Instituto Complutense de Análisis Económico
UNIVERSIDAD COMPLUTENSE
FACULTAD DE ECONOMICAS
Campus de Somosaguas
28223 MADRID

Teléfono 394 26 11 - FAX 294 26 13

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Rafael Caballero
Emilio Cerdá
María del Mar Muñoz
Lourdes Rey

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RELATIONS AMONG SEVERAL EFFICIENCY CONCEPTS IN STOCHASTIC MULTIPLE OBJECTIVE PROGRAMMING*

R. Caballero[1], E. Cerdá[2], M.M. Muñoz[1], and L. Rey[1].

[1]Departamento de Economía Aplicada (Matemáticas)
Universidad de Málaga.

[2]Departamento de Fundamentos del Análisis Económico.
Universidad Complutense de Madrid.

ABSTRACT

In this paper, the resolution of stochastic multiple objective programming problems is studied. The existence of random parameters in the objective functions has yielded to the definition of several efficient solution concepts for such problems in the literature. We will focus our attention in the study of some of these concepts, namely, minimun risk and β probability. Once these concepts are defined, the relations among the sets of efficient solutions obtained are studied.

RESUMEN

En este artículo se estudia la resolución de problemas de programación estocástica multiobjetivo. La existencia de parámetros aleatorios en las funciones objetivo ha dado lugar a varios conceptos de solución eficiente para estos problemas que aparecen en diferentes trabajos. Nos centramos en el estudio de algunos de estos conceptos, concretamente, mínimo riesgo y probabilidad β. Tras la definición de estos conceptos, se estudian las relaciones entre los conjuntos de soluciones eficientes obtenidos.

Keywords: Multiple Objective Programming, Stochastic Programming, Efficiency.

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1. Introduction

Let us consider the following Stochastic Multiple Objective Programming Problem:

Problem SMP:

$$\min_{\mathbf{x}} \mathbf{z}(\mathbf{x},\boldsymbol{\xi}) = (z_1(\mathbf{x},\boldsymbol{\xi}), z_2(\mathbf{x},\boldsymbol{\xi}), \ldots, z_q(\mathbf{x},\boldsymbol{\xi}))^t$$
$$\text{s.t. } \mathbf{x} \in \mathbf{D}$$

,where $\mathbf{x}\in\mathbf{R}^n$ is the vector of decision variables of the problem and $\boldsymbol{\xi}$ is a random vector defined on a set $\mathbf{E}\subset\mathbf{R}^k$. We assume that the family of events $\mathbf{F}$ is given and that for every $A\epsilon\mathbf{F}$ the probability of A, P(A), is known. We also assume that the distribution of probability, P, is independent of the decision variables $x_1, x_2, \ldots, x_n$.

We assume that the functions $z_1(\mathbf{x},\boldsymbol{\xi}), z_2(\mathbf{x},\boldsymbol{\xi}), \ldots, z_q(\mathbf{x},\boldsymbol{\xi})$ are defined in the space $\mathbf{R}^n\times\mathbf{E}$. We also assume that the set $\mathbf{D}\subset\mathbf{R}^n$ is compact, convex and nonempty and that it is a deterministic set or it has been transformed to its deterministic equivalent by the criterion of chance constraints.

In this paper we solve the problem in two stages. In the first stage we transform the stochastic problem into another multiple objective problem that is its deterministic equivalent, according to some of the criteria of transformation of the stochastic objectives. In the second stage the set of efficient solutions of the deterministic multiple objective problem obtained in stage one is calculated. For a stochastic problem it is possible to obtain different sets of efficient solutions, one for each of the criteria used to obtain the deterministic equivalent. The choice of the criterion will depend on the characteristics of the decision process that generate the problem.

In this paper we consider two different concepts of efficiency : Minimum risk efficiency for levels $u_1, u_2, \ldots, u_q$ and efficiency with probabilities $\beta_1, \beta_2, \ldots, \beta_q$ and we analyse the relationship between these two concepts.

