = r_1$, r_2 and $\rho_2 = c_h$, where it has upward discontinuities: $\hat{A}(\alpha; c_l, \gamma, \sigma) < \lim_{a \longrightarrow \alpha^-} \hat{A}(a; c_l, \gamma, \sigma)$. When both reservation prices are higher than the high cost firms' monopoly price, $\rho_t > p_h^m$, both t, investment does not depend on the reservation prices. When within a type of equilibrium at least one of the reservation prices is lower than the high cost firms' monopoly price, $\rho_t < p_h^m$, some t, investment falls with that reservation price. When due to a rise in a reservation price causes high cost firms to become active for a period, investment falls discontinuously. Investment falls with idiosyncratic cost volatility.

Assume that the period 1 realized measure of low cost firms equals the expected measure of low cost firms: $\tilde{\mu} = \int_0 \mu(a(i))\, di$.[9] Given symmetry it follows that:

$$\tilde{\mu} = \int_0^1 \mu(a(i))\, di = \mu(a) \tag{14}$$

, i.e., $\tilde{\mu} = \tilde{\mu}(a)$, with $\tilde{\mu}'' < 0 < \tilde{\mu}'$; $\tilde{m} = m(a, \gamma)$, i.e., $\tilde{m} = \tilde{m}(a, \gamma)$, with $\tilde{m}_{aa} < 0 < \tilde{m}_a$, and $\tilde{m}_{a\gamma} < 0$.

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[8] That is, $\mu'(0) > 1/[\Lambda(p_l^m, c_l; p_l^m, p_h^m c_h, v_l) - \Lambda(p_h^m, c_h; p_l^m, p_h^m, c_l, v_h)]$.

[9] This assumption is justifiable because the investment trials are made independently.

Using (5) and (14), equation (4) defines the period 2 search best response function:

$$\rho_2 = R^2(a;c_l,\gamma,\sigma) \qquad (15)$$

which is of the form $R^2(.): [0,+\infty)\times[0,+\infty)\times(-\infty,+\infty)\times(0,+\infty) \longrightarrow [p_l^m,+\infty)$, is continuously differentiable, falls with investment, rises with the search cost, the low cost level, and with idiosyncratic cost volatility if the period 2 proportion of low cost firms falls with idiosyncratic cost volatility, $\text{sgn}\{\partial R^2/\partial\gamma\} = -\text{sgn}(m_\gamma)$.

Using (10), (11), (14), and (15), equation (7) defines the period 1 search best response function:

$$\rho_1 = R^1(a;c_l,\gamma,\sigma) \qquad (16)$$

which is of the form $R^1(.): [0,+\infty)\times[0,+\infty)\times(-\infty,+\infty)\times(0,+\infty) \longrightarrow [p_l^m,+\infty)$, is continuously differentiable, falls with investment, rises with the search cost, the low cost level, and idiosyncratic cost volatility.

Equilibrium is given by equations (13), (15) and (16).

To discuss stability consider the following adjustment process, consisting of a succession of rounds, each composed of two stages. In the first stage of each round, firms choose an investment level which is a best response to the period 1 and period 2 reservation prices chosen by the consumers in the previous round. In the second stage, consumers choose period 1 and period 2 reservation prices which are best responses to the investment level chosen by firms in the first stage of that round. A steady state $\{a^*,\rho_1^*,\rho_2^*\}$ of the adjustment process is an equilibrium: $\{a^*,\rho_1^*,\rho_2^*\} = \{A(\rho_1^*,\rho_2^*;c_l,\gamma), R^1(a^*;c_l,\gamma,\sigma), R^2(a^*;c_l,\gamma,\sigma)\}$. An equilibrium $\{a^*,\rho_1^*,\rho_2^*\}$ is ***locally asymptotically stable*** for the adjustment process, if there exists a neighborhood of $\{a^*,\rho_1^*,\rho_2^*\}$ such that for any initial point on the neighborhood, the adjustment process converges to $\{a^*,\rho_1^*,\rho_2^*\}$; otherwise an equilibrium is ***unstable.*** Let $\varepsilon(y,x)$ denote the elasticity of y with respect to x.

Proposition 1: (i) Equilibrium exists. **(ii)** Equilibrium is unique if globally:

$$\varepsilon(A,\rho_1)\,\varepsilon(R^1,a) + \varepsilon(A,\rho_2)\,\varepsilon(R^2,a) < 1 \qquad (17)$$

(iii) Equilibria for which (22) holds locally, are ***locally asymptotically stable,*** otherwise they are ***unstable.***

Proof: (i) Let $a_1 := (R^1)^{-1}(r_1;c_l,\gamma,\sigma)$, and $a_2 := (R^1)^{-1}(r_2;c_l,\gamma,\sigma)$, $a_3 := (R^2)^{-1}(c_h;c_l,\gamma,\sigma)$. Replacing (13) and (14) on (11) one gets the mapping: $a = \hat{A}(a;c_l,\delta,\sigma) := A(R^1(a;c_l,\delta,\sigma),R^2(a;c_l,\delta,\sigma);c_l,\delta)$, $\hat{A}(.): [\underline{a},\bar{a}]\times[0,+\infty)\times(-\infty,+\infty)\times(0,\bar{\sigma}) \longrightarrow [\underline{a},\bar{a}]$, which is continuous with respect to a, except at a_1, a_2, and a_3, where it has upward discontinuities ($A_{\rho i} < 0$, and $R^i_a < 0$). Thus, by Tarski's fixed point theorem $\hat{A}(.)$, has a fixed point for every (c_l,δ,σ) on $[0,+\infty)\times(-\infty,+\infty)\times(0,\bar{\sigma})$.

(ii) Let $g(a;c_l,\gamma,\sigma) := \hat{A}(a;c_l,\gamma,\sigma) - a$, g: $\hat{A}(.): [\underline{a},\bar{a}]\times(0,1)\times(0,v_{ll})\times(0,\bar{\sigma}) \longrightarrow \Re$ and $g(.)$ is continuous with respect to a, except at a_1, a_2, and a_3., Since $g(\underline{a};c_l,\gamma,\sigma) > 0$, $g(\bar{a};c_l,\gamma,\sigma) < 0$, a sufficient condition for uniqueness is $\partial g/\partial a < 0$, i.e., $\partial g/\partial a = (\partial A/\partial\rho_1)(\partial R^1/\partial a) + (\partial A/\partial\rho_2)(\partial R^2/\partial a) - 1 < 0$,

(iii) See Pereira (98) ¶

4 Comparative Statics

In this section I do the local comparative statics. I show that rise in either idiosyncratic or industry wide cost volatility or the search cost can reduce investment, and rise the reservation prices.

Proposition 2: At locally stable equilibria: **(i)** If the period 2 reservation price is not between the high cost level and the high cost firms' monopoly price, $\rho_2 \notin [c_h, p_h^m]$, or, if the period 2 proportion of low cost firms is non-increasing in idiosyncratic cost volatility,

$$m_\gamma = \mu v_{ll}' + (1-\mu)v_{hl}' \le 0 \qquad (18)$$

then a rise in ***idiosyncratic cost volatility*** reduces investment and increases both reservation prices; otherwise a change in idiosyncratic cost volatility can have a potentially ambiguous impact on investment and the reservation prices. **(ii)** A rise in ***industry wide cost volatility*** reduces investment and increases both reservation prices. **(iii)** A rise in the search cost reduces investment and increases both reservation prices.