2. Minimum risk efficiency for levels $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_q$.

This concept of solution, defined by Stancu-Minasian and Tigan (1984), considers efficient solutions of the Problem SMP to the efficient solutions of the multiple objective deterministic problem that is obtained when we apply the minimum risk criterium to each of the objective functions of the problem. For applying this criterion it is necessary to fix a level of minimum satisfaction for each of the stochastic objectives $u_1, u_2, \ldots, u_q$, $u_k \in\mathbf{R}$, k=1,2...,q. When these values are fixed, the minimum risk problem, the deterministic equivalent of Problem SMP, consists of maximizing the probability that each of the stochastic objectives does not surpass the fixed satisfaction level, in such a way that the deterministic equivalent to Problem SMP is:

Problem MR(**u**)

$$\max_{\mathbf{x}} (P(z_1(\mathbf{x},\boldsymbol{\xi}) \leq u_1), \ldots, P(z_q(\mathbf{x},\boldsymbol{\xi}) \leq u_q))^t$$
$$\text{s.t. } \mathbf{x} \in \mathbf{D}$$

For this problem, Stancu-Minasian and Tigan (1984) define the concept of vectorial solution minimum risk of level $\mathbf{u}$ for the Problem SMP in the following way:

Definition 1.

$\mathbf{x}\in$D is a vectorial solution minimum risk of level $\mathbf{u}$ if it is an efficient solution to Problem MR($\mathbf{u}$).

From now on,we shall call these solutions as efficient minimum risk solutions of levels $u_1, u_2, ..., u_q$.Denote by $E_{MR}(\mathbf{u})$, the set of efficient solutions to the Problem MR($\mathbf{u}$).

The multiple objective deterministic equivalent problem that is obtained applying this criterion, Problem MR($\mathbf{u}$),depends,in general,on the fixed vector of satisfaction levels $\mathbf{u}$, in such a way that, in general, given $\mathbf{u}$, $\mathbf{u}' \in \mathbf{R}^q$,if $\mathbf{u}\neq\mathbf{u}'$,then the sets of efficient minimum risk solutions of levels $\mathbf{u}$ and $\mathbf{u}'$ will be different: $E_{MR}(\mathbf{u})\neq E_{MR}(\mathbf{u}')$.

3. Efficient solutions with probabilities $\beta_1, \beta_2, ..., \beta_q$.

The concept of efficiency with probabilities $\beta_1, \beta_2, ..., \beta_q$, is a generalization of a concept defined previously by Goicoechea, Hansen and Duckstein (1982), the concept of stochastic nondominated solution of level β that they define in the following way:

Definition 2: Stochastic nondominated solution of level β.

Let $z_k(\mathbf{x})$ be a value belonging to the rank of the random variable $z_k(\mathbf{x},\boldsymbol{\xi})$,$k = 1, 2, ..., q$.

$\mathbf{x}\in\mathbf{D}$ is a stochastic nondominated solution of level $\beta \in (0,1)$ if:

(i) $P\{z_k(\mathbf{x},\boldsymbol{\xi}) \leq z_k(\mathbf{x})\} = \beta$, $\forall k \in \{1, 2, ..., q\}$

(ii)There does not exist a vector $\mathbf{y}\in\mathbf{D}$, such that:

* $P(z_k(\mathbf{y},\xi) \leq z_k(\mathbf{y})) = \beta$, $\forall$ k=1,2,...,q.

* $\exists l \in \{1, 2, ..., q\}$,such that $z_l(\mathbf{y})<z_l(\mathbf{x})$

* $z_k(\mathbf{y}) \leq z_k(\mathbf{x})$,$\forall k \in \{1, 2, ..., q\}$, $k \neq l$

From this definition, given the Stochastic Multiple Objective Programming Problem, if we apply the Kataoka criterion to each of the stochastic objective functions of the problem with a probability β,we obtain the following problem:

$$\min_{(\mathbf{x}^t,\mathbf{u}^t)} \mathbf{u} = (u_1, u_2, ..., u_q)^T$$
$$s.t. P\{z_k(\mathbf{x},\boldsymbol{\xi}) \leq u_k\} = \beta, \text{ k=1,2,...,q}$$
$$\mathbf{x} \in \mathbf{D}$$

and we find that the set of efficient solutions to this problem is the set of nondominated solutions of level β previously defined, because for each k$\in$ $\{1, 2, ..., q\}$ the variable u_k will be a function $z_k(\mathbf{x})$ that is obtained from the equality $P\{z_k(\mathbf{x},\xi) \leq u_k\} = \beta$.In this way we find that the set of nondominated solutions of level β is obtained from the application of the Kataoka criterion to each of the objective functions of the stochastic multiple objective problem,fixing the same level of probability for all the stochastic functions.