Proof: Differentiating the system:

$$\begin{cases} a - A(\rho_1,\rho_2;c_l,\gamma) = 0 \\ \rho_1 - R^1(a;c_l,\gamma,\sigma) = 0 \\ \rho_2 - R^1(a;c_l,\gamma,\sigma) = 0 \end{cases}$$

gives

$$\begin{bmatrix} \frac{\partial a}{\partial\gamma} & \frac{\partial a}{\partial c_l} & \frac{\partial a}{\partial\sigma} \\ \frac{\partial\rho_1}{\partial\gamma} & \frac{\partial\rho_1}{\partial c_l} & \frac{\partial\rho_1}{\partial\sigma} \\ \frac{\partial\rho_2}{\partial\gamma} & \frac{\partial\rho_2}{\partial c_l} & \frac{\partial\rho_2}{\partial\sigma} \end{bmatrix} = M^{-1}\begin{bmatrix} A_\gamma + A_{\rho_1}R^1_\gamma + A_{\rho_2}R^2_\gamma & A_{c_l} + A_{\rho_1}R^1_{c_l} + A_{\rho_2}R^2_{c_l} & A_{\rho_2}R^2_\sigma + A_{\rho_1}R^1_\sigma \\ R^1_\gamma[1 - A_{\rho_2}R^2_a] + A_{\rho_2}R^2_\gamma R^1_a + A_\gamma R^1_a & R^1_{c_l}[1 - A_{\rho_2}R^2_a] + A_{\rho_2}R^2_{c_l}R^1_a + A_{c_l}R^1_a & A_{\rho_2}R^1_a R^2_\sigma + R^1_\sigma[1 - A_{\rho_2}R^2_a] \\ R^2_\gamma[1 - A_{\rho_1}R^1_a] + A_{\rho_1}R^1_\gamma R^2_a + A_\gamma R^2_a & R^2_{c_l}[1 - A_{\rho_1}R^1_a] + A_{\rho_1}R^1_{c_l}R^2_a + A_{c_l}R^2_a & A_{\rho_1}R^1_\sigma R^2_a + R^2_\sigma[1 - A_{\rho_1}R^1_a] \end{bmatrix}$$

with $M = 1 - A_{\rho_2}R^2_a - A_{\rho_1}R^1_a > 0$ given (17). Thus: $\partial a/\partial c_l < 0$, $\partial\rho_1/\partial c_l > 0$, $\partial\rho_2/\partial c_l > 0$; $\partial a/\partial\sigma < 0$, $\partial\rho_1/\partial\sigma > 0$, $\partial\rho_2/\partial\sigma > 0$; and if $R^2_\gamma > 0$ or $A_{\rho_2} = 0$: $\partial a/\partial\gamma > 0$, $\partial\rho_1/\partial\gamma > 0$, $\partial\rho_2/\partial\gamma > 0$. ¶

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Appendix

Using Lemmas 1-3 (12) can be written as:

$$\begin{cases}
\mu'(a)\left[\pi(p_l^m;c_l)-\pi(\rho_1;c_h)\right] + m_a(a,\gamma)\left[\pi(p_l^m;c_l)-\pi(\rho_2;c_h)\right] - 1 = 0 & r_1 \le \rho_1 \le p_h^m, c_h \le \rho_2 \le p_h^m \\
\mu'(a)\left[\pi(p_l^m;c_l)-\pi(\rho_1;c_h)\right] + m_a(a,\gamma)\left[\pi(p_l^m;c_l)-\pi(p_h^m;c_h)\right] - 1 = 0 & r_1 \le \rho_1 \le p_h^m, p_h^m < \rho_2 \\
\mu'(a)\left[\pi(p_l^m;c_l)-\pi(p_h^m;c_h)\right] + m_a(a,\gamma)\left[\pi(p_l^m;c_l)-\pi(\rho_2;c_h)\right] - 1 = 0 & p_h^m < \rho_1, c_h \le \rho_2 \le p_h^m \\
\mu'(a)\left[\pi(p_l^m;c_l)-\pi(p_h^m;c_h)\right] + m_a(a,\gamma)\left[\pi(p_l^m;c_l)-\pi(p_h^m;c_h)\right] - 1 = 0 & p_h^m < \rho_1, \rho_2 \\
\mu'(a)\left[\pi(p_l^m;c_l)-\pi(\rho_1;c_l)\right] + \left(\frac{m_a(a,\gamma)}{\tilde{m}(a,\gamma)}\right)\pi(p_l^m;c_l) - 1 = 0 & r_2 \le \rho_1 < p_h^m, \rho_2 < c_h \\
\mu'(a)\left[\pi(p_l^m;c_l)-\pi(p_h^m;c_l)\right] + \left(\frac{m_a(a,\gamma)}{\tilde{m}(a,\gamma)}\right)\pi(p_l^m;c_l) - 1 = 0 & p_h^m < \rho_1, \rho_2 < c_h \\
\left(\frac{\mu'(a)}{\tilde{\mu}(a)}\right)\pi(p_l^m;c_l) + \left(\frac{\mu'(a)}{\tilde{\mu}(a)}\right)\left[v_{ll}\pi(p_l^m;c_l) + (1 - v_{ll})\pi(\rho_2;c_h)\right] - 1 = 0 & \rho_1 < r_1, \rho_2 \le p_h^m \\
\left(\frac{\mu'(a)}{\tilde{\mu}(a)}\right)\pi(p_l^m;c_l) + \left(\frac{\mu'(a)}{\tilde{\mu}(a)}\right)\left[v_{ll}\pi(p_l^m;c_l) + (1 - v_{ll})\pi(p_h^m;c_h)\right] - 1 = 0 & \rho_1 < r_1, p_h^m < \rho_2 \\
\left(\frac{\mu'(a)}{\tilde{\mu}(a)}\right)\pi(p_l^m;c_l) + \left[\left(\frac{\mu'(a)}{\tilde{\mu}(a)}\right)(1 - v_{ll}) + \left(\frac{\tilde{m}_a(a,\gamma)}{\tilde{m}(a,\gamma)}\right)v_{ll}\right]\pi(p_l^m;c_l) - 1 = 0 & \rho_1 < r_2, \rho_2 < c_h
\end{cases}$$

which defines implicitly $a = A(\rho_1,\rho_2;c_l,\gamma)$ and shows that when due to a fall in a reservation price the model switches between types of equilibria, the marginal benefit of investment rises discontinuously.

Using (5) and (14), (4) can be written as

$$\begin{cases}
\tilde{m}(a,\gamma)\left[S(p_l^m) - S(\rho_2)\right] - \sigma = \quad \Leftarrow \quad \rho_2 \le p_h^m \\
\tilde{m}(a,\gamma)\left[S(p_l^m) - S(p_h^m)\right] + \left[S(p_h^m) - S(\rho_2)\right] - \sigma = 0 \quad \Leftarrow \quad \rho_2 > p_h^m
\end{cases}$$

which defines implicitly $\rho_2 = R^2(a;c_l,\gamma,\sigma)$. Straightforward differentiation shows that R^2 is strictly decreasing in a, and strictly increasing in c_l, σ and $\mathrm{sgn}\left\{\partial R^2/\partial\gamma\right\} = -\mathrm{sgn}(m_\gamma)$.