From this concept, it is possible to generalize the idea, considering different levels of probability for the objective functions of the problem in the following way:

Problem K(β)

$$\min_{(\mathbf{x}^t,\mathbf{u}^t)} \mathbf{u} = (u_1, u_2, ..., u_q)^t$$
$$s.t. P\{z_k(\mathbf{x},\boldsymbol{\xi}) \leq u_k\} = \beta_k, \text{ k=1,2,...,q}$$
$$\mathbf{x}\in\mathbf{D}$$

Definition 3.

Let $\mathbf{x}\in\mathbf{D}$. We say that $\mathbf{x}$ is an efficient solution with probabilities $\beta_1, \beta_2, ..., \beta_q$ if there exists $\mathbf{u}\in\mathbf{R}^q$ such that $(\mathbf{x}^t,\mathbf{u}^t)^t$ is an efficient solution of the Problem K(β).

Denote by $\mathbf{E}_K(\beta) \subseteq \mathbf{R}^n$ the set of efficient solutions with probabilities $\beta = (\beta_1, \beta_2, ..., \beta_q)^t$.

Note that the concept of efficient solution with probabilities $\beta_1, \beta_2, ..., \beta_q$ is defined for the vectors $\mathbf{x}$, though the solutions of the problem to solve are vectors $(\mathbf{x}^t,\mathbf{u}^t)^t \in \mathbf{R}^{n+q}$.

As in the case of minimum risk,this concept of efficiency is associated with some levels of probability previously fixed, and therefore the deterministic multiple objective problem in which efficient solutions with probabilities $\beta_1, \beta_2, ..., \beta_q$ are obtained (Problem K(β), depends, in general,on the fixed vector of probabilities,$\beta = (\beta_1, \beta_2, ..., \beta_q)^t$.Then, in general, given $\beta, \beta' \in \mathbf{R}^q$,if $\beta \neq \beta'$,the set of efficient solutions for β is different to the one obtained for β':$\mathbf{E}_K(\beta) \neq \mathbf{E}_K(\beta')$.

4. Relations among the efficient minimum risk solutions of level $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_q$ and the efficient solutions with probabilities $\beta_1, \beta_2, ..., \beta_q$.

From the Problem SMP, let us consider the following problems:

Problem MR($\mathbf{u}$):

$$\max_{\mathbf{x}} \left(P\{z_1(\mathbf{x},\xi) \leq u_1\}, ..., P\{z_q(\mathbf{x},\boldsymbol{\xi}) \leq u_q\}\right)^t$$
$$s.t. \mathbf{x} \in \mathbf{D}$$

and

Problem K(β):

$$\min_{(\mathbf{x}^t,\mathbf{u}^t)} \mathbf{u}=(u_1, u_2, ..., u_q)^t$$
$$s.t.\ P\{z_k(\mathbf{x},\xi) \leq u_k\} = \beta_k$$
$$\mathbf{x} \in \mathbf{D}$$

corresponding to the deterministic programs from which we obtain the efficient solutions minimum risk of levels $u_1, u_2, ..., u_q$ and to the efficient solutions with probabilities $\beta_1, \beta_2, ..., \beta_q$ for the Problem SMP.Now we are going to analyse the relations among the sets of efficient points of these two problems.

We assume that the feasible sets of both problems, $\mathbf{D}\subset \mathbf{R}^n$ and

$$\{(\mathbf{x}^t,\mathbf{u}^t)^t \in \mathbf{D}\times\mathbf{R}^q \mid P\{z_k(\mathbf{x},\boldsymbol{\xi}) \leq u_k\} = \beta_k\}$$

are closed ,bounded and nonempties,and therefore both problems have efficient solutions.

We also assume that for every k$\in$ $\{1,2,...,q\}$,and for every $\mathbf{x}\in\mathbf{D}$, the distribution function of the random variable $z_k(\mathbf{x},\xi)$ is continuous and strictly increasing.These hypotheses imply that for every probability β_k,there exists a unique real number u_k,such that $P\{z_k(\mathbf{x},\xi) \leq u_k\} = \beta_k$.