Using (10), (11), (14), and (15), (7) can be written as

$$\begin{cases}
\tilde{\mu}(a)\left\{\left[S(p_l^m) - S(\rho_1)\right] + (v_{ll} - v_{hl})\left[S(p_l^m) - S(\rho_2)\right]\right\} - \sigma = 0 \Leftarrow \rho_1, \rho_2 < p_h^m \\
\tilde{\mu}(a)\left\{\left[S(p_l^m) - S(\rho_1)\right] + (v_{ll} - v_{hl})\left[S(p_l^m) - S(p_h^m)\right]\right\} - \sigma = 0 \Leftarrow \rho_1 < p_h^m < \rho_2 \\
\tilde{\mu}(a)\left[S(p_l^m) - S(p_h^m)\right] + \left|S(p_h^m) - S(\rho_1)\right| + \tilde{\mu}(a)(v_{ll} - v_{hl})\left[S(p_l^m) - S(\rho_2)\right] - \sigma = 0 \Leftarrow \rho_2 < p_h^m < \rho_1 \\
\tilde{\mu}(a)\left\{\left[S(p_l^m) - S(p_h^m)\right] + (v_{ll} - v_{hl})\left[S(p_l^m) - S(p_h^m)\right]\right\} + \left|S(p_h^m) - S(\rho_1)\right| - \sigma = 0 \Leftarrow p_h^m < \rho_1, \rho_2
\end{cases}$$

or using $R^2(.)$ and defining $\varpi(a,\gamma,\sigma) := \left\{1 - \tilde{\mu}(a)(v_{ll}(\gamma) - v_{hl}(\gamma))/\left[v_{hl}(\gamma) + \tilde{\mu}(\alpha)(v_{ll}(\gamma) - v_{hl}(\gamma))\right]\right\}\sigma > 0$

$$\begin{cases}
\tilde{\mu}(a)\left[S(p_l^m) - S(\rho_1)\right] - \varpi(a,\gamma,\sigma) = 0 \Leftarrow \rho_1, \rho_2 < p_h^m \\
\tilde{\mu}(a)\left\{\left[S(p_l^m) - S(\rho_1)\right] + (v_{ll} - v_{hl})\left[S(p_l^m) - S(p_h^m)\right]\right\} - \sigma = 0 \Leftarrow \rho_1 < p_h^m < \rho_2 \\
\tilde{\mu}(a)\left[S(p_l^m) - S(p_h^m)\right] + \left|S(p_h^m) - S(\rho_1)\right| - \varpi(a,\gamma,\sigma) = 0 \Leftarrow \rho_2 < p_h^m < \rho_1 \\
\tilde{\mu}(a)\left\{\left[S(p_l^m) - S(p_h^m)\right] + (v_{ll} - v_{hl})\left[S(p_l^m) - S(p_h^m)\right]\right\} + \left|S(p_h^m) - S(\rho_1)\right| - \sigma = 0 \Leftarrow p_h^m < \rho_1, \rho_2
\end{cases}$$

which defines implicitly $\rho_l = R^1(a;c_l,\gamma,\sigma)$. Differentiating ϖ: $\varpi_\gamma = \sigma\left[\tilde{\mu}(v_{ll} - v_{hl})v'_{hl}\right]/\tilde{m}^2 > 0$, $\varpi_a = -\sigma\left[\tilde{\mu}'(v_{ll} - v_{hl})v_{hl}\right]/\tilde{m}^2 < 0$, $\varpi_\sigma = \left\{1 - \tilde{\mu}(v_{ll} - v_{hl})/\left[v_{hl} + \tilde{\mu}(v_{ll} - v_{hl})\right]\right\} > 0$. Consider ρ_l, $\rho_2 < p_h^m$ (the others cases are similar). Then: $\partial\rho_l/\partial a = -\left\{\tilde{\mu}'\left[S(p_l^m) - S(\rho_l)\right] - \varpi_a\right\}/\tilde{\mu}x(\rho_l) < 0$, $\partial\rho_l/\partial\gamma = \varpi_\gamma/\tilde{\mu}x(\rho_l) > 0$, $\partial\rho_l/\partial\sigma = \varpi_\sigma/\tilde{\mu}x(\rho_l) > 0$, $\partial\rho_l/\partial c_l = \tilde{\mu}x(p_l^m)\left(\partial p_l^m/\partial c_l\right)/\tilde{\mu}x(\rho_l) > 0$, and therefore R^1 is strictly decreasing in a and strictly increasing in c_l, γ, and σ.