Let $\mathbf{E}_{MR}(\mathbf{u})$ be the set of efficient solutions to the Problem MR(**u**), and $\mathbf{E}_K(\beta)$ the set of efficient solutions to the Problem K(β). The following theorem relates both sets.

Theorem 1

Assume that the distribution function of the random variable $z_k(\mathbf{x},\xi)$ is continuous and strictly increasing.Then **x** is an efficient solution to Problem MR(**u**) if and only if $(\mathbf{x}^t,\mathbf{u}^t)^t$ is an efficient solution to Problem K(β), with **u** and β such that:

$$P\{z_k(\mathbf{x},\xi) \leq u_k\} = \beta_k \text{ , } \forall k \in \{1,2,...,q\}$$

Proof

We demonstrate the theorem by *reductio ad absurdum.*

(a) If **x** is an efficient solution to Problem MR(**u**), then $(\mathbf{x}^t,\mathbf{u}^t)^t$ is an efficient solution to Problem K(β).

It is clear that $(\mathbf{x}^t,\mathbf{u}^t)^t$ is a feasible solution to Problem K(β).

Let us suppose that $(\mathbf{x}^t,\mathbf{u}^t)^t$ is not efficient in Problem K(β).Then there exists a feasible vector $(\mathbf{x}'^t,\mathbf{u}'^t)^t$ that dominates to $(\mathbf{x}^t,\mathbf{u}^t)^t$, and therefore it is verified that:

$$\begin{array}{l} \mathbf{x}' \in \mathbf{D} \\ P\{z_k(\mathbf{x}',\xi) \leq u'_k\} = \beta_k = P\{z_k(\mathbf{x},\xi) \leq u_k\} \text{ ,} \forall k \in \{1,2,...,q\} \\ u'_k \leq u_k \text{ , } \forall k = 1,2,...q \text{ and } u'_s < u_s \text{,for some s} \in \{1,2,...,q\} \end{array}$$

In accordance with the properties of the distribution function of the random variable $z_k(\mathbf{x},\xi)$,we have that if $u'_k \leq u_k$ and $u'_s < u_s$ then :

$$\begin{array}{rcl} P\{z_k(\mathbf{x}',\xi) \leq u'_k\} & \leq & P\{z_k(\mathbf{x}',\xi) \leq u_k\} \\ P\{z_s(\mathbf{x}',\xi) \leq u'_s\} & < & P\{z_s(\mathbf{x}',\xi) \leq u_s.\} \end{array}$$

Therefore,

$$\begin{array}{rcl} P\{z_k(\mathbf{x},\xi) \leq u_k\} & = & P\{z_k(\mathbf{x}',\xi) \leq u'_k\} \leq P\{z_k(\mathbf{x}',\xi) \leq u_k\} \\ P\{z_s(\mathbf{x},\xi) \leq u_s\} & = & P\{z_s(\mathbf{x}',\xi) \leq u'_s\} < P\{z_s(\mathbf{x}',\xi) \leq u_s\} \end{array}$$

and **x** is not an efficient solution to Problem MR(**u**) , which contradicts the hypothesis.

(b) If $(\mathbf{x}^t,\mathbf{u}^t)^t$ is an efficient solution to Problem K(β), then **x** is an efficient solution to Problem MR(**u**).

It is clear that $\mathbf{x}\in\mathbf{D}$.

Suppose that **x** is not an efficient solution to Problem MR(**u**), then there exists a feasible vector $\mathbf{x}' \in\mathbf{D}$, and it is verified that :

$$\begin{array}{rcl} \beta_k & = & P\{z_k(\mathbf{x},\xi) \leq u_k\} \leq P\{z_k(\mathbf{x}',\xi) \leq u_k\} \text{ ,k=1,2,...,q.} \\ \beta_s & = & P\{z_s(\mathbf{x},\xi) \leq u_s\} < P\{z_s(\mathbf{x}',\xi) \leq u_s\} \text{ , for some s} \in \{1,2,...,q\} \end{array}$$

For the properties of the distribution function we know that there exist $u'_1,u'_2,...,u'_q$, with

$u'_k \leq u_k, \forall k \in \{1,2,...,q\}$, and there exists at least one s$\in$ $\{1,2,...,q\}$, such that $u'_s < u_s$, verifying that:

$$\begin{array}{rcl} \beta_k & = & P\{z_k(\mathbf{x},\xi) \leq u_k\} = P\{z_k(\mathbf{x}',\xi) \leq u'_k\} \text{ ,} \forall k \in \{1,2,...,q\} \\ \beta_s & = & P\{z_s(\mathbf{x},\xi) \leq u_s\} = P\{z_s(\mathbf{x}',\xi) \leq u'_s\} \text{ } s \in \{1,2,...,q\} \end{array}$$

which is in contradiction with the hypothesis.

Corollary 1.

$$\cup_{\mathbf{u}\in R^q}(\mathrm{E}_{MR}(\mathbf{u})) = \cup_{\beta\in B}(\mathrm{E}_K(\beta))$$

, with $\beta = \{\beta = (\beta_1,\beta_2,...,\beta_q) \in \mathbf{R}^q \mid \beta_k \in (0,1),$ k=1,2,...,q$\}$.

The proof of this corollary is immediate from the previous results.

From the results obtained we have that the unions of the sets of efficient points of both problems coincide.Moreover, if $\mathbf{x}\in\mathbf{D}$ is an efficient solution to Problem K(β), for some fixed probabilities $\beta = (\beta_1,\beta_2,...,\beta_q)^t$,from theorem 1, we know that it is also an efficient minimum risk solution of levels $u_1,u_2,...,u_q$,maintaining for the satisfaccion levels and the probabilities the relation that appears in the theorem , and vice versa.This result permits us to perform the analysis of these efficient solutions by one of the two concepts and, from theorem 1, to obtain the level or the probability for which it is efficient in accordance with the other.

5. Application of the Cantelli inequality to the distribution function of the stochastic objective

In previous sections we have studied the concept of efficient minimum risk solution of levels $u_1,u_2,...,u_q$ and the concept of efficient solution with probabilities $\beta_1,\beta_2,...,\beta_q$ .We also have obtained the relationship betwwen the two concepts.In this section we are going to present an approach to try to study some cases in which it is very difficult, if not impossible, to obtain these solutions.Let us note that in order to obtain efficient solutions to Problems MR(**u**) and K(β), it is necessary to know the probability distribuion of the stochastic objectives of Problem SMP, and this is not always possible.We propose using Cantelli inequality (Rao (1973)) in order to obtain some insight in these cases.

Cantelli inequality

Let ξ be a random variable, with expected value $E(\xi)$ and finite variance σ^2_ξ.Then:

$$\begin{array}{rcl} P\{\xi - E(\xi) \leq \lambda\} & \leq & \dfrac{\sigma^2_\xi}{(\sigma^2_\xi + \lambda^2)} \text{ ,if } \lambda < 0 \\ P\{\xi - E(\xi) \leq \lambda\} & \geq & 1 - \dfrac{\sigma^2_\xi}{(\sigma^2_\xi + \lambda^2)} = \dfrac{\lambda^2}{(\sigma^2_\xi + \lambda^2)} \text{ , if } \lambda \geq 0 \end{array}$$

Let us suppose that we know the expected value of the random variable $z_k(\mathbf{x},\xi)$, $E\{z_k(\mathbf{x},\xi)\}$, and its variance $Var\{z_k(\mathbf{x},\xi)\}$ .Suppose also that the feasible set of the deterministic equivalent ,**D**, is such that the variance is finite and its value is different from zero for all feasible **x**.

In this case, if we apply the Cantelli inequality to the distribution function of the k objective , for $u_k \geq E\{z_k(\mathbf{x},\boldsymbol{\xi})\}$,taking $\lambda = u_k - E\{z_k(\mathbf{x},\boldsymbol{\xi})\} \geq 0$, we obtain:

$$\begin{aligned} P(z_k(\mathbf{x},\boldsymbol{\xi}) &\leq u_k) = P(z_k(\mathbf{x},\boldsymbol{\xi}) - E\{z_k(\mathbf{x},\boldsymbol{\xi})\} \leq u_k - E\{z_k(\mathbf{x},\boldsymbol{\xi})\} \geq \\ &\geq \frac{(u_k - E\{z_k(\mathbf{x},\boldsymbol{\xi})\})^2}{Var(z_k(\mathbf{x},\boldsymbol{\xi})) + (u_k - E\{z_k(\mathbf{x},\boldsymbol{\xi})\})^2} \end{aligned}$$

If we substitute the distribution functions of the objectives by these bounds in Problem MR(**u**), we obtain the following new problem:

Problem AMR(**u**):

$$\max_{\mathbf{x}} \left(\frac{(u_1 - E\{z_1(\mathbf{x},\boldsymbol{\xi})\})^2}{Var(z_1(\mathbf{x},\boldsymbol{\xi})) + (u_1 - E\{z_1(\mathbf{x},\boldsymbol{\xi})\})^2}, \ldots, \frac{(u_q - E\{z_q(\mathbf{x},\boldsymbol{\xi}\})^2}{Var(z_q(\mathbf{x},\boldsymbol{\xi})) + (u_q - E\{z_q(\mathbf{x},\boldsymbol{\xi})\})^2} \right)^t$$

$$s.t. \quad E\{z_k(\mathbf{x},\boldsymbol{\xi})\} \leq u_k \; k = 1,2,\ldots,q$$
$$\mathbf{x} \in \mathbf{D}$$

It is clear that the set of efficient solutions to Problem AMR(**u**) , that we denote by $\mathbf{E}_{AMR}(\mathbf{u})$, in general does not coincide with the set of efficient solutions of Problem MR(**u**). That´s to say that $\mathbf{E}_{AMR}(\mathbf{u}) \neq \mathbf{E}_{MR}(\mathbf{u})$, and the set $\mathbf{E}_{AMR}(\mathbf{u})$ can only be taken as an approximation of the set $\mathbf{E}_{MR}(\mathbf{u})$.

On the other side ,if we define the set:

$$S = \left\{ (\mathbf{x}^t, \mathbf{u}^t)^t \in \mathbf{R}^{n+q} \mid \sqrt{\frac{\beta_k}{1-\beta_k}} \sqrt{Var\{z_k(\mathbf{x},\boldsymbol{\xi})\}} + E\{z_k(\mathbf{x},\boldsymbol{\xi})\} \leq u_k \text{ ,k=1,2,...,q, } \mathbf{x} \in \mathbf{D} \right\}$$

,from the Cantelli inequality ,it can be proved that:

$$S \subset \{(\mathbf{x}^t, \mathbf{u}^t)^t \mid P\{z_k(\mathbf{x},\boldsymbol{\xi}) \leq u_k\} \geq \beta_k \text{ , k=1,2,...,q, } \mathbf{x} \in \mathbf{D}\}$$

and we state the problem:

Problem AK($\boldsymbol{\beta}$) :

$$\min_{(\mathbf{x},\mathbf{u})} \mathbf{u} = (u_1, u_2, \ldots, u_q)^t$$

$$s.t. \; E\{z_k(\mathbf{x},\boldsymbol{\xi})\} + \sqrt{\frac{\beta_k}{1-\beta_k}} \sqrt{Var\{z_k(\mathbf{x},\boldsymbol{\xi})\}} \leq u_k \text{ , k=1,2,...,q}$$
$$\mathbf{x} \in \mathbf{D} \subset \mathbf{R}^n$$

As in the minimum risk case, the set of efficient solutions to Problem AK($\boldsymbol{\beta}$), denoted by $\mathbf{E}_{AK}(\boldsymbol{\beta})$ will be different from the set of efficient solutions to Problem K($\boldsymbol{\beta}$).It is $\mathbf{E}_{AK}(\boldsymbol{\beta}) \neq \mathbf{E}_{K}(\boldsymbol{\beta})$,in general, but we can take the first one as an approximation of the second one.

The following theorem gives us the relation between the efficient solutions of Problems AMR(**u**) and AK($\boldsymbol{\beta}$).

Theorem 2

x is an efficient solution to Problem AMR(**u**) if and only if $(\mathbf{x}^t,\mathbf{u}^t)^t$ is an efficient solution to Problem AK($\boldsymbol{\beta}$), with **u** and $\boldsymbol{\beta}$ such that:

$$u_k = \sqrt{\frac{\beta_k}{1-\beta_k}} \sqrt{Var\{z_k(\mathbf{x},\boldsymbol{\xi})\}} + E\{z_k(\mathbf{x},\boldsymbol{\xi})\} \quad (1)$$

or equivalently

$$\beta_k = \frac{(u_k - E\{z_k(\mathbf{x},\boldsymbol{\xi})\})^2}{Var\{z_k(\mathbf{x},\boldsymbol{\xi})\} + (u_k - E\{z_k(\mathbf{x},\boldsymbol{\xi})\})^2} \text{ , for } k = 1,2,\ldots,q. \quad (2)$$

Proof..

We demonstrate the theorem by *reductio ad absurdum*.

(a)Let **x** be an efficient solution to Problem AMR(**u**), $\mathbf{x} \in \mathbf{E}_{AMR}(\mathbf{u})$.Suppose that $(\mathbf{x}^t,\mathbf{u}^t)^t$ is not efficient for Problem AK($\boldsymbol{\beta}$), with $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_q)^t$, and β_k given by (2) , $\forall k = 1,2,\ldots,q$.

It is clear that $(\mathbf{x}^t,\mathbf{u}^t)^t$ is a feasible solution to Problem AK($\boldsymbol{\beta}$).

There exists a solution $(\mathbf{x}'^t,\mathbf{u}'^t)^t$ such that:

$$\mathbf{x}' \in \mathbf{D}$$
$$\sqrt{\frac{\beta_k}{1-\beta_k}} \sqrt{Var\{z_k(\mathbf{x}',\boldsymbol{\xi})\}} + E\{z_k(\mathbf{x}',\boldsymbol{\xi})\} \leq u'_k, \forall k \in \{1,2,\ldots,q\}$$

with $u'_k \leq u_k$, for each k$\in \{1,2,\ldots,q\}$, and $u'_s < u_s$, for some s$\in \{1,2,\ldots,q\}$.

Then:

$$\sqrt{\frac{\beta_k}{1-\beta_k}} \sqrt{Var\{z_k(\mathbf{x}',\boldsymbol{\xi})\}} + E\{z_k(\mathbf{x}',\boldsymbol{\xi})\} \leq u_k ,\forall k = 1,2,\ldots,q$$

$$\sqrt{\frac{\beta_s}{1-\beta_s}} \sqrt{Var\{z_s(\mathbf{x}',\boldsymbol{\xi})\}} + E\{z_s(\mathbf{x}',\boldsymbol{\xi})\} < u_s \text{ , for some s} \in \{1,2,\ldots,q\}$$

It is clear that $E\{z_k(\mathbf{x}',\boldsymbol{\xi})\} \leq u_k ,\forall k \in \{1,2,\ldots,q\}$.

We obtain that:

$$\beta_k \leq \frac{(u_k - E\{z_k(\mathbf{x}',\boldsymbol{\xi})\})^2}{Var\{z_k(\mathbf{x}',\boldsymbol{\xi})\} + (u_k - E\{z_k(\mathbf{x}',\boldsymbol{\xi})\})^2} \text{ , } k = 1,2,\ldots,q.$$
$$\beta_s < \frac{(u_s - E\{z_s(\mathbf{x}',\boldsymbol{\xi})\})^2}{Var\{z_s(\mathbf{x}',\boldsymbol{\xi})\} + (u_s - E\{z_s(\mathbf{x}',\boldsymbol{\xi})\})^2} \text{ , s} \in \{1,2,\ldots,q\}.$$

and according to the values of β_k and β_s,we have that **x** is not an efficient solution to Problem AMR(**u**) , which is in contradiction with the hypothesis.

(b)Let $(\mathbf{x}^t,\mathbf{u}^t)^t$ be an efficient solution to Problem AK($\boldsymbol{\beta}$), with $\boldsymbol{\beta}$ given , and with u_k given by (1),$\forall k = 1,2,\ldots,q$.

It is clear than **x** is a feasible solution to Problem AMR(**u**).

Suppose that **x** is not an efficient solution to Problem AMR(**u**). Then there exists a vector $\mathbf{x}' \in \mathbf{D}$, verifying that $E\{z_k(\mathbf{x}',\boldsymbol{\xi})\} \leq u_k$

for each $k = 1,2,\ldots,q$, and such that $\mathbf{x}'$ dominates **x** , that is to say:

$$\frac{(u_k - E\{z_k(\mathbf{x},\boldsymbol{\xi})\})^2}{Var\{z_k(\mathbf{x},\boldsymbol{\xi})\} + (u_k - E\{z_k(\mathbf{x},\boldsymbol{\xi})\})^2} \leq \frac{(u_k - E\{z_k(\mathbf{x}',\boldsymbol{\xi})\})^2}{Var\{z_k(\mathbf{x}',\boldsymbol{\xi})\} + (u_k - E\{z_k(\mathbf{x}',\boldsymbol{\xi})\})^2}.$$

,and there exists an s$\in \{1,2,...,q\}$ such that :

$$\frac{(u_s - E\{z_s(\mathbf{x},\boldsymbol{\xi})\})^2}{Var\{z_s(\mathbf{x},\boldsymbol{\xi})\} + (u_s - E\{z_s(\mathbf{x},\boldsymbol{\xi})\})^2} \leq \frac{(u_s - E\{z_s(\mathbf{x}',\boldsymbol{\xi})\})^2}{Var\{z_s(\mathbf{x}',\boldsymbol{\xi})\} + (u_s - E\{z_s(\mathbf{x}',\boldsymbol{\xi})\})^2}.$$

We know that:

$$\beta_k = \frac{(u_k - E\{z_k(\mathbf{x},\boldsymbol{\xi})\})^2}{Var\{z_k(\mathbf{x},\boldsymbol{\xi})\} + (u_k - E\{z_k(\mathbf{x},\boldsymbol{\xi})\})^2}$$

which is equivalent to:

$$u_k = \sqrt{\frac{\beta_k}{1-\beta_k}}\sqrt{Var\{z_k(\mathbf{x},\boldsymbol{\xi})\}} + E\{z_k(\mathbf{x},\boldsymbol{\xi})\}$$

for every k$\in \{1,2,...,q\}$.

We have that:

$$\beta_k \leq \frac{(u_k - E\{z_k(\mathbf{x}',\boldsymbol{\xi})\})^2}{Var\{z_k(\mathbf{x}',\boldsymbol{\xi})\} + (u_k - E\{z_k(\mathbf{x}',\boldsymbol{\xi})\})^2}, k = 1,2,...,q.$$

which implies that:

$$u_k \geq \sqrt{\frac{\beta_k}{1-\beta_k}}\sqrt{Var\{z_k(\mathbf{x}',\boldsymbol{\xi})\}} + E\{z_k(\mathbf{x}',\boldsymbol{\xi})\}, k = 1,2,...,q.$$

and we have that:

$$\beta_s < \frac{(u_s - E\{z_s(\mathbf{x}',\boldsymbol{\xi})\})^2}{Var\{z_s(\mathbf{x}',\boldsymbol{\xi})\} + (u_s - E\{z_s(\mathbf{x}',\boldsymbol{\xi})\})^2}, s \in \{1,2,...,q\}.$$

and therefore:

$$u_s > \sqrt{\frac{\beta_s}{1-\beta_s}}\sqrt{Var\{z_s(\mathbf{x}',\boldsymbol{\xi})\}} + E\{z_s(\mathbf{x}',\boldsymbol{\xi})\}, \text{ for some s} \in \{1,2,...,q\}.$$

which contradicts the hypothesis that $(\mathbf{x}^t,\mathbf{u}^t)^t$ is an efficient solution to the Problem AK(β).

6. References

Ben Abdelaziz F. (1992). *L 'efficacité en programmation multiobjectifs stochastique.* Ph.D. Thesis, Université Laval, Quebec, Canada.

Goicoechea, A., D.R. Hansen and L. Duckstein (1982). *Multiobjective Decision Analysis with Engineering and Business Applications.* John Wiley.

Rao, C.R. (1973). *Linear Statistical Inference and its Applications.* Second Edition. John Wiley.

Stancu-Minasian I.M. (1984). *Stochastic Programming with Multiple Objective Functions.* D. Reidel Publishing Company.

Stancu-Minasian I.M. and S. Tigan (1984). *"The vectorial Minimum Risk Problem".* Proceedings of the Colloquium on Approximation and Optimization. Cluj-Napoca.

Stancu-Minasian, I.M. (1992). *"Stochastic Programming with Multiple Fractile Criteria".* Rev. Roumaine Math. Pures Appl. 37,10, pp. 939-941.

Szidarovszky,F., M.E. Gershon and L. Duckstein (1986). *Techniques for Multiobjective Decision Making in Systems Management.* Elsevier